Proof-theoretic analysis by iterated reflection

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Abstract

Progressions of iterated reflection principles can be used as a tool for the ordinal analysis of formal systems. We discuss various notions of proof-theoretic ordinals and compare the information obtained by means of the reflection principles with the results obtained by the more usual proof-theoretic techniques. In some cases we obtain sharper results, e.g., we define proof-theoretic ordinals relevant to logical complexity Π_1^0 and, similarly, for any class Π_n^0.

We provide a more general version of the fine structure relationships for iterated reflection principles (due to U. Schmerl [25]). This allows us, in a uniform manner, to analyze main fragments of arithmetic axiomatized by restricted forms of induction, including $I\Sigma_n$, $I\Sigma_n^-$, $I\Pi_n^-$ and their combinations.

We also obtain new conservation results relating the hierarchies of uniform and local reflection principles. In particular, we show that (for a sufficiently broad class of theories T) the uniform Σ_1-reflection principle for T is Σ_2-conservative over the corresponding local reflection principle. This bears some corollaries on the hierarchies of restricted induction schemata in arithmetic and provides a key tool for our generalization of Schmerl's theorem.

1 Introduction

Proof-theoretic ordinals: a discussion. Since the fundamental work of Gentzen in the late 30's it was understood that formal theories of sufficient expressive power can, in several natural ways, be associated ordinals. Informally, these ordinals can be thought of as a kind of quantitative measure of the "proof-theoretic strength" of a theory. On a more formal level, proof-theoretic ordinal of a theory is one of its main metamathematical characteristics, and its knowledge

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usually reveals much other information about the theory under consideration, in particular, the information of computational character. Thus, the calculation of proof-theoretic ordinals, or *ordinal analysis*, has become one of the central aims in the study of formal systems (cf. [23]).

Perhaps, the most traditional approach to ordinal analysis is the definition of the proof-theoretic ordinal of a theory T as the supremum of order types of primitive recursive well-ordering relations, whose well-foundedness is provable in T. Since the well-foundedness is a Π^1_1-concept, this ordinal is sometimes called the Π^1_1-*ordinal* of T (denoted $|T|_{\Pi^1_1}$). This definition provides a stable and convenient measure of proof-theoretic strength of theories. Moreover, it is not dependent on any special concepts of natural well-orderings, which is typical for the other proof-theoretic ordinals discussed below. However, it has the following drawbacks: 1) it is only applicable to theories in which the well-foundedness is expressible, e.g., it does not (directly) apply to first order Peano arithmetic; 2) it is a fairly rough measure. It is well-known that the Π^1_1-ordinal does not allow to distinguish between a theory and any of its extensions by true Σ^1_1-axioms: all such extensions share one and the same Π^1_1-ordinal.

An alternative approach to ordinal analysis makes use of the notion of *provably total computable function* of a theory T, that is, a Σ^0_1-definable function, whose totality (formulated as an arithmetical Π^0_2-sentence) is provable in T. The class of p.t.c.f. is an important computational characteristic of a theory. In typical cases, for sufficiently strong theories, such classes can be characterized recursion-theoretically using transfinite subrecursive hierarchies of fast growing functions. This yields what is usually called the proof-theoretic Π^0_2-*ordinal* of T. There are several choices for such hierarchies and the resulting ordinals depend on what hierarchy you actually take as a scale. Most popular are the Fast Growing (or the Schwichtenberg–Wainer) hierarchy, the Hardy hierarchy, and the Slow Growing hierarchy. However, under certain conditions, there are fixed relationships between these hierarchies, which allows to translate the results obtained for one of them into the others.

Π^0_2-analysis, although sharper, is more problematic than the Π^1_1 one. A well-known difficulty here is that the common hierarchies, most notably the Slow Growing one, depend on a particular choice of the ordinal notation system and a particular fundamental sequences assignment. Such a choice always presents a certain degree of arbitrariness and technical complication.[1] However, there is a payoff: first of all, Π^0_2-analysis allows to extract computational information from proofs. Another important aspect of Π^0_2-analysis is that it is closely related to constructing independent finite combinatorial principles for formal theories.

In this paper the Fast Growing hierarchy is taken as basic. This hierarchy corresponds to a natural jump hierarchy of subrecursive function classes (the

[1] It is a long standing open question, whether a natural ordinal notation system can be canonically chosen for sufficiently large constructive ordinals. It has to be noted, however, that the standard proof-theoretic methods, in practical cases, usually allow to define natural ordinal notation systems for suitable initial segments of the constructive ordinals, that is, they simultaneously allow for Π^1_1- and Π^0_2-analyses of a theory, whenever they work. Pohlers [22] calls this property *profoundness* of the ordinal analysis.

Kleene hierarchy) and therefore is less ad hoc. Besides, it is more robust than the others, in particular, it is possible to give a formulation of the Fast Growing hierarchy which is independent of the fundamental sequences assignments (see Section 3 for the details). Whenever we speak about Π^0_2-ordinals of theories, we always mean the ordinals measured by this hierarchy.

Proof-theoretic Π^0_2-analysis deals with independent principles of complexity Π^0_2 (sentences, expressing the totality of fast-growing functions), but it fails to distinguish between the theories only different in true Π^0_1-axioms. However, the most prominent independent principle — Gödel's consistency assertion $\mathsf{Con}(T)$ for an axiom system T — has logical complexity Π^0_1. So, for example, $|\mathsf{PA} + \mathsf{Con}(\mathsf{PA})|_{\Pi^0_2} = |\mathsf{PA}|_{\Pi^0_2} = \epsilon_0$. In general, theories having the same Π^0_2-ordinal can be of quite different consistency strength.

Historically, there have been proposals to define proof-theoretic ordinals relevant to logical complexity Π^0_1. This level of logical complexity is characteristic for the 'consistency strength' of theories and thus plays a role, e.g., in connection with Hilbert's program. On the other hand, an independent interest in Π^0_1-analysis is its relationship with the concept of *relative interpretability* of formal theories. By the results of Orey, Feferman and Hájek, this notion (for large classes of theories) is equivalent to Π^0_1-conservativity. The proposals to define general notions of proof-theoretic Π^0_1-ordinals, however, generally fell victim to just criticism, see [13]. To refresh the reader's memory, we discuss one such proposal below.

Indoctrinated by Hilbert's program, Gentzen formulated his ordinal analysis of Peano arithmetic as a proof of consistency of PA by transfinite induction up to ϵ_0. Accordingly, a naive attempt at generalization was to define the Π^0_1-ordinal of a system T as the order type of the shortest primitive recursive well-ordering $\prec$ such that the corresponding scheme of transfinite induction $TI(\prec)$ proves $\mathsf{Con}(T)$.

This definition is inadequate for several reasons. The first objection is that the formula $\mathsf{Con}(T)$ may not be canonical, that is, it really depends on the chosen *provability predicate* for T rather than T itself. Feferman [7] gave examples of Σ_1-provability predicates externally numerating PA and satisfying Löb's derivability conditions such that the corresponding consistency assertions are not PA-provably equivalent. In Appendix B we consider another example of this sort, for which the two provability predicates correspond to sufficiently *natural* proof systems axiomatizing PA. This indicates that the intended Π^0_1-ordinal of a theory T can, in fact, be a function of its *provability predicate* (and possibly some additional data), rather than just of the set of axioms of T taken externally.[2] Two possible ways to avoid this problem are: 1) to restrict the attention to specific natural theories, for which the canonical provability predicates are known; 2) to stipulate that theories always come together with their own fixed provability predicates. In other words, if two deductively equivalent axiom systems are formalized with different provability predicates, they should

[2]This is also typical for the other attempts to define proof-theoretic ordinals "from above" (cf. Appendix B for a discussion).

be considered as different. As remarked above, the second option appears to be better than the first, and we stick to it in this paper.

The second objection is that the primitive recursive well-ordering may be sufficiently pathological, and then $TI(\prec)$ can already prove $\mathsf{Con}(T)$ for some well-ordering $\prec$ of type ω (as shown by Kreisel). This problem can be avoided, if we only consider *natural* primitive recursive well-orderings, which are known for certain initial segments of the constructive ordinals. This would make the definition work at least for certain classes of theories, whose ordinals are not too large. Notice that essentially the same problem appears in the definition of the proof-theoretic Π^0_2-ordinal described above.

The third objection is that, although $\mathsf{Con}(T)$ is a Π^0_1-formula, the logical complexity of the schema $TI(\prec)$ is certainly higher. Kreisel noticed that the formulation of Gentzen's result would be more informative, if one restricts the complexity of transfinite induction formulas to primitive recursive, or open, formulas (we denote this schema $TI_{p.r.}(\prec)$). That is, Gentzen's result can be recast as a reduction of ω-induction of arbitrary arithmetical complexity to open transfinite induction up to ϵ_0.

This formulation allows to rigorously attribute to, say, PA the natural ordinal (notation system up to) ϵ_0. However, for other theories T this approach is not yet fully satisfactory, for it is easy to observe that $TI_{p.r.}(\prec)$ has logical complexity Π^0_2, which is higher than Π^0_1. So, the definition of $|T|_{\Pi^0_1}$ as the infimum of order types of natural primitive recursive well-orderings $\prec$ such that

$$\mathsf{PRA} + TI_{p.r.}(\prec) \vdash \mathsf{Con}(T),$$

in fact, reduces a Π^0_1-principle to a Π^0_2-principle. The opposite reduction, however, is not possible. Thus, the ordinals obtained are not necessarily 'the right ones'. For example, in this sense the ordinal of $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ happens to be the same number ϵ_0, whereas any decent Π^0_1-analysis should separate the system from PA. One can attempt to push down the complexity of $TI_{p.r.}(\prec)$ by formulating it as a transfinite induction *rule* and disallowing nested applications of the rule, but in the end this would look less natural than the approach proposed in this paper.

Proof-theoretic analysis by iterated reflection. The aim of this paper is to present another approach to proof-theoretic Π^0_1- and, in general, Π^0_n-analysis for any $n \geq 1$. The treatment of arbitrary n is not substantially different from the treatment of $n = 1$. For $n = 2$ our definition is shown to agree with the usual Π^0_2-analysis w.r.t. the Fast Growing hierarchy.[3] The apparent advantage of the method is that for the 'problematic' cases, such as $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$, one obtains meaningful ordinal assignments. For example, we will see that $|\mathsf{PA} + \mathsf{Con}(\mathsf{PA})|_{\Pi^0_1} = \epsilon_0 \cdot 2$, which is well above the Π^0_1- and Π^0_2-ordinal ϵ_0 of PA, as expected.

A basic idea of the proof-theoretic Π^0_n-analysis is that of *conservative approximation* of a given theory T by formulas of complexity Π^0_n whose behavior is

[3]This can be considered as an evidence supporting our definition for the other n.

well-understood. Many properties of T, e.g., its class of p.t.c.f., can be learned from the known properties of the conservative approximations. As suitable approximations we take progressions of transfinitely iterated reflection principles (of relevant logical complexity). In particular, progressions of iterated consistency assertions, which are equivalent to iterated Π^0_1-reflection principles, provide suitable approximations of complexity Π^0_1.

The choice of the reflection formulas as the approximating ones has the following two advantages. First of all, the hierarchies of reflection principles are natural analogs of the jump hierarchies in recursion theory (this analogy is made more precise in Section 3 of this paper). So, in a sense, they are more elementary than the other candidate schemata, such as transfinite induction. Second, and more important, they allow for a convenient *calculus*. That is, the proof-theoretic ordinals for many theories can be determined by rather direct calculations, once some basic rules of handling iterated reflection principles are established. The key tool for this kind of calculations is Schmerl's formula [25], which is generalized and provided a new proof in this paper.

The idea of using iterated reflection principles for the classification of axiomatic systems goes back to the old works of Turing [28] and Feferman [8]. Given a base theory T, one constructs a transfinite sequence of extensions of T by iteratedly adding formalized consistency statements, roughly, according to the following clauses:

(T1) $T_0 = T$;

(T2) $T_{\alpha+1} = T_\alpha + \mathsf{Con}(T_\alpha)$;

(T3) $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$, for α a limit ordinal.

By Gödel's Incompleteness Theorem, whenever the initial theory T is sound[4], the theories T_α form a strictly increasing transfinite sequence of sound Π^0_1-axiomatized extensions of T. Choosing for T some reasonable minimal fragment of arithmetic (in this paper we work over the elementary arithmetic EA) this sequence can be used to associate an ordinal $|U|_{\Pi^0_1}$ to any theory U extending EA as follows:

$$|U|_{\Pi^0_1} := \sup\{\alpha : \mathsf{EA}_\alpha \subseteq U\}.$$

This definition provides interesting information only for those theories U which can be well approximated by the sequence EA_α. For such U one should be able to show that for $\alpha = |U|_{\Pi^0_1}$ the theory EA_α axiomatizes *all* arithmetical Π^0_1-consequences of U, that is,

$$U \equiv_{\Pi_1} \mathsf{EA}_\alpha. \tag{1}$$

(Here and below $T \equiv_{\Pi_n} U$ means that the theories T and U prove the same Π_n-sentences.) Thus, (1) can be viewed as an exact reduction of U to a purely Π^0_1-axiomatized theory EA_α, and in this sense $|U|_{\Pi^0_1}$ is called the proof-theoretic

[4] That is, if all theorems of T hold in the standard model of arithmetic.

Π^0_1-ordinal of U. Theories U satisfying equivalence (1) are called Π^0_1-*regular*. Verifiability of (1) within, say, EA implies

$$\mathsf{EA} \vdash \mathsf{Con}(U) \leftrightarrow \mathsf{Con}(\mathsf{EA}_\alpha),$$

and thus, $|U|_{\Pi^0_1}$ can also be thought of as the ordinal measuring the consistency strength of the theory U.

The program as described above, however, encounters several technical difficulties. One familiar difficulty is the fact that the clauses (T1)–(T3) do not uniquely define the sequence of theories T_α, that is, the theory T_α depends on the formal representation of the ordinal α within arithmetic rather than on the ordinal itself.

For the analysis of this problem Feferman [8] considered families of theories of the form $(T_c)_{c\in\mathcal{O}}$ satisfying (T1)–(T3) along every path within $\mathcal{O}$, where $\mathcal{O}$ is Kleene's universal system of ordinal notation. Using an idea of Turing, he showed that every true Π^0_1-sentence is provable in T_c for a suitable ordinal notation $c \in \mathcal{O}$ with $|c| = \omega + 1$. It follows that there are two ordinal notations $a, b \in \mathcal{O}$ with $|a| = |b| = \omega + 1$ such that T_a proves $\mathsf{Con}(T_b)$, and this observation seems to break down the program of associating ordinals to theories as described above, at least in the general case.

However, a possibility remains that for *natural* (mathematically meaningful) theories U one can exhaust all Π^0_1-consequences of U using only specific *natural* ordinal notations, and a careful choice of such notations should yield proper ordinal bounds. This idea has been developed in the work of U. Schmerl [25], who showed among other things that for natural ordinal notations

$$\mathsf{PA} \equiv_{\Pi_1} \mathsf{PRA}_{\epsilon_0}.$$

This essentially means that $|\mathsf{PA}|_{\Pi^0_1} = \epsilon_0$, which coincides with the ordinal associated to PA through other proof-theoretic methods.

The significant work of Schmerl, however, attracted less attention than it, in our opinion, deserved. Partially this could be explained by a rather special character of the results, as they were stated in his paper. At present, twenty years later, thanks to the development of provability logic and formal arithmetic, we know much more about the structure of the fragments of PA, as well as about the properties of provability predicates. One of the goals of this paper is to revise and put in the right context this work of Schmerl. We provide a simpler approach to defining and treating iterated reflection principles, which helps to overcome some technical problems and allows for further development of these methods.

Plan of the paper. In Section 2 we define progressions of iterated reflection principles and note some basic facts about them. This allows to rigorously define Π^0_n-ordinals of theories following the ideas presented in the introduction.

In Section 3 we relate, in a very general setup, the hierarchy of iterated Π^0_2-reflection principles and the Fast Growing hierarchy. This shows that our approach, for the particular case of logical complexity Π^0_2, agrees with the usual proof-theoretic Π_2^0-analysis and provides the expected kind of information about the classes of provably total computable functions. Proofs of some technical lemmata are postponed till Appendix A.

Section 4 can be read essentially independently from the previous parts of the paper. It presents a new conservation result relating the uniform and local reflection schemata. In particular, it is shown that uniform Π_2-reflection principle is Σ_2-conservative over the local Σ_1-reflection principle. This yields as an immediate corollary the result in [12] on the relation between parametric and parameter-free induction schemata: $I\Sigma_n$ is Σ_{n+2}-conservative over $I\Sigma_n^-$. The results of that section also provide a clear proof of a particular case of Schmerl's formula, which already has some meaningful corollaries for fragments of PA. At the same time it serves as a basis for a generalization given in the further sections.

Section 5, aiming at a proof of Schmerl's formula, presents a few lemmata to the effect that some conservation results for noniterated reflection principles can be directly extended to iterated ones. In Section 6 we formulate and prove (a generalization of) Schmerl's formula. In particular, we formulate a general relationship between Π_1^0- and Π_2^0-ordinals of Π_2^0-regular theories: Π_1^0-ordinal of a theory is one ω-power higher than its Π_2^0-ordinal.

In Section 7 we apply the general methodology to calculate proof-theoretic ordinals of main fragments of PA, including forms of parameter-free induction and their combinations with the parametric ones. This section is mostly meant as an illustration of the possibilities of our techniques. In our opinion, the most interesting examples treated are Π_2^0-irregular theories such as parameter-free Π_1-induction schema $I\Pi_1^-$, $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$, and the like.

Appendix A supplies technical lemmata for the results of Section 3. Appendix B discusses an attempt to define proof-theoretic ordinals "from above", and the role of nonstandard provability predicates. In Appendix C ordinal Π_1^0-ordinals of the weak systems (not proving the totality of the superexponentiation function) are treated.

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2 Constructing iterated reflection principles

In defining iterated reflection principles we closely follow [3]. Our present approach is slightly more general, but the proofs of basic lemmas remain essentially the same, so we just fix the terminology and indicate some basic ideas.

Iterated consistency assertions. We deal with first order theories formulated in a language containing that of arithmetic. Our basic system is Kalmar elementary arithmetic EA (or $I\Delta_0(\exp)$, cf. [11]). For convenience we assume that a symbol for the exponentiation function 2^x is explicitly present in the language of EA. EA^+ denotes the extension of EA by an axiom stating the totality of the superexponential function 2^x_x (or $I\Delta_0 + \mathrm{supexp}$). EA^+ is the minimal extension of EA where the cut-elimination theorem for first order logic is provable. Hence, it will often play the role of a natural metatheory for various arguments in this paper.

Elementary formulas are bounded formulas in the language of EA. A theory T is *elementary presented* if it is equipped with a *numeration*, that is, an elementary formula $\mathsf{Ax}_T(x)$ defining the set of axioms of T in the standard model of arithmetic.

By an *elementary linear ordering* $(D, \prec)$ we mean a pair of elementary formulas $x \in D$ and $x \prec y$ such that EA proves that the relation $\prec$ linearly orders the domain D. An elementary *well-ordering* is an elementary linear ordering, which is well-founded in the standard model.

Given an elementary linear ordering $(D, \prec)$, we use Greek variables α, β, γ, etc. to denote the elements of D (and the corresponding ordinals). Since D is elementary definable, these variables can also be used within EA.

An elementary formula $\mathsf{Ax}_T(\alpha, x)$ *numerates* a family of theories $(T_\alpha)_{\alpha \in D}$, if for each α the formula $\mathsf{Ax}_T(\bar{\alpha}, x)$ defines the set of axioms of T_α in the standard model. If such a formula Ax_T exists, the family $(T_\alpha)_{\alpha \in D}$ is called *uniformly elementary presented.*

¿From $\mathsf{Ax}_T(\alpha, x)$, as well as from a numeration of an individual theory, the (parametric) provability predicate $\Box_T(\alpha, x)$ and the consistency assertion $\mathsf{Con}(T_\alpha)$ are constructed in a standard way. Specifically, there is a canonical Σ^0_1-formula $P(X, x)$, with a set parameter X and a number parameter x, expressing the fact that x codes a formula logically provable from the set of (non-logical) axioms coded by X. Then $\Box_T(\alpha, x) := P(\{u : \mathsf{Ax}_T(\alpha, u)\}, x)$ and $\mathsf{Con}(T_\alpha) := \neg\Box_T(\alpha, \bot)$. Notice that $\Box_T(\alpha, x)$ is first order Σ_1, and $\mathsf{Ax}_T(\alpha, u)$ occurs in $\Box_T(\alpha, x)$ as a subformula (replacing the occurrences of the form $u \in X$ in $P(X, x)$).

As usual, we write $\Box_T(\alpha, \varphi)$ instead of $\Box_T(\alpha, \ulcorner\varphi\urcorner)$, and $\Box_T(\alpha, \varphi(\dot{x}))$ instead of $\Box_T(\alpha, \ulcorner\varphi(\dot{x})\urcorner)$. Here $\ulcorner\varphi(\dot{x})\urcorner$ denotes the standard elementary function (and the corresponding EA-definable term) that maps a number n to the code of the formula $\varphi(\bar{n})$.

Now we present progressions of iterated consistency assertions. Somewhat generalizing [3], we distinguish between explicit and implicit progressions. Both are defined by formalizing (in two different ways) the following variant of conditions (T1)–(T3): for all $\alpha \in D$,

$$T_\alpha \equiv T + \{\mathsf{Con}(T_\beta) : \beta \prec \alpha\}.$$

Suppose we are given an "initial" elementary presented theory T and an elementary linear ordering $(D, \prec)$. An elementary formula $\mathsf{Ax}_T(\alpha, x)$ *explicitly* *numerates* a progression based on iteration of consistency along $(D, \prec)$ if

$$\mathsf{EA} \vdash \mathsf{Ax}_T(\alpha, x) \leftrightarrow (\mathsf{Ax}_T(x) \vee \exists \beta \prec \alpha \ x = \ulcorner \mathsf{Con}(T_{\dot\beta}) \urcorner). \tag{2}$$

A formula $\mathsf{Ax}_T(\alpha, x)$ *implicitly numerates* such a progression if

$$\mathsf{EA} \vdash \Box_T(\alpha, x) \leftrightarrow P(\{u : \mathsf{Ax}_T(u) \vee \exists \beta \prec \alpha \ u = \ulcorner \mathsf{Con}(T_{\dot\beta}) \urcorner\}, x). \tag{3}$$

Obviously, every explicit numeration is implicit, but the converse is generally false. It is often technically more convenient to deal with implicit numerations, for it allows one to disregard the superfluous information about the exact axiomatization of theories T_α (see, e.g., our proof of Theorem 1).

An *explicit/implicit progression* based on iteration of consistency is any family of theories $(T_\alpha)_{\alpha \in D}$ presented by an explicit/implicit numeration. If $(D, \prec)$ is an elementary *well*-ordering and the initial theory T is Σ_1-sound, then any implicit progression based on iteration of consistency is a strictly increasing sequence of Σ_1-sound theories satisfying (T1)–(T3).[5]

Notice that the definition of explicit and implicit numerations is self-referential. This raises the questions about the existence and uniqueness of such progressions.

Lemma 2.1 (existence) *For any elementary linear ordering $(D, \prec)$ and any initial theory T, there is an explicit progression based on iteration of consistency along $(D, \prec)$.*

Proof. The definition (2) has the form of a fixed point equation. Indeed, the formula $\mathsf{Con}(T_\beta)$ is constructed effectively from $\mathsf{Ax}_T(\beta, x)$, essentially by replacing x by u and substituting the result into $\neg P(X, \bot)$ for $u \in X$. Hence, there is an elementary definable term con that outputs the Gödel number of $\mathsf{Con}(T_{\overline{\beta}})$ given the Gödel number of $\mathsf{Ax}_T(\overline{\beta}, x)$. Then the equation (2) can be rewritten as follows:

$$\mathsf{EA} \vdash \mathsf{Ax}_T(\alpha, x) \leftrightarrow \big(\mathsf{Ax}_T(x) \vee \exists \beta \leq x \ (\beta \prec \alpha \wedge x = \mathsf{con}(\ulcorner \mathsf{Ax}_T(\dot\beta, x) \urcorner))\big). \tag{4}$$

Fixed point lemma guarantees that an elementary solution $\mathsf{Ax}_T(\alpha, x)$ exists. To see that the solution satisfies (2) it only has to be noted that, assuming the Gödel numbering we use is standard, provably in EA for any β,

$$\beta \leq \ulcorner \overline{\beta} \urcorner \leq \ulcorner \mathsf{Ax}_T(\overline{\beta}, x) \urcorner \leq \ulcorner \mathsf{Con}(T_{\overline{\beta}}) \urcorner,$$

q.e.d.

In the following, the words *progression based on iteration of consistency* will always refer to implicit progressions. We note the obvious *monotonicity* property of such progressions:

[5] This is essentially the only place in all the development below, where well-foundedness matters. Actually, for the progressions based on iteration of consistency, well-foundedness w.r.t. the Σ_2-definable subsets would be sufficient.

Lemma 2.2 $\mathsf{EA} \vdash \alpha \prec \beta \to (\Box_T(\alpha, x) \to \Box_T(\beta, x))$.

The next lemma shows that any progression based on iteration of consistency is uniquely defined by the initial theory and the elementary linear ordering.

Lemma 2.3 (uniqueness) *Let U and V be elementary presented extensions of EA, $(D, \prec)$ an elementary linear ordering, $(U_\alpha)_{\alpha \in D}$ and $(V_\alpha)_{\alpha \in D}$ progressions based on iteration of consistency with the initial theories U and V, respectively. Then*

$$\mathsf{EA} \vdash \forall x\, (\Box_U(x) \leftrightarrow \Box_V(x)) \quad \Longrightarrow \quad \mathsf{EA} \vdash \forall \alpha \forall x\, (\Box_U(\alpha, x) \leftrightarrow \Box_V(\alpha, x)).$$

The uniqueness property is a robust background for further treatment of recursive progressions. In particular, it allows one to consistently use combined expressions like $(T_\alpha)_\beta$ for the composition of progressions (along the same ordering). The proof of Lemma 2.3 employs a trick coming from the work of U. Schmerl [25], which will also be used for the other results below.

Lemma 2.4 (reflexive induction) *For any elementary linear ordering $(D, \prec)$, any theory T is closed under the following* reflexive induction *rule:*

$$\forall \alpha\, (\Box_T(\forall \beta \prec \dot\alpha\, A(\beta)) \to A(\alpha)) \ \vdash \ \forall \alpha\, A(\alpha).$$

Proof. Assuming $T \vdash \forall \alpha\, (\Box_T(\forall \beta \prec \dot\alpha\, A(\beta)) \to A(\alpha))$ we derive:

$$\begin{array}{rcl} T \vdash \Box_T \forall \alpha\, A(\alpha) & \to & \forall \alpha\, \Box_T \forall \beta \prec \dot\alpha\, A(\beta) \\ & \to & \forall \alpha\, A(\alpha). \end{array}$$

Löb's theorem for T then yields $T \vdash \forall \alpha\, A(\alpha)$, q.e.d.

Proof of Lemma 2.3 then easily follows by reflexive induction in EA. We refer the reader to [3] for the details. It is important to realize that, although the reflexive induction has formally nothing to do with the well-foundedness, it allows one to prove certain properties of progressions *as if* they were proved by transfinite induction, in agreement with the underlying intuition. This makes the uses of reflexive induction in this context quite simple and natural.

Remark 2.5 All the above results continue to hold, if one replaces EA by EA^+ or any other sound elementary presented extension of EA.

Iterated reflection principles. Let T be an elementary presented theory. The *local* reflection principle for T is the schema

$$\mathsf{Rfn}(T): \qquad \Box_T \varphi \to \varphi, \quad \varphi \text{ a sentence.}$$

The *uniform* reflection principle is the schema

$$\mathsf{RFN}(T): \qquad \forall x\, (\Box_T \varphi(\dot x) \to \varphi(x)), \quad \varphi(x) \text{ a formula.}$$

Partial reflection principles are obtained from the above schemata by imposing a restriction that φ belongs to one of the classes Γ of the arithmetical hierarchy (denoted $\mathsf{Rfn}_\Gamma(T)$ and $\mathsf{RFN}_\Gamma(T)$, respectively). See [26, 14, 3] for some basic information about reflection principles.

We shall also consider the following *metareflection rule*:

$$\mathsf{RR}_{\Pi_n}(T): \qquad \frac{\varphi}{\mathsf{RFN}_{\Pi_n}(T+\varphi)}.$$

We let Π_m-$\mathsf{RR}_{\Pi_n}(T)$ denote the above rule with the restriction that φ is a Π_m-sentence.

For $n \geq 1$, $\Pi_n(\mathbb{N})$ denotes the set of all true Π_n-sentences. $\mathsf{True}_{\Pi_n}(x)$ denotes a canonical truthdefinition for Π_n-sentences, that is, a Π_n-formula naturally defining the set of Gödel numbers of $\Pi_n(\mathbb{N})$-sentences in EA.

Let T be an elementary presented theory containing EA. The set of axioms of the theory $T+\Pi_n(\mathbb{N})$ can be defined, e.g., by the Π_n-formula $\mathsf{Ax}_T(x)\vee\mathsf{True}_{\Pi_n}(x)$. Then the formula

$$\Box_T^{\Pi_n}(x) := P(\{u : \mathsf{Ax}_T(u) \vee \mathsf{True}_{\Pi_n}(u)\}, x)$$

naturally represents the Σ_{n+1}-complete provability predicate for $T+\Pi_n(\mathbb{N})$, and $\mathsf{Con}^{\Pi_n}(T) := \neg\Box_T^{\Pi_n}\bot$ is the corresponding consistency assertion. (For $n=0$ we stipulate that these formulas coincide with $\Box_T$ and $\mathsf{Con}(T)$, respectively.)

Relativized provability predicates $\Box_T^{\Pi_n}$, as well as the usual provability predicate $\Box_T$, satisfy Löb's derivability conditions. $\Box_T^{\Pi_n}$ is EA-provably Σ_{n+1}-complete, that is,

$$\mathsf{EA} \vdash \forall x\, (\sigma(x) \to \Box_T^{\Pi_n}\sigma(\dot{x})),$$

for any Σ_{n+1}-formula $\sigma(x)$. Besides, for $n \geq 0$ the following relationships are known (see [4]).

Lemma 2.6 *For any elementary presented theory T containing EA, the following schemata are equivalent over EA:*

(i) $\mathsf{Con}^{\Pi_n}(T)$;

(ii) $\mathsf{RFN}_{\Pi_{n+1}}(T)$;

(iii) $\mathsf{RFN}_{\Sigma_n}(T)$.

This shows that the uniform reflection principles are generalizations of the consistency assertions to higher levels of the arithmetical hierarchy. (Notice that the schema $\mathsf{RFN}_{\Pi_1}(T)$ is equivalent to the standard consistency assertion $\mathsf{Con}(T)$.)

Relativized local reflection principles are generally not equivalent to any of the previously considered schemata. They are defined as follows:

$$\mathsf{Rfn}_{\Sigma_m}^{\Pi_n}(T): \qquad \Box_T^{\Pi_n}\varphi \to \varphi, \quad \text{for } \varphi \in \Sigma_m,$$

and similarly for the local Π_m-reflection principle. Notice that the relativized analog of, say, $\mathsf{Rfn}_{\Sigma_m}(T)$ is actually $\mathsf{Rfn}_{\Sigma_{m+n}}^{\Pi_n}(T)$.

Progressions based on iteration of reflection principles are defined in an analogy with (2). If $\Phi(T)$ is any of the reflection schemata for T introduced above, then the progressions based on iteration of Φ along $(D, \prec)$ will be denoted $(\Phi(T)_\alpha)_{\alpha \in D}$. Thus, $T_\alpha \equiv \mathsf{Con}(T)_\alpha$; we also write $(T)^n_\alpha$ short for $\mathsf{RFN}_{\Pi_n}(T)_\alpha$.

Theories $\Phi(T)_\alpha$ are defined by formalizing the condition

$$\Phi(T)_\alpha \equiv T + \{\Phi(\Phi(T)_\beta) : \beta \prec \alpha\}.$$

This can be done as follows. Since the instances of the reflection principles are elementarily recognizable, with each of the above schemata Φ one can naturally associate an elementary formula $\Phi\text{-}\mathsf{code}(e, x)$ expressing that *e is the code of a Σ_1-formula $\Box_U(v)$, and x is the code of an instance of $\Phi(U)$ formulated for $\Box_U$*. Then $\mathsf{Ax}_T(\alpha, x)$ is called an explicit numeration of a progression based on iteration of Φ, if

$$\mathsf{EA} \vdash \mathsf{Ax}_T(\alpha, x) \leftrightarrow (\mathsf{Ax}_T(x) \vee \exists \beta \prec \alpha\ \Phi\text{-}\mathsf{code}(\ulcorner \Box_T(\dot\beta, v)\urcorner, x)),$$

and an implicit numeration, if

$$\mathsf{EA} \vdash \Box_T(\alpha, x) \leftrightarrow P(\{u : \mathsf{Ax}_T(u) \vee \exists \beta \prec \alpha\ \Phi\text{-}\mathsf{code}(\ulcorner \Box_T(\dot\beta, v)\urcorner, u)\}, x).$$

Then the analogs of existence, monotonicity, and uniqueness lemmas hold for such progressions too, with similar proofs. We omit them.

Π^0_n-ordinals. Let an elementary well-ordering $(D, \prec)$ be fixed. All the definitions below are to be understood relative to this ordering. We define:

$$|T|_{\Pi^0_n} := \sup\{\alpha : (\mathsf{EA})^n_\alpha \subseteq T\}.$$

If the ordering $(D, \prec)$ is too short, that is, if for all $\alpha \in D$, $(\mathsf{EA})^n_\alpha \subseteq T$, we can set $|T|_{\Pi^0_n} := \infty$.

A theory T is *Π^0_n-regular*, if there is an $\alpha \in D$ such that

$$T \equiv_{\Pi_n} (\mathsf{EA})^n_\alpha. \tag{5}$$

Notice that Π^0_n-regular theories are Π^0_n-sound, because $(\mathsf{EA})^n_\alpha$ is. If the equivalence (5) is provable in a (meta)theory U, then T is called *U-provably Π^0_n-regular*. For $U \subseteq T$ in this case we have $\alpha = |T|_{\Pi^0_n}$, because the formalization of (5) implies

$$U \vdash \mathsf{RFN}_{\Pi_n}(T) \leftrightarrow \mathsf{RFN}_{\Pi_n}((\mathsf{EA})^n_\alpha),$$

and $T \nvdash \mathsf{RFN}_{\Pi_n}(T)$ then yields $(\mathsf{EA})^n_{\alpha+1} \not\subseteq T$.

Comparing this definition with the above discussion of proof-theoretic Π^0_1-ordinals we notice that it lacks the mentioned drawbacks 1) and 3).

Ad 1). Indeed, EA is a natural finitely axiomatizable theory, so it has a canonical provability predicate. Uniqueness theorem then guarantees that the progression $(\mathsf{EA})^n_\alpha$ for $\alpha \in D$ is uniquely defined. (There is no mentioning of the provability predicates *for T* in the definition of $|T|_{\Pi^0_n}$.)

Ad 2). As defined above, the Π^0_n-ordinal of a theory T is a function of a pre-fixed elementary well-ordering $(D, \prec)$. We shall see later that the analysis of natural theories requires imposing some additional natural structure on the well-ordering. (This situation is only slightly better than a restriction to concrete natural well-orderings.) As expected in view of the general problem of natural ordinal notations, at present we do not have an answer to the question what kind of structure is needed for the analysis of arbitrary theories. None of the existing definitions of proof-theoretic ordinals of logical complexity below Π^1_1 is free from this drawback, and this may even be unavoidable.

Ad 3). Equation (5) provides an exact reduction of a given Π^0_n-regular theory T to a purely Π^0_n-axiomatized theory $(\mathsf{EA})^n_\alpha$. This is the main advantage of the considered definition.

3 Iterated Π_2-reflection and the fast growing hierarchy

In this section we relate the hierarchies of iterated uniform Π_2-reflection principles and the hierarchies of fast growing functions. This shows that, under very general assumptions, the proof-theoretic analysis by iterated Π_2-reflection principles over EA provides essentially the same information as the usual Π^0_2-ordinal analysis. For the natural ordinal notation system up to ϵ_0 similar results can be deduced from the work of R. Sommer [27]. Our present approach is somewhat more general and also seems to be technically simpler, so we opted for an independent presentation.

Let $\mathcal{E}$ denote the class of elementary functions. For any set of functions $\mathcal{K}$, $\mathbf{C}(\mathcal{K})$ denotes the closure of $\mathcal{K} \cup \mathcal{E}$ under composition; $\mathbf{E}(\mathcal{K})$ denotes is the elementary closure of $\mathcal{K}$, that is, the closure of $\mathcal{K} \cup \mathcal{E}$ under composition and bounded recursion. If all the functions from $\mathcal{K}$ are monotone and have elementary graphs, then $\mathbf{C}(\mathcal{K}) = \mathbf{E}(\mathcal{K})$, see [2].

Let an elementary well-ordering $(D, \prec)$ be fixed. Throughout this section we assume that there is an element $0 \in D$ satisfying $\mathsf{EA} \vdash \forall \alpha\, (0 = \alpha \vee 0 \prec \alpha)$.

A hierarchy of functions F_α for $\alpha \in D$ is defined recursively as follows:

$$F_\alpha(x) := \max\{2^x_x + 1\} \cup \{F^{(v)}_\beta(u) + 1 : \beta \prec \alpha,\ \beta, u, v \leq x\}. \qquad (*)$$

Since $(D, \prec)$ is well-founded, all F_α are well-defined. The functions F_α generate the hierarchy of function classes

$$\mathcal{F}_\alpha := \mathbf{E}(\{F_\beta : \beta \prec \alpha\}).$$

One easily verifies that for the initial elements $\alpha \in D$ the classes $\mathcal{F}_\alpha$ coincide with the classes of the familiar Grzegorczyk hierarchy: $\mathcal{F}_0 = \mathcal{E}$, $\mathcal{F}_1 = \mathcal{E}'$, ... , $\mathcal{F}_\omega$ = primitive recursive functions, ... The further classes are a natural extension of the Grzegorczyk hierarchy into the transfinite. Notice that this hierarchy is defined for an arbitrary (not necessarily natural) well-ordering and does not depend on the assignments of fundamental sequences.

A slight modification of this hierarchy has recently been proposed by Weiermann and studied in detail by Möllerfeld [17]. Building on some previous results, see [24] for an overview, he relates this hierarchy to some other natural hierarchies of function classes. Since our hierarchy has to be reasonably representable in EA, in some respects we need a sharper treatment than in [17].

Proofs of the following two lemmas will be given in the Appendix.

Lemma 3.1 *$F_\alpha(x) = y$ is an elementary relation of α, x, and y.*

Notice that a priori we only know that this relation is recursive. Let $F_\alpha(x) \simeq y$ be a natural elementary formula representing it.

Lemma 3.2 *The following properties are verifiable in EA:*

(i) $(x_1 \leq x_2 \wedge F_\alpha(x_1) \simeq y_1 \wedge F_\alpha(x_2) \simeq y_2) \to y_1 \leq y_2$;

(ii) $(\beta \prec \alpha \wedge F_\beta(x) \simeq y_1 \wedge F_\alpha(x) \simeq y_2) \to y_1 \leq y_2$;

(iii) A natural formalization of $(*)$ *(axioms (F1)–(F3) from Appendix A).*

Let $F_\alpha\!\downarrow$ denote the formula $\forall x \exists y F_\alpha(x) \simeq y$, and let S_α denote the theory $\mathsf{EA} + \{F_{\bar\beta}\!\downarrow : \beta \prec \alpha\}$. Obviously, the theories $(S_\alpha)_{\alpha \in D}$ are uniformly elementary presented, e.g., one can define

$$\mathsf{Ax}_S(\alpha, x) :\leftrightarrow \mathsf{Ax}_{\mathsf{EA}}(x) \vee \exists \beta < x\, (\beta \prec \alpha \wedge x = \ulcorner \forall u \exists v\, F_{\dot\beta}(u) \simeq v \urcorner).$$

Our aim is to prove that $(S_\alpha)_{\alpha \in D}$ is deductively equivalent to the progression of iterated uniform Π_2-reflection principles over EA.

Notice that $S_0 \equiv \mathsf{EA}$, and S_α contains EA^+ for $\alpha \succ 0$.

Theorem 1 *Provably in EA^+, $\forall \alpha\ S_\alpha \equiv (\mathsf{EA})^2_\alpha$.*

As a corollary we obtain the following statement.

Corollary 3.3 *For all $\alpha \in D$, $\mathcal{F}((\mathsf{EA})^2_\alpha) = \mathcal{F}_\alpha$.*

Proof. Obviously, symbols for all functions F_β for $\beta \prec \alpha$ can be introduced into the language of S_α. The corresponding definitional extension of S_α admits a purely universal axiomatization, because the graphs of all F_β are elementary. By Herbrand's theorem

$$\mathcal{F}(S_\alpha) = \mathbf{C}(\{F_\beta : \beta \prec \alpha\}),$$

which coincides with the class $\mathcal{F}_\alpha$, since all functions F_β are monotone and have elementary graphs, q.e.d.

Proof of Theorem 1. By the uniqueness lemma it is sufficient to establish within EA^+ that $(S_\alpha)_{\alpha \in D \setminus \{0\}}$ is an implicit progression based on iteration of uniform Π_2-reflection principles over EA^+. So, we show the following main

Lemma 3.4 *Provably in* EA^+,

$$\forall\alpha\ S_\alpha \equiv \mathit{EA} + \{\mathsf{RFN}_{\Pi_2}(S_\beta) : \beta \prec \alpha\}.$$

Proof. We formalize the proofs of the following two lemmas in EA^+. (Notice that the arguments are local, that is, they do not use any form of transfinite induction on α.)

Lemma 3.5 *Provably in* EA,

$$\forall\beta\ \mathit{EA} + \mathsf{RFN}_{\Pi_2}(S_\beta) \vdash F_{\overline{\beta}}\!\downarrow.$$

Proof. Let $F^{(u)}_\beta(x) \simeq y$ abbreviate

$$\exists s \in \mathit{Seq}\,[(s)_0 = x \wedge \forall i < u\ F_\beta((s)_i) \simeq (s)_{i+1} \wedge (s)_u = y].$$

¿From the assumption $\gamma \prec \overline{\beta}$ within $\mathsf{EA} + \mathsf{RFN}_{\Pi_2}(S_\beta)$ one can derive:

1. $\Box_{S_\beta} F_{\dot\gamma}\!\downarrow$ (by the definition of S_β)
2. $\forall u\, \Box_{S_\beta} \forall x \exists y\ F^{(\dot u)}_{\dot\gamma}(x) \simeq y$ (by elementary induction on u from 1.)
3. $\forall x, u \exists y F^{(u)}_\gamma(x) \simeq y$ (by $\mathsf{RFN}_{\Pi_2}(S_\beta)$ from 2.)

This shows that

$$\mathsf{EA} + \mathsf{RFN}_{\Pi_2}(S_\beta) \vdash \forall\gamma \prec \overline{\beta}\ \forall x, u \exists y F^{(u)}_\gamma(x) \simeq y. \tag{6}$$

On the other hand, by elementary induction on x one obtains

$$\mathsf{EA} \vdash \forall x\, \exists \gamma_0 \leq x\, \forall \gamma \leq x\, (\gamma \prec \beta \to \gamma \preceq \gamma_0).$$

Reasoning inside EA, from (6) for this particular γ_0 one obtains a y such that $F^{(x)}_{\gamma_0}(x) \simeq y$. We claim that y is as required. By (i) and (ii) of Lemma 3.2 and by (6), for all $u, \gamma \leq x$ such that $\gamma \prec \beta$ one has $\exists z \leq y\, F^{(u)}_\gamma(x) \simeq z$. By property (F3) from Appendix A we then obtain $F_\beta(x) \simeq y$, q.e.d.

Lemma 3.6 *Provably in* EA^+,

$$\forall\beta \prec \alpha\ S_\alpha \vdash \mathsf{RFN}_{\Pi_2}(S_\beta).$$

Proof. Let S^*_β denote the definitional extension of S_β by function symbols for all the functions $\{F_\gamma : \gamma \prec \beta\}$. Clearly, S^*_β is a conservative extension of S_β, moreover, this can be shown in EA^+ uniformly in β. Thus, it is sufficient to prove the lemma for the theories S^*_β.

First of all, by a standard result (cf. [9] and [2], Proposition 5.11) based on the monotonicity of the functions F_β we obtain that S^*_α proves induction for bounded formulas in the extended language (and this is, obviously, formalizable).

Second, since Herbrand's theorem is formalizable in EA^+, we have that, for any elementary formula $\sigma(y,x)$,

$$S^*_\beta \vdash \exists y \sigma(y,\bar{n}) \Rightarrow S^*_\beta \vdash \sigma(t,\bar{n}),$$

for some closed term t in S^*_β. So, it is sufficient to establish in S^*_α the reflection principle for S^*_β for open formulas (in the language of S^*_β). This proof is very similar to the proof of Theorem 2 in [2], so we only sketch it.

The proof involves two main ingredients. First, we need a natural evaluation function for terms in the language of S^*_β, that is, a function $\mathsf{eval}_\beta(e,x)$ satisfying

$$\mathsf{eval}_\beta(\ulcorner t \urcorner, \langle \vec{x} \rangle) = t(\vec{x}),$$

for all such terms $t(\vec{x})$.

It is not difficult to see that $\mathsf{eval}_\beta \in \mathcal{F}_\alpha$ and is naturally definable in S^*_α. Essentially, it is sufficient to check that eval_β is bounded by an elementary function in F_β. Indeed, by monotonicity of all functions F_γ, every term $t(x)$ is bounded by some iterate of a function F_γ, for a suitable $\gamma \prec \beta$, which means that

$$\mathsf{eval}_\beta(e,x) \leq F^{(n(e))}_{\gamma(e)}(x),$$

where $\gamma(e)$ and $n(e)$ are elementary functions. For the natural coding of terms we can additionally assume that $n(e) < e$, for all e. Then we can estimate the evaluation function as follows:

$$F^{(n(e))}_{\gamma(e)}(x) \leq F^{(\max(e,x))}_{\gamma(e)}(\max(e,x)) \leq F_\beta(\max(e,x,\gamma(e))).$$

Therefore, eval_β is elementary in F_β and belongs to $\mathcal{F}_\alpha$. As a corollary we obtain that S^*_α proves $\Delta_0(\mathsf{eval}_\beta)$-induction.

The second ingredient is a proof in S^*_α of the reflection principle for S^*_β for open formulas. This is done straightforwardly by the induction on the length of a cut-free S^*_β-derivation using eval_β. This induction has $\Delta_0(\mathsf{eval}_\beta)$-form, hence it is formalizable in S_α. The whole argument (involving cut-elimination) is then formalizable in EA^+. Details can be found in [2], Section 7. This completes the proof of Lemma 3.6 and Theorem 1, q.e.d.

Remark 3.7 One can show that the hierarchy $(T)^2_\alpha$, for a given Π_2-axiomatized sound extension T of EA, corresponds to the so-called *Kleene hierarchy* over the class $\mathcal{F}(T)$. The Kleene hierarchy is essentially obtained by adding a canonical universal function for the previous class at successor stages and taking the unions at limit stages. For $T = \mathsf{EA}$ a standard result (elaborated for the kind of hierarchies considered here in [17]) shows that the Kleene hierarchy coincides with the classes $\mathcal{F}_\alpha$ introduced above.

4 Uniform reflection is not much stronger than local reflection

In this section we establish a relationship between the uniform reflection schema and a suitable version of reflection rule. This allows us to prove that for every elementary presented theory T containing EA the theory $\mathsf{EA} + \mathsf{RFN}_{\Sigma_1}(T)$ is Σ_2-conservative over $\mathsf{EA} + \mathsf{Rfn}_{\Sigma_1}(T)$. Since the arithmetical complexity of the schema $\mathsf{Rfn}_{\Sigma_1}(T)$ is $\mathcal{B}(\Sigma_1)$, a somewhat unexpected aspect of this result is that the Σ_2-consequences of $\mathsf{RFN}_{\Sigma_1}(T)$ can be axiomatized by a set of formulas of lower arithmetical complexity.

A relativization of this theorem allows us to obtain an alternative proof of the results of Kaye, Paris and Dimitracopoulos [12] on the partial conservativity of the parametric induction schemata over the parameter-free ones. At the same time, we also obtain for free the well-known result of Parsons on the partial conservativity of induction schemata over the induction rules over EA. Further, this result leads to a more general version of Schmerl's theorem [25], which plays an important role in the present approach to proof-theoretic analysis.

Theorem 2 *Let T be an elementary presented theory containing EA, and let U be a Π_{n+1}-axiomatized extension of EA ($n \geq 1$). Then $U + \mathsf{RFN}_{\Sigma_n}(T)$ is Π_n-conservative over $U + \Pi_n\text{-}\mathsf{RR}_{\Pi_n}(T)$.*

Proof. For the proof of this theorem it is convenient to give a sequential formulation of $\Pi_n\text{-}\mathsf{RR}_{\Pi_n}(T)$. Let $\Pi_n\text{-}\mathsf{RR}^G_{\Pi_n}(T)$ denote the following inference rule in the formalism of Tait calculus:

$$\frac{\Gamma, \varphi(s)}{\Gamma, \neg\mathsf{Prf}_T(t, \ulcorner\neg\varphi(\dot{s})\urcorner)},$$

for all terms t, s and formulas $\varphi(a) \in \Pi_n$, where $\ulcorner\psi(\dot{s})\urcorner$ denotes the result of substitution of a term s in the term $\ulcorner\psi(\dot{a})\urcorner$. $\Pi_n\text{-}\mathsf{RR}^g_{\Pi_n}(T)$ will denote the same rule with the restriction that Γ consists of Π_n-formulas.

The following lemma states that the terms $\ulcorner\psi(\dot{s})\urcorner$ have a natural commutation property.

Lemma 4.1 *For any term $s(\vec{x})$, where the list $\vec{x}$ exhausts all the variables of s, and any formula $\varphi(a)$ (where s is substitutable in φ for a) there holds:*

$$\mathsf{EA} \vdash \forall\vec{x}\, \big(\Box_T\varphi(s(\dot{\vec{x}})) \leftrightarrow \Box_T\varphi(\dot{s})\big).$$

Proof. Obviously,

$$\mathsf{EA} \vdash s(\vec{x}) = y \to (\varphi(s(\vec{x})) \leftrightarrow \varphi(y)).$$

Hence, by the provable Σ_1-completeness and Löb's conditions

$$\begin{aligned} \mathsf{EA} \vdash s(\vec{x}) = y \quad &\to \quad \Box_T(s(\dot{\vec{x}}) = \dot{y}) \\ &\to \quad \Box_T\big(\varphi(s(\dot{\vec{x}})) \leftrightarrow \varphi(\dot{y})\big) \\ &\to \quad \big(\Box_T\varphi(s(\dot{\vec{x}})) \leftrightarrow \Box_T\varphi(\dot{y})\big) \end{aligned} \tag{7}$$

On the other hand, by the definition of $\ulcorner\varphi(\dot{s})\urcorner$,

$$\mathsf{EA} \vdash s(\vec{x}) = y \to \big(\Box_T\varphi(\dot{s}) \leftrightarrow \Box_T\varphi(\dot{y})\big),$$

which together with (7) yields the claim, q.e.d.

Under the standard interpretation of a sequent as the disjunction of the formulas occurring in it the following lemma holds.

Lemma 4.2 *The rule* Π_n-$\mathsf{RR}^G_{\Pi_n}(T)$ *is equivalent to the schema* $\mathsf{RFN}_{\Sigma_n}(T)$.

Proof. For a reduction of Π_n-$\mathsf{RR}^G_{\Pi_n}(T)$ to $\mathsf{RFN}_{\Sigma_n}(T)$ consider an arbitrary Σ_n-formula $\sigma(a)$. In the formalism of Tait calculus $\neg\neg\sigma$ happens to be graphically the same as σ, so we can derive:

$$\cfrac{\cfrac{\cfrac{\sigma(a), \neg\sigma(a)}{\sigma(a), \neg\mathsf{Prf}_T(b, \ulcorner\sigma(\dot{a})\urcorner)}}{\sigma(a), \forall y \neg\mathsf{Prf}_T(y, \ulcorner\sigma(\dot{a})\urcorner)}}{\forall x\,(\Box_T\sigma(\dot{x}) \to \sigma(x)).}$$

For a reduction of Π_n-$\mathsf{RR}^G_{\Pi_n}(T)$ to $\mathsf{RFN}_{\Sigma_n}(T)$ notice that for any terms s, t and any Π_n-formula φ we have:

$$\begin{aligned} \mathsf{EA} + \mathsf{RFN}_{\Sigma_n}(T) \vdash \varphi(s) &\to \neg\Box_T\neg\varphi(\dot{s}) \\ &\to \forall y \neg\mathsf{Prf}_T(y, \ulcorner\neg\varphi(\dot{s})\urcorner) \\ &\to \neg\mathsf{Prf}_T(t, \ulcorner\neg\varphi(\dot{s})\urcorner), \end{aligned}$$

q.e.d.

Let $\Diamond_T^{\Pi_n}\varphi$ denote $\neg\Box_T^{\Pi_n}\neg\varphi$. Notice that for any φ, $\Diamond_T^{\Pi_n}\varphi$ is EA-equivalent to $\mathsf{RFN}_{\Sigma_n}(T+\varphi)$.

Lemma 4.3 *For $n \geq 1$, the following rules are equivalent (and even congruent in the terminology of [2]):*

(i) Π_n-$\mathsf{RR}_{\Pi_n}(T)$,

(ii) Π_n-$\mathsf{RR}^g_{\Pi_n}(T)$,

(iii) $\dfrac{\Gamma, \varphi(s)}{\Gamma, \Diamond_T^{\Pi_{n-1}}\varphi(\dot{s})}$, *for* $\Gamma \cup \{\varphi\} \subseteq \Pi_n$.

Proof. Reduction of (ii) to (iii) is obvious, because

$$\begin{aligned} \mathsf{EA} \vdash \Diamond_T^{\Pi_{n-1}}\varphi(\dot{s}) &\to \Diamond_T\varphi(\dot{s}) \\ &\to \neg\mathsf{Prf}_T(t, \ulcorner\neg\varphi(\dot{s})\urcorner). \end{aligned}$$

For a reduction of (iii) to (i) we reason as follows. Let $\vec{x}$ denote the list of all the free variables in Γ and s. Notice that, if $\Gamma \subseteq \Pi_n$, then the universal closure of $\bigvee\Gamma \vee \varphi(s) \in \Pi_n$ and we can construct the following derivation:

1. $\bigvee\Gamma(\vec{x}) \vee \varphi(s(\vec{x}))$

2. $\forall \vec{x}\left(\bigvee \Gamma(\vec{x}) \vee \varphi(s(\vec{x}))\right)$

3. $\Diamond_T^{\Pi_{n-1}} \forall \vec{x}\left(\bigvee \Gamma(\vec{x}) \vee \varphi(s(\vec{x}))\right)$ (by Π_n-$\mathsf{RR}_{\Pi_n}(T)$)

4. $\forall \vec{x}\, \Diamond_T^{\Pi_{n-1}}\left(\bigvee \Gamma(\dot{\vec{x}}) \vee \varphi(s(\dot{\vec{x}}))\right)$ (by Löb's conditions from 3)

5. $\Diamond_T^{\Pi_{n-1}}(\bigvee \Gamma(\dot{\vec{x}})) \to \bigvee \Gamma(\vec{x})$ (by provable Σ_n-completeness of $\Box_T^{\Pi_{n-1}}$)

6. $\bigvee \Gamma(\vec{x}) \vee \Diamond_T^{\Pi_{n-1}} \varphi(s(\dot{\vec{x}}))$ (by 4, 5 and Löb's conditions)

7. $\bigvee \Gamma(\vec{x}) \vee \Diamond_T^{\Pi_{n-1}} \varphi(\dot{s})$ (by Lemma 4.1)

In order to reduce (i) to (ii), for any Π_n-formula φ and any Σ_{n-1}-formula $\sigma(x)$ we reason as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\sigma(x), \sigma(x) \quad \varphi}{\varphi \wedge \neg\sigma(x), \sigma(x)}}{\Diamond_T(\varphi \wedge \neg\sigma(\dot{x})), \sigma(x)}}{\neg\Box_{T+\varphi}\sigma(\dot{x}), \sigma(x)}}{\forall x\, (\Box_{T+\varphi}\sigma(\dot{x}) \to \sigma(x))} \quad \text{(by } \Pi_n\text{-}\mathsf{RR}^g_{\Pi_n}(T) \text{ and logic)}$$

This gives the required proof of an arbitrary instance of $\mathsf{RFN}_{\Sigma_{n-1}}(T+\varphi)$ from a derivation of φ, q.e.d.

Resuming the proof of Theorem 2 we show that the standard cut-elimination procedure can be considered as a reduction of $\mathsf{RFN}_{\Sigma_n}(T)$ to Π_n-$\mathsf{RR}^g_{\Pi_n}(T)$. Consider a cut-free derivation of a sequent of the form

$$\neg U, \neg\mathsf{RFN}_{\Sigma_n}(T), \Pi, \tag{8}$$

where Π is a set of Π_n-formulas, $\neg U$ is a finite set of negated axioms of U, and $\neg\mathsf{RFN}_{\Sigma_n}(T)$ is a finite set of negated instances of $\mathsf{RFN}_{\Sigma_n}(T)$ of the form

$$\exists y \exists x\, [\mathsf{Prf}_T(y, \ulcorner\neg\varphi(\dot{x})\urcorner) \wedge \varphi(x)]$$

for some $\varphi(x) \in \Pi_n$. Let $R_\varphi(x,y)$ denote the formula in square brackets. We can also assume that the axioms of U have the form $\forall x_1 \ldots \forall x_m \neg A(x_1, \ldots, x_m)$ for some Π_n-formulas $A(\vec{x})$.

By the subformula property, any formula occurring in the derivation of a sequent Γ of the form (8) either (a) is a Π_n-formula, or (b) has the form $\neg\mathsf{RFN}_{\Sigma_n}(T)$, $\exists x R_\varphi(t,x)$ or $R_\varphi(t,s)$, for appropriate terms s,t, or (c) has the form

$$\exists x_{i+1} \ldots \exists x_m A(t_1, \ldots, t_i, x_{i+1}, \ldots, x_m)$$

for some $i < n$ and terms $t_1, \ldots, t_i$. Let Γ^- denotes the result of deleting all formulas of types (b) and (c) from Γ.

Lemma 4.4 *If a sequent Γ of the form (8) is cut-free provable, then Γ^- is provable from the axioms of U (considered as initial sequents) using the logical rules, including Cut, and the rule Π_n-$\mathsf{RR}^g_{\Pi_n}(T)$.*

Proof goes by induction on the height of the derivation d of Γ. It is sufficient to consider the cases that a formula of the form (b) or (c) is introduced by the last inference in d. Besides, it is sufficient to only consider the formulas of the form $R_\varphi(t,s)$ and $\exists x_m A(t_1,\dots,t_{m-1},x_m)$, because in all other cases after the application of $(\cdot)^-$ the premise and the conclusion of the rule coincide.

So, assume that the derivation d has the form

$$\frac{\mathsf{Prf}_T(t,\ulcorner\neg\varphi(\dot{s})\urcorner),\Delta \quad \varphi(s),\Delta}{R_\varphi(t,s),\Delta}\ (\wedge)$$

where $\varphi\in\Pi_n$. Then by the induction hypothesis we obtain Π_n-$\mathsf{RR}^g_{\Pi_n}(T)$-derivations of the sequents

$$\mathsf{Prf}_T(t,\ulcorner\neg\varphi(\dot{s})\urcorner),\Delta^- \tag{9}$$

and

$$\varphi(s),\Delta^-. \tag{10}$$

Since Δ^- consists of Π_n-formulas, the rule Π_n-$\mathsf{RR}^g_{\Pi_n}(T)$ is applicable to (10), and we obtain a derivation of

$$\neg\mathsf{Prf}_T(t,\ulcorner\neg\varphi(\dot{s})\urcorner),\Delta^-.$$

Applying the Cut-rule with the sequent (9) we obtain the required derivation of Δ^-. If the last inference in d has the form

$$\frac{A(t_1,\dots,t_{m-1},t_m),\Delta}{\exists x_m A(t_1,\dots,t_{m-1},x_m),\Delta}\ (\exists)$$

then by the induction hypothesis we obtain a Π_n-$\mathsf{RR}^g_{\Pi_n}(T)$-derivation of

$$A(t_1,\dots,t_{m-1},t_m),\Delta^-.$$

Then a derivation of

$$\exists x_1\dots\exists x_m A(x_1,\dots,x_m),\Delta^-$$

is obtained by several applications of the rule $(\exists)$. The sequent Δ^- is now derived applying the Cut-rule with the axiom sequent $\forall x_1\dots\forall x_m\neg A(x_1,\dots,x_m)$, q.e.d.

Theorem 2 now follows immediately from Lemmas 4.4 and 4.3, q.e.d.

Proposition 4.5 *If T is a Π_{n+1}-axiomatized extension of EA, then $T+\mathsf{RFN}_{\Sigma_n}(T)$ is Π_n-conservative over $(T)^n_\omega$.*

This statement has been obtained (by other methods) for $T=\mathsf{PRA}$ in [25], and for $n=1$ and $T=\mathsf{EA}$ in [1].

Proof. It is sufficient to notice that $(T)^n_\omega$ is closed under the rule Π_n-$\mathsf{RR}_{\Pi_n}(T)$, q.e.d.

Proposition 4.6 *If U is a Π_{n+1}-axiomatized extension of EA, then $U + \mathsf{RFN}_{\Sigma_n}(T)$ is a Σ_{n+1}-conservative extension of $U + \mathsf{Rfn}_{\Sigma_n}^{\Pi_{n-1}}(T)$. In particular, if U is Π_2-axiomatized, then $U + \mathsf{RFN}_{\Sigma_1}(T)$ is Σ_2-conservative over $U + \mathsf{Rfn}_{\Sigma_1}(T)$.*

Proof. Assume $U + \mathsf{RFN}_{\Sigma_n}(T) \vdash \sigma$ for a sentence $\sigma \in \Sigma_{n+1}$, then

$$U + \neg\sigma + \mathsf{RFN}_{\Sigma_n}(T) \vdash \bot,$$

and by Theorem 2

$$U + \neg\sigma + \Pi_n\text{-}\mathsf{RR}_{\Pi_n}(T) \vdash \bot.$$

Notice that the rule Π_n-$\mathsf{RR}_{\Pi_n}(T)$ is obviously reducible to the schema $\mathsf{Rfn}_{\Sigma_n}^{\Pi_{n-1}}(T)$. Hence, we obtain

$$U + \neg\sigma + \mathsf{Rfn}_{\Sigma_n}^{\Pi_{n-1}}(T) \vdash \bot,$$

and by deduction theorem

$$U + \mathsf{Rfn}_{\Sigma_n}^{\Pi_{n-1}}(T) \vdash \sigma,$$

q.e.d.

Thus, from our characterization of parameter-free induction schemata (cf. [2] or Section 7 of this paper) we directly obtain an interesting conservation result due to Kaye, Paris and Dimitracopoulos [12] (by a model-theoretic proof).

Corollary 4.7 *For $n \geq 1$, $I\Sigma_n$ is a Σ_{n+2}-conservative extension of $I\Sigma_n^-$.*

¿From Proposition 4.5 and the characterization of induction rules in terms of reflection principles (cf. [2]) one can also obtain the following theorem of Parsons [21].

Corollary 4.8 *For $n \geq 1$, $I\Sigma_n$ is Π_{n+1}-conservative over $I\Sigma_n^R$.*

Proposition 4.5 also implies that $I\Sigma_1$ is Π_2-conservative over $(\mathsf{EA})^2_\omega$. Together with Corollary 3.3 this implies that $\mathcal{F}(I\Sigma_1)$ coincides with $\mathcal{F}_\omega$, that is, with the class of primitive recursive functions. This well-known result was originally established by C. Parsons [20], G. Takeuti and G. Mints by other methods.

5 Extending conservation results to iterated reflection principles

The definition of progressions based on iteration of reflection principles allows one to directly "extend by continuity" some basic conservation results for reflection principles to their transfinite iterations. In particular, this leads to a useful generalization of Schmerl's fine structure theorem, which will be discussed in the next section. Here we just state a number of such easy extension results. Throughout this section we fix an elementary well-ordering $(D, \prec)$ and an initial elementary presented theory T.

The following proposition generalizes Statement 3 of Theorem 1 in [3].

Proposition 5.1 *The following statements are provable in EA:*

(i) $\forall\alpha\ \mathsf{Rfn}_{\Sigma_1}(T)_\alpha \subseteq \mathsf{Rfn}(T)_\alpha$;

(ii) $\forall\alpha\ \mathsf{Rfn}(T)_\alpha \subseteq_{\mathcal{B}(\Sigma_1)} \mathsf{Rfn}_{\Sigma_1}(T)_\alpha$.

Proof. We give an informal argument by reflexive induction on α. Since both (i) and (ii) are formalized as Π_2-formulas, we may actually argue in $\mathsf{EA} + B\Sigma_1$ (and then use Π_2-conservativity of the latter over EA). Denote $V^\alpha := \mathsf{Rfn}_{\Sigma_1}(T)_\alpha$ and $U^\alpha := \mathsf{Rfn}(T)_\alpha$.

(i) By the definition of (implicit) progressions, modulo provable equivalence every axiom of V^α is either an axiom of T, and in this case there is nothing to prove, or it is an instance of the schema $\mathsf{Rfn}_{\Sigma_1}(V^\beta)$ for some $\beta \prec \alpha$, that is, it has the form $\Box_{V^\beta}\sigma \to \sigma$ for a sentence $\sigma \in \Sigma_1$. By the reflexive induction hypothesis we have

$$\mathsf{EA} \vdash \Box_{V^\beta}\sigma \to \Box_{U^\beta}\sigma,$$

whence

$$\mathsf{EA} \vdash (\Box_{U^\beta}\sigma \to \sigma) \to (\Box_{V^\beta}\sigma \to \sigma).$$

Thus, $U^\alpha \vdash \Box_{V^\beta}\sigma \to \sigma$, by the definition of U^α, that is, every axiom of V^α is provable in U^α, as required. ($B\Sigma_1$ then implies that every *theorem* of V^α is provable in U^α. In the following we shall not mention such uses of $B\Sigma_1$.)

(ii) Assume $U^\alpha \vdash \delta$, where δ is a $\mathcal{B}(\Sigma_1)$-sentence. By the definition of U^α and the formalized deduction theorem there exist $\beta_1, \ldots, \beta_m \prec \alpha$ and sentences $\varphi_1, \ldots, \varphi_m$ such that

$$T \vdash \bigwedge_{i=1}^{m} (\Box_{U^{\beta_i}}\varphi_i \to \varphi_i) \to \delta.$$

By provable monotonicity, stipulating $\beta := \max_\prec\{\beta_1, \ldots, \beta_m\}$, we obtain

$$T \vdash \bigwedge_{i=1}^{m} (\Box_{U^\beta}\varphi_i \to \varphi_i) \to \delta. \tag{11}$$

Lemmas 4 and 5 in [3] then yield $\mathcal{B}(\Sigma_1)$-sentences $\psi_1, \ldots, \psi_m$ such that

$$T \vdash \bigwedge_{i=1}^{m} (\Box_{U^\beta}\psi_i \to \psi_i) \to \delta. \tag{12}$$

(Note that these sentences together with a proof of (12) are constructed elementarily from the proof (11).) The reflexive induction hypothesis implies

$$\mathsf{EA} \vdash \Box_{U^\beta}\psi_i \to \Box_{V^\beta}\psi_i,$$

whence

$$T \vdash \bigwedge_{i=1}^{m} (\Box_{V^\beta}\psi_i \to \psi_i) \to \delta.$$

Thus, $V^\alpha \vdash \delta$, by the definition of V^α, q.e.d.

The following generalization of Statements 1 and 2 of Theorem 1 in [3] is similarly proved.

Proposition 5.2 *For all $n > 1$, the following statements are provable in EA:*

(i) $\forall\alpha\ \mathsf{Rfn}(T)_\alpha \equiv_{\Sigma_n} \mathsf{Rfn}_{\Sigma_n}(T)_\alpha$;

(ii) $\forall\alpha\ \mathsf{Rfn}(T)_\alpha \equiv_{\Pi_n} \mathsf{Rfn}_{\Pi_n}(T)_\alpha$.

We also state without proof an obvious relativization of 5.1.

Proposition 5.3 *For all $n \geq 1$, the following statements are provable in EA:*

$$\forall\alpha\ \mathsf{Rfn}^{\Pi_n}(T)_\alpha \equiv_{\mathcal{B}(\Sigma_{n+1})} \mathsf{Rfn}^{\Pi_n}_{\Sigma_{n+1}}(T)_\alpha.$$

The next proposition generalizes Proposition 4.6. Its proof is based on a formalization of Theorem 2, which is possible in EA^+, but not in EA itself. (The nonelementarity is only due to the application of cut-elimination in that proof.)

Proposition 5.4 *If T is a Π_{n+1}-axiomatized extension of EA, then the following is provable in EA^+:*

$$\forall\alpha\ \mathsf{RFN}_{\Sigma_n}(T)_\alpha \equiv_{\Sigma_{n+1}} \mathsf{Rfn}^{\Pi_{n-1}}_{\Sigma_n}(T)_\alpha.$$

Proof. Inclusion $(\supseteq)$ is obvious. We give a proof of $(\subseteq_{\Sigma_{n+1}})$ for $n = 1$ (for $n > 1$ the proof is no different, but the notation would be heavier to read). Denote $U^\alpha := \mathsf{RFN}_{\Sigma_1}(T)_\alpha$; $V^\alpha := \mathsf{Rfn}_{\Sigma_1}(T)_\alpha$. We give an informal argument by reflexive induction on α in EA^+.

Assume $U^\alpha \vdash \sigma$, where $\sigma \in \Sigma_2$. By the definition of U^α, for some $\beta \prec \alpha$ we have

$$T + \mathsf{RFN}_{\Sigma_1}(U^\beta) \vdash \sigma.$$

Notice that by the monotonicity of V^β

$$T + \mathsf{RFN}_{\Sigma_1}(V^\beta) \vdash \mathsf{RFN}_{\Sigma_1}(\mathsf{EA}),$$

that is, $T + \mathsf{RFN}_{\Sigma_1}(V^\beta)$ contains EA^+. On the other hand, by the reflexive induction hypothesis

$$\mathsf{EA}^+ \vdash \forall x \in \Sigma_1\ (\Box_{U^\beta}(x) \leftrightarrow \Box_{V^\beta}(x)),$$

whence

$$\mathsf{EA}^+ \vdash \mathsf{RFN}_{\Sigma_1}(U^\beta) \leftrightarrow \mathsf{RFN}_{\Sigma_1}(V^\beta).$$

Therefore, $\mathsf{RFN}_{\Sigma_1}(U^\beta)$ is contained in $T + \mathsf{RFN}_{\Sigma_1}(V^\beta)$, and thus,

$$T + \mathsf{RFN}_{\Sigma_1}(V^\beta) \vdash \sigma.$$

It follows that

$$T + \neg\sigma + \mathsf{RFN}_{\Sigma_1}(V^\beta) \vdash \bot,$$

and by (formalized) Theorem 2

$$T + \neg\sigma + \Pi_1\text{-}\mathsf{RR}_{\Pi_1}(V^\beta) \vdash \bot,$$

and

$$T + \neg\sigma + \mathsf{Rfn}_{\Sigma_1}(V^\beta) \vdash \bot.$$

Thus,

$$T + \mathsf{Rfn}_{\Sigma_1}(V^\beta) \vdash \sigma,$$

that is, $V_\alpha \vdash \sigma$, q.e.d.

6 Schmerl's formula

Our approach to Schmerl's formula borrows a general result from [3] relating the hierarchies of iterated local reflection principles and of iterated consistency assertions over an arbitrary initial theory T. This result and the results below hold under the assumption that $(D, \prec)$ is a *nice* elementary well-ordering. A nice well-ordering is an elementary well-ordering equipped with elementary terms representing the ordinal constants and functions $0, 1, +, \cdot, \omega^x$. These functions should provably in EA satisfy some minimal obvious axioms NWO listed in [3]. Besides, there should be an elementary EA-provable isomorphism between natural numbers (with the usual order) and the ordinals $\prec \omega$. Under these assumptions on the class of well-orderings we have the following theorem [3].

Proposition 6.1 *EA proves that, for all α, β such that $\alpha \succeq 1$, there holds*

$$(\mathsf{Rfn}(T)_\alpha)_\beta \equiv_{\Pi_1} T_{\omega^\alpha \cdot (1+\beta)}.$$

In particular, for all $\alpha \succeq 1$, $\mathsf{Rfn}(T)_\alpha \equiv_{\Pi_1} T_{\omega^\alpha}$.

A proof is obtained by elementary reflexive induction and for the nontrivial inclusion ($\subseteq_{\Pi_1}$) it only uses provability logic. Since the relativized provability predicates satisfy the same provability logic, essentially the same argument also yields the following theorem.

Proposition 6.2 *For any $n \geq 1$, provably in EA,*

$$\forall \alpha \succeq 1\ \forall \beta \quad (\mathsf{Rfn}^{\Pi_{n-1}}(T)_\alpha)_\beta \equiv_{\Pi_n} (T)^n_{\omega^\alpha \cdot (1+\beta)}.$$

Proof. One only has to notice that provably in EA, for all α,

$$\mathsf{Con}^{\Pi_{n-1}}(T)_\alpha \equiv \mathsf{RFN}_{\Pi_n}(T)_\alpha \equiv (T)^n_\alpha,$$

which can be verified by a straightforward reflexive induction on α, q.e.d.

Now we can combine this result with Proposition 5.4 and obtain a generalization of Schmerl's theorem. (U. Schmerl [25] proves this statement for specially defined progressions corresponding to $(\mathsf{PRA})^n_\alpha$).

Theorem 3 (Schmerl's formula) *For all $n \geq 1$, if T is an elementary presented Π_{n+1}-axiomatized extension of EA, the following holds provably in EA^+:*

$$\forall \alpha \succeq 1\ (T)^{n+1}_{\alpha} \equiv_{\Pi_n} (T)^n_{\omega^\alpha}.$$

Proof. It is sufficient to notice that for all $\alpha \succeq 1$,

$$(T)^{n+1}_{\alpha} \equiv_{\Sigma_{n+1}} \mathsf{Rfn}^{\Pi_{n-1}}_{\Sigma_n}(T)_{\alpha} \equiv_{\Pi_n} (T)^n_{\omega^\alpha},$$

by Propositions 5.4 and 6.2, respectively, q.e.d.

Next we observe an easy but useful extension lemma.

Lemma 6.3 *Let U, V be elementary presented extensions of EA, and Γ be one of the classes Σ_{k+1}, Π_{k+1} or $\mathcal{B}(\Sigma_k)$, for $k \geq n \geq 1$. Then*

$$\mathsf{EA} \vdash U \subseteq_{\Gamma} V \quad \Longrightarrow \quad \mathsf{EA} \vdash \forall \alpha\ (U)^n_{\alpha} \subseteq_{\Gamma} (V)^n_{\alpha}.$$

Proof. Reasoning by reflexive induction on α assume $U_\alpha \vdash \varphi$ for a sentence $\varphi \in \Gamma$. Then for some $\beta \prec \alpha$,

$$U + \mathsf{RFN}_{\Pi_n}((U)^n_{\beta}) \vdash \varphi.$$

By reflexive induction hypothesis

$$U + \mathsf{RFN}_{\Pi_n}((V)^n_{\beta}) \vdash \varphi,$$

whence

$$U \vdash \neg\mathsf{RFN}_{\Pi_n}((V)^n_{\beta}) \vee \varphi.$$

The latter formula is in Γ (modulo logical equivalence), therefore

$$V \vdash \neg\mathsf{RFN}_{\Pi_n}((V)^n_{\beta}) \vee \varphi,$$

and thus $V_\alpha \vdash \varphi$, q.e.d.

Remark 6.4 This lemma also holds for EA^+ in place of EA, with the same proof.

The ordinal functions $\omega_n(\alpha)$ are introduced as usual:

$$\begin{cases} \omega_0(\alpha) & := \quad \alpha \\ \omega_{k+1}(\alpha) & := \quad \omega^{\omega_k(\alpha)} \end{cases}$$

Denote $\omega_n := \omega_n(1)$, $\quad \epsilon_0 := \sup\{\omega_n : n < \omega\}$.

Our next theorem generalizes Theorem 3 to mixed hierarchies (for $T = \mathsf{PRA}$ established by Schmerl).

Theorem 4 *For all $n, m \geq 1$, if T is an elementary presented Π_{n+1}-axiomatized extension of EA, the following statements hold provably in EA^+:*

(i) $\forall \alpha \succeq 1\ (T)^{n+m}_\alpha \equiv_{\Pi_n} (T)^n_{\omega_m(\alpha)}$;

(ii) $\forall \alpha \succeq 1\ ((T)^{n+m}_\alpha)^n_\beta \equiv_{\Pi_n} (T)^n_{\omega_m(\alpha)\cdot(1+\beta)}$.

Proof. Part (i) follows by m-fold application of Theorem 3. For a proof of (ii) notice that by (i) and 5.4

$$(T)^{n+m}_\alpha \equiv_{\Pi_{n+1}} (T)^{n+1}_{\omega_{m-1}(\alpha)} \equiv_{\Sigma_{n+1}} \mathsf{Rfn}^{\Pi_{n-1}}_{\Sigma_n}(T)_{\omega_{m-1}(\alpha)}.$$

Lemma 6.3 and Proposition 6.2 imply that

$$((T)^{n+m}_\alpha)^n_\beta \equiv_{\Pi_{n+1}} ((T)^{n+1}_{\omega_{m-1}(\alpha)})^n_\beta \equiv_{\Sigma_{n+1}} (\mathsf{Rfn}^{\Pi_{n-1}}_{\Sigma_n}(T)_{\omega_{m-1}(\alpha)})^n_\beta \equiv_{\Pi_n} (T)^n_{\omega^{\omega_{m-1}(\alpha)}\cdot(1+\beta)},$$

which yields the required formula

$$((T)^{n+m}_\alpha)^n_\beta \equiv_{\Pi_n} (T)^n_{\omega_m(\alpha)\cdot(1+\beta)},$$

q.e.d.

This theorem directly applies to theories T like EA or PRA. Following Schmerl's idea it is also possible to derive from it similar formulas for iterated reflection principles over PA, which we obtain in the next section.

Corollary 6.5 *If T is a Π^0_2-regular theory, then it is Π^0_1-regular and $|T|_{\Pi^0_1}$ is one ω-power higher than $|T|_{\Pi^0_2}$.*

Proof. If T is a Π^0_2-regular theory and $|T|_{\Pi^0_2} = \alpha$, then $T \equiv_{\Pi_2} (\mathsf{EA})^2_\alpha$. By Theorem 3 $(\mathsf{EA})^2_\alpha \equiv_{\Pi_1} \mathsf{EA}_{\omega^\alpha}$, which means that $T \equiv_{\Pi_1} \mathsf{EA}_{\omega^\alpha}$, that is, T is Π^0_1-regular and $|T|_{\Pi^0_1} = \omega^\alpha$, q.e.d.

7 Ordinal analysis of fragments

Now we have at our disposal all necessary tools to give an ordinal analysis of arithmetic and its fragments. The general methodology bears similarities with the traditional Π^1_1-ordinal analysis, see [23]. To determine the ordinal of a given formal system T one first finds a suitable *embedding* of T into the hierarchy of reflection principles over EA. Then one applies Schmerl's formula for a *reduction* of the reflection principles axiomatizing T to iterated reflection principles of lower complexity. ¿From this point of view the use of Schmerl's formula substitutes the use of cut-elimination for ω-logic. Notice that, although the meaning of the ordinals is different, ordinal bounds are essentially the same in both approaches. Also notice that in the present approach the embedding part is more informative. In particular, this allows to obtain upper and lower bounds for proof-theoretic ordinals simultaneously.

The following embedding results are known ($n \geq 1$).

(E1) Leivant and Ono [15, 18] show that $I\Sigma_n$ is equivalent to $\mathsf{RFN}_{\Pi_{n+2}}(\mathsf{EA})$ over EA, that is,

$$I\Sigma_n \equiv (\mathsf{EA})_1^{n+2}.$$

Notice that this is sharper than the related results in [25] and the original result by Kreisel and Lévy [14] stating that

$$\mathsf{PA} \equiv \mathsf{EA} + \mathsf{RFN}(\mathsf{EA}).$$

(E2) For the closures of EA under Σ_n- and Π_{n+1}-induction rules we have the following characterization (cf. [2]):

$$I\Sigma_n^R \equiv I\Pi_{n+1}^R \equiv (\mathsf{EA})_\omega^{n+1}.$$

(E3) Parameter-free induction schemata have been characterized in [5]:

(a) $I\Sigma_n^- \equiv \mathsf{EA} + \mathsf{Rfn}_{\Sigma_{n+1}}^{\Pi_n}(\mathsf{EA})$.

(b) $I\Pi_{n+1}^- \equiv \mathsf{EA} + \mathsf{Rfn}_{\Sigma_{n+2}}^{\Pi_n}(\mathsf{EA})$.

(c) $\mathsf{EA}^+ + I\Pi_1^- \equiv \mathsf{EA}^+ + \mathsf{Rfn}_{\Sigma_2}(\mathsf{EA}) \equiv \mathsf{EA}^+ + \mathsf{Rfn}_{\Sigma_2}(\mathsf{EA}^+)$.

Over EA the schema $I\Pi_1^-$ is equivalent to the local Σ_2-reflection principle for EA formulated for the predicate of *cut-free provability*, see [5] and Appendix C for more details.

Remark 7.1 The upper bound results only require the embeddings (E1)–(E3) from left to right, that is, the provability of the induction principles from suitable forms of the reflection principles. All these embeddings are very easy to prove using a trick by Kreisel [14]. For example, in order to prove $I\Sigma_n \subseteq (\mathsf{EA})_1^{n+2}$ let $\sigma(x, a)$ be any Σ_n-formula and consider the formula $\psi(x, a) := \sigma(0, a) \wedge \forall u\, (\sigma(u, a) \to \sigma(u + 1, a)) \to \sigma(x, a)$. ψ is logically equivalent to a Π_{n+2}-formula. By an elementary induction on m it is easy to see that, for all m, k, $\mathsf{EA} \vdash \psi(\overline{m}, \overline{k})$, and this fact is formalizable in EA. Hence $\mathsf{EA} \vdash \forall x, a\, \Box_{\mathsf{EA}}\psi(\dot{x}, \dot{a})$, and applying uniform Π_{n+2}-reflection yields

$$\mathsf{EA} + \mathsf{RFN}_{\Pi_{n+2}}(\mathsf{EA}) \vdash \forall x, a\, \psi(x, a),$$

which is equivalent to an instance of $I\Sigma_1$. The proofs of the corresponding embeddings in (E2) and (E3) are rather similar.

The embeddings (E1)–(E3) from right to left are increasingly more difficult to prove (but, if one is only intersted in the upper bound results, this is not strictly necessary). The simplest argument for PA, due to Kreisel and Lévy [14], goes from an EA-proof of a formula φ to a cut-free derivation in Tait calculus of a sequent consisting of φ and some negated instances of the axioms of EA. All formulas occurring in this derivation are Π_n, where n is the maximum of the logical complexity of φ and 2 (since all the axioms of EA are Π_1). Then one can use a truthpredicate for Π_n-formulas and prove by induction on the depth of the cut-free derivation that all sequents occurring in it are true. This induction argument is clearly formalizable in PA.

Remark 7.2 All the embedding results mentioned in (E1)–(E3) can be naturally formalized in EA. This is obvious for the finitely axiomatized systems in (E1) and can be checked for (E2) and (E3) as well.

Now we obtain some corollaries, starting from the simplest analysis of $I\Sigma_n$ and PA.

Proposition 7.3 *For all $n \geq 1$,*

(i) $I\Sigma_n \equiv_{\Pi_2} (\mathit{EA})^2_{\omega_n} \equiv_{\Pi_1} \mathit{EA}_{\omega_{n+1}}$*, hence* $|I\Sigma_n|_{\Pi^0_1} = \omega_{n+1}$*;*

(ii) $\mathit{PA} \equiv_{\Pi_n} (\mathit{EA})^n_{\epsilon_0}$*, hence* $|\mathit{PA}|_{\Pi^0_n} = \epsilon_0$*.*

Proof. Statement (i) follows from (E1) and Theorem 4. Statement (ii) similarly follows from the equivalence

$$\mathsf{PA} \equiv \bigcup_{m\geq 0} (\mathsf{EA})^{n+m}_1 \equiv_{\Pi_n} \bigcup_{m\geq 0} (\mathsf{EA})^n_{\omega_m(1)},$$

which holds for any $n \geq 1$, q.e.d.

Remark 7.4 Part (i) of this proposition is formalizable in EA^+. Part (ii) will be formalizable in EA^+ under some additional assumptions on the choice of a nice well-ordering. For example, we can extend NWO by a binary function symbol for $\omega_n(\alpha)$ and a constant symbol ϵ_0 with the obvious defining axioms. The extended theory will be denoted $\mathrm{NWO}(\epsilon_0)$, and we require that the nice well-ordering interprets these axioms by elementary functions.

For the induction rules from (E2) we obtain the same bounds ($n \geq 1$).

Proposition 7.5 $I\Sigma^R_n \equiv_{\Pi_2} (\mathit{EA})^2_{\omega_n} \equiv_{\Pi_1} \mathit{EA}_{\omega_{n+1}}$.

For parameter-free induction schemata we have, by the conservation results for local reflection principles in [2, 5] and Section 4 of this paper

Proposition 7.6 $(\mathit{EA})^{n+2}_1 \equiv_{\Sigma_{n+2}} I\Sigma^-_n \equiv_{\mathcal{B}(\Sigma_{n+1})} I\Pi^-_{n+1}$.

It follows that, for $n \geq 1$, the theories $I\Sigma^-_n$, $I\Pi^-_{n+1}$ and $I\Sigma_n$ have the same Π^0_2- and Π^0_1-ordinals.

In [5] the theory $I\Sigma_n + I\Pi^-_{n+1}$ is analyzed ($n \geq 1$). On the basis of (E3)(b) it is shown that

$$I\Sigma_n + I\Pi^-_{n+1} \equiv_{\Pi_{n+1}} (I\Sigma_n)^{n+1}_\omega \equiv ((\mathsf{EA})^{n+2}_1)^{n+1}_\omega.$$

Applying Theorem 4 yields the following corollary, which determines its Π^0_2- and Π^0_1-ordinals.

Proposition 7.7 $I\Sigma_n + I\Pi^-_{n+1} \equiv_{\Pi_2} (\mathit{EA})^2_{\omega_n(2)} \equiv_{\Pi_1} \mathit{EA}_{\omega_{n+1}(2)}$.

Now we consider the exceptional case of the parameter-free Π_1-induction schema. We first analyze the system $\mathsf{EA}^+ + I\Pi_1^-$, the weaker theory $\mathsf{EA} + I\Pi_1^-$ will be treated in Appendix C.

Notice that by (E3)(c) the theory $\mathsf{EA}^+ + I\Pi_1^-$ is certainly not Π_2-conservative (not even Π_1-conservative) over EA^+. Yet, the next proposition shows that its class of provably total computable functions coincides with that of EA^+. This means that $\mathsf{EA}^+ + I\Pi_1^-$ is not Π_2-regular (its Π^0_2-ordinal equals 1).

Proposition 7.8 *(i)* $\mathcal{F}(\mathsf{EA}^+ + I\Pi_1^-) = \mathcal{F}(\mathsf{EA}^+) = \mathcal{F}_1$;

(ii) $\mathcal{F}(\mathsf{EA} + I\Pi_1^-) = \mathcal{F}_0 = \mathcal{E}$.

Proof. By (E3)(c), $\mathsf{EA}^+ + I\Pi_1^-$ is contained in $\mathsf{EA}^+ + \mathsf{Rfn}(\mathsf{EA}^+)$ and similarly

$$\mathsf{EA} + I\Pi_1^- \subseteq \mathsf{EA} + \mathsf{Rfn}(\mathsf{EA}).$$

Feferman [7] noticed that $\mathsf{Rfn}(T)$ is provable in T together with all true Π_1-sentences. Yet, it is equally well-known that adding any amount of true Π_1-axioms to a sound theory does not increase its class of provably total computable functions. This proves both parts (i) and (ii), q.e.d.

Proposition 7.9 *$\mathsf{EA}^+ + I\Pi_1^-$ is Π^0_1-regular and $|\mathsf{EA}^+ + I\Pi_1^-|_{\Pi^0_1} = \omega^2$.*

Proof. Recall that by Proposition 6.1 (for this particular case established by Goryachev [10]) $T + \mathsf{Rfn}(T)$ is Π_1-conservative over T_ω. Hence, by Theorem 4,

$$\mathsf{EA}^+ + I\Pi_1^- \equiv_{\Pi_1} (\mathsf{EA}^+)_\omega \equiv ((\mathsf{EA})^2_1)_\omega \equiv_{\Pi_1} \mathsf{EA}_{\omega^2}.$$

Thus, the theory is Π^0_1-regular with the ordinal ω^2, q.e.d.

Now we consider the extensions of PA by reflection principles. The following proposition holds for nice well-orderings satisfying $\mathrm{NWO}(\epsilon_0)$. Notice that the function ϵ_0^α can be expressed in $\mathrm{NWO}(\epsilon_0)$ by the term $\omega^{\epsilon_0\cdot\alpha}$.

Proposition 7.10 *For each $n \geq 1$, provably in EA^+,*

(i) $(\mathsf{PA})^n_\alpha \equiv_{\Pi_n} (\mathsf{EA})^n_{\epsilon_0\cdot(1+\alpha)}$;

(ii) $(\mathsf{PA})^{n+1}_\alpha \equiv_{\Pi_n} (\mathsf{PA})^n_{\epsilon_0^\alpha}$, *if* $\alpha \succeq 1$.

Proof. By formalized Proposition 7.3(ii), provably in EA^+,

$$\mathsf{PA} \equiv_{\Pi_{n+1}} (\mathsf{EA})^{n+1}_{\epsilon_0}.$$

By Lemma 6.3 and Theorem 4(ii) we then obtain:

$$(\mathsf{PA})^n_\alpha \equiv_{\Pi_{n+1}} ((\mathsf{EA})^{n+1}_{\epsilon_0})^n_\alpha \equiv_{\Pi_n} (\mathsf{EA})^n_{\epsilon_0\cdot(1+\alpha)}.$$

Formula (ii) follows from (i) and Theorem 3, because for $\alpha \succeq 1$ one has

$$(\mathsf{EA})^{n+1}_{\epsilon_0\cdot(1+\alpha)} \equiv_{\Pi_n} (\mathsf{EA})^n_{\omega^{\epsilon_0\cdot(1+\alpha)}} \equiv (\mathsf{EA})^n_{\epsilon_0\cdot(1+\epsilon_0^\alpha)} \equiv_{\Pi_n} (\mathsf{PA})^n_{\epsilon_0^\alpha},$$

q.e.d.

As a particular case we obtain that $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ is a Π^0_1-regular theory with the ordinal $\omega \cdot 2$.

Corollary 7.11 $PA + \mathsf{Con}(PA) \equiv_{\Pi_1} EA_{\epsilon_0 \cdot 2}$.

Since $\mathsf{Con}(\mathsf{PA})$ is a true Π_1-sentence, $|\mathsf{PA} + \mathsf{Con}(\mathsf{PA})|_{\Pi_2^0} = \epsilon_0$, therefore $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ is not Π_2^0-regular.

As another example of this sort let us compute the ordinals of the following Π_1^0-regular, but Π_2^0-irregular, theories.

Proposition 7.12 *(i)* $|I\Sigma_1 + \mathsf{Con}(PA)|_{\Pi_1^0} = \epsilon_0 + \omega^\omega$;

(ii) $|EA^+ + \mathsf{Con}(I\Sigma_1)|_{\Pi_1^0} = \omega^\omega + \omega$.

Proof. For (i) notice that by Theorem 4 provably in EA^+,

$$(I\Sigma_1)^1_{\epsilon_0} \equiv ((\mathsf{EA})^3_1)^1_\omega \equiv_{\Pi_1} \mathsf{EA}_{\omega_2(1)\cdot(1+\epsilon_0)} \equiv \mathsf{EA}_{\epsilon_0} \equiv_{\Pi_1} \mathsf{PA}.$$

Therefore,

$$\mathsf{EA}^+ \vdash \mathsf{Con}(\mathsf{PA}) \leftrightarrow \mathsf{Con}((I\Sigma_1)^1_{\epsilon_0}).$$

It follows that

$$I\Sigma_1 + \mathsf{Con}(\mathsf{PA}) \equiv I\Sigma_1 + \mathsf{Con}((I\Sigma_1)^1_{\epsilon_0}) \equiv (I\Sigma_1)^1_{\epsilon_0+1}.$$

For the latter theory we have

$$(I\Sigma_1)^1_{\epsilon_0+1} \equiv ((\mathsf{EA})^3_1)^1_{\epsilon_0+1} \equiv_{\Pi_1} \mathsf{EA}_{\omega_2\cdot(\epsilon_0+1)} \equiv \mathsf{EA}_{\epsilon_0+\omega^\omega}.$$

For (ii) reasoning in a similar way we obtain

$$\mathsf{EA}^+ + \mathsf{Con}(I\Sigma_1) \equiv ((\mathsf{EA})^2_1)_{\omega^\omega+1} \equiv_{\Pi_1} \mathsf{EA}_{\omega\cdot(1+\omega^\omega+1)} \equiv \mathsf{EA}_{\omega^\omega+\omega},$$

q.e.d.

8 Conclusion and further work

This paper demonstrates the use of reflection principles for the ordinal analysis of fragments of Peano arithmetic. More importantly, reflection principles provide a uniform definition of proof-theoretic ordinals for any arithmetical complexity Π_n^0, in particular, for the complexity Π_1^0.

The results of this paper are further developed in our later paper [6], where the notion of *graded provability algebra* is introduced. It provides an abstract algebraic framework for proof-theoretic analysis and links canonical ordinal notation systems with certain algebraic models of provability logic. We hope that this further development will shed additional light on the problem of canonicity of ordinal notations.

9 Appendix A

Let $(D, \prec)$ be an elementary well-ordering. Define

$$\begin{aligned} \alpha[x] &:= \max\nolimits_{\prec}\{\beta \leq x : \beta \prec \alpha\} \\ \beta \prec_x \alpha &:\Leftrightarrow (\beta \leq x \wedge \beta \prec a). \end{aligned}$$

Recall that the functions F_α are defined as follows:

$$F_\alpha(x) := \max\{2^x_x + 1\} \cup \{F^{(v)}_\beta(u) + 1 : \beta \prec \alpha,\ u, v, \beta \leq x\}.$$

For technical convenience we also define $F_{-1}(x) = 2^x$ and $\alpha[x] = -1$, if there is no $\beta \prec_x \alpha$.

Lemma 9.1 *For all α, β, x, y,*

(i) $x \leq y \to F_\alpha(x) \leq F_\alpha(y)$;

(ii) $\beta \prec \alpha \to F_\beta(x) \leq F_\alpha(x)$.

Proof. Part (i) is obvious. Part (ii) follows from the fact that

$$\gamma \prec_x \beta \prec \alpha \Rightarrow \gamma \prec_x \alpha,$$

q.e.d.

Lemma 9.2 *For all α, x, $F_\alpha(x) = F^{(x)}_{\alpha[x]}(x) + 1$.*

Proof. This is obvious for $\alpha[x] = -1$. Otherwise, from Part (i) of the previous lemma we obtain

$$u, v \leq x \to F^{(v)}_\beta(u) \leq F^{(x)}_\beta(x).$$

Part (ii) a nd Part (i) by an elementary induction on y then yield

$$\beta \prec \alpha \to F^{(y)}_\beta(x) \leq F^{(y)}_\alpha(x).$$

Hence, if $u, v \leq x$ and $\beta \prec_x \alpha$, then $\beta \prec_x \alpha[x]$ or $\beta = \alpha[x]$, and in both cases

$$F^{(v)}_\beta(u) \leq F^{(x)}_{\alpha[x]}(x),$$

q.e.d.

We now define evaluation trees. An *evaluation tree* is a finite tree labeled by tuples of the form $\langle \alpha, x, y \rangle$ satisfying the following conditions:

1. If there is no $\beta \prec_x \alpha$, then $\langle \alpha, x, y \rangle$ is a terminal node and $y = 2^x_x + 1$.
2. Otherwise, there are x immediate successors of $\langle \alpha, x, y \rangle$, and their labels have the form

$$\langle \alpha[x], x, y_1 \rangle,\ \langle \alpha[x], y_1, y_2 \rangle, \ldots , \langle \alpha[x], y_{x-1}, y_x \rangle$$

and $y = y_x + 1$.

Obviously, the relation x *is a code of an evaluation tree* is elementary.

Lemma 9.3 *(i) If a node of an evaluation tree is labeled by $\langle \alpha, x, y \rangle$, then $F_\alpha(x) = y$.*

(ii) If $F_\alpha(x) = y$, then there is an evaluation tree with the root labeled by $\langle \alpha, x, y \rangle$.

Proof. Part (i) is proved by transfinite induction on α. If $\alpha[x] = -1$, the statement is obvious. Otherwise, $\alpha[x] \prec \alpha$, hence by the induction hypothesis at the immediate successor nodes one has $F_{\alpha[x]}(y_i) = y_{i+1}$ for all $i < x$ (where we also put $y_0 := x$). It follows that $y_x = F^{(x)}_{\alpha[x]}(x)$, whence $y = y_x + 1 = F_\alpha(x)$.

Part (ii) obviously follows from the definition of F_α, q.e.d.

Now we observe that, for any evaluation tree T, whose root is labeled by $\langle \alpha, x, y \rangle$, the value $\max(\alpha, y)$ is a common bound to the following parameters:

(a) each γ, u, v such that $\langle \gamma, u, v \rangle$ occurs in T;

(b) the number of branches at every node of T;

(c) the depth of T.

Ad (a): If $\langle \gamma, u, v \rangle$ is an immediate successor of $\langle \alpha, x, y \rangle$, then $\gamma = \alpha[x] \leq x$, and $u, v \leq y$ by the monotonicity of F.

Statement (b) follows from (a) and the fact that the number of branches at a node $\langle \gamma, u, v \rangle$ equals u.

Statement (c) follows from the observation that if $\langle \gamma, u, v \rangle$ is an immediate successor of $\langle \alpha, x, y \rangle$, then $y > v$.

An immediate corollary of (a)–(c) is that the code of the evaluation tree T is bounded by the value $g(\max(\alpha, y))$, for an elementary function g. Hence we obtain

Proof of Lemma 3.1. Using Lemma 9.3 the relation $F_\alpha(x) = y$ can be expressed by formalizing the statement that *there is an evaluation tree with the code $\leq g(\max(\alpha, y))$, whose root is labeled by $\langle \alpha, x, y \rangle$.* All quantifiers here are bounded, hence the relation $F_\alpha(x) = y$ is elementary, q.e.d.

Inspecting the definition of the relation $F_\alpha(x) = y$ notice that the proofs of the monotonicity properties and bounds on the size of the tree only required elementary induction (transfinite induction is not used). Hence, these properties together with the natural defining axioms for F_α can be verified in EA. This yields a proof of Lemma 3.2. Here we just formally state the required properties of F_α formalizable in EA.

F1. $(\forall \beta \leq x \neg \beta \prec \alpha) \to [F_\alpha(x) \simeq y \leftrightarrow y = 2^x_x + 1]$

F2. $F_\alpha(x) \simeq y \wedge \beta \leq x \wedge \beta \prec \alpha \to \exists z \leq y \, \exists u, v \leq x \; F^{(v)}_\beta(u) \simeq z$

F3. $\forall \beta, u, v \leq x (\beta \prec \alpha \to \exists y \leq z \; F^{(v)}_\beta(u) \simeq y) \to \exists y \leq z \; F_\alpha(x) \simeq y$.

Here, as usual, $F^{(x)}_\beta(x) \simeq y$ abbreviates

$$\exists s \in Seq \, [(s)_0 = x \wedge \forall i < x \; F_\beta((s)_i) \simeq (s)_{i+1} \wedge (s)_x = y].$$

10 Appendix B

One can roughly classify the existing definitions of proof-theoretic ordinals in two groups, which I call the definitions 'from below' and 'from above'. Informally speaking, proof-theoretic ordinals defined from below measure the strength of the principles of certain complexity Γ that are *provable* in a given theory T. In contrast, the ordinals defined from above measure the strength of certain characteristic for T *unprovable* principles of complexity Γ. For example, $\mathsf{Con}(T)$ is such a characteristic principle of complexity Π^0_1.

The standard Π^1_1- and Π^0_2-ordinals are defined from below, and so are the Π^0_n-ordinals introduced in this paper. The notorious ordinal of the shortest natural primitive recursive well-ordering $\prec$ such that $TI_{p.r.}(\prec)$ proves $\mathsf{Con}(T)$ (apart from the already discussed feature of logical complexity mismatch) is a typical definition from above.

All the usual definitions of proof-theoretic ordinals can also be reformulated in the form 'from above'. Let a natural elementary well-ordering be fixed. For the case of Π^0_n-ordinals the corresponding approach would be to let

$$|T|^{\vee}_{\Pi^0_n} := \min\{\alpha : \mathsf{EA}^+ + \mathsf{RFN}_{\Pi_n}((\mathsf{EA})^n_\alpha) \vdash \mathsf{RFN}_{\Pi_n}(T)\}.$$

(Notice that for $n > 1$ the theory on the left hand side of $\vdash$ can be replaced by $(\mathsf{EA})^n_{\alpha+1}$.)

In a similar manner one can transform the definition of the Π^0_2-ordinal via the Fast Growing hierarchy into a definition 'from above'. The class of p.t.c.f. of T has a natural indexing, e.g., we can take as indices of a function f the pairs $\langle e, p\rangle$ such that e is the usual Kleene index (= the code of a Turing machine) of f, and p is the code of a T-proof of the Π^0_2-sentence expressing the totality of the function $\{e\}$. With this natural indexing in mind one can write out a formula defining the universal function $\varphi_T(e, x)$ for the class of unary functions in $\mathcal{F}(T)$. Then the Π^0_2-sentence expressing the totality of φ_T would be the desired characteristic principle. (It is not difficult to show that the totality of φ_T formalized in this way is EA^+-equivalent to $\mathsf{RFN}_{\Pi_2}(T)$.) The Π^0_2-ordinal of T can then be defined as follows:

$$|T|^{\vee}_{\Pi^0_2} := \min\{\alpha : \varphi \in \mathcal{F}_{\alpha+1}\}.$$

Notice that the proof-theoretic ordinals of T defined 'from above' not only depend on the externally taken set of theorems of T, but also on the way T is formalized, that is, essentially on the provability predicate or the *proof system* for T. For example, in the above definition the universal function $\varphi_T(e, x)$ depends on the Gödel numbering of proofs in T. In practice, for most of the natural(ly formalized) theories the ordinals defined 'from below' and those 'from above' coincide:

Proposition 10.1 *If T is EA^+-provably Π^0_n-regular and contains EA^+, then* $|T|^{\vee}_{\Pi^0_n} = |T|_{\Pi^0_n}$.

Proof. Let $\alpha = |T|_{\Pi^0_n}$. By provable regularity,

$$\mathsf{EA}^+ \vdash \mathsf{RFN}_{\Pi_n}(T) \leftrightarrow \mathsf{RFN}_{\Pi_n}((\mathsf{EA})^n_\alpha),$$

hence $\mathsf{EA}^+ + (\mathsf{EA})^n_{\alpha+1} \vdash \mathsf{RFN}_{\Pi_n}(T)$. On the other hand, by Gödel's theorem

$$\mathsf{EA}^+ + (\mathsf{EA})^n_\alpha \subseteq T \nvdash \mathsf{RFN}_{\Pi_n}(T),$$

q.e.d.

The following example demonstrates that, nonetheless, there are reasonable (and naturally formalized) proof systems for which these ordinals are different, so sometimes the ordinal defined from above bears essential additional information.

Consider some standard formulation of PA, it has a *natural* provability predicate $\Box_{\mathsf{PA}}$. The system PA^* is obtained from PA by adding Parikh's inference rule:

$$\frac{\Box_{\mathsf{PA}}\varphi}{\varphi},$$

where φ is any sentence. For the reasons of semantical correctness, Parikh's rule is admissible in PA, so PA^* proves the same theorems as PA. However, as is well known, the equivalence of the two systems cannot be established within PA (otherwise, PA^* would have a speed-up over PA bounded by a p.t.c.f. in PA, which was disproved by Parikh [19]).[6] Below we analyze the situation from the point of view of the proof-theoretic ordinals.

Notice that PA^* is a reasonable proof system, and it has a natural Σ_1 provability predicate $\Box_{\mathsf{PA}^*}$. P. Lindström [16] proves the following relationship between the provability predicates in PA and PA^*:

Lemma 10.2 *$EA \vdash \forall x\ (\Box_{PA^*}(x) \leftrightarrow \exists n\ \Box_{PA}\Box^n_{PA}(\dot{x}))$, where $\Box^n_{PA}$ means n times iterated $\Box_{PA}$.*

Notation: The right hand side of the equivalence should be understood as the result of substituting in the external $\Box_{\mathsf{PA}}$ the elementary term for the function $\lambda n, x.\ \ulcorner\Box^n_{\mathsf{PA}}(\bar{x})\urcorner$.

Proof (sketch). The implication ($\leftarrow$) holds, because PA^* is provably closed under Parikh's rule, that is,

$$\mathsf{PA} \vdash \Box^n_{\mathsf{PA}}\varphi \quad \Rightarrow \quad \mathsf{PA}^* \vdash \varphi,$$

by n applications of the rule, and this is obviously formalizable.

The implication ($\rightarrow$) holds, because the predicate $\exists n\ \Box_{\mathsf{PA}}\Box^n_{\mathsf{PA}}(\dot{x})$ is provably closed under PA, *modus ponens* and Parikh's rule:

$$\begin{aligned}\mathsf{PA} \vdash \varphi \quad &\Rightarrow \quad \mathsf{PA} \vdash \Box^1_{\mathsf{PA}}\varphi,\\ \mathsf{PA} \vdash \Box^n_{\mathsf{PA}}\varphi, \quad \mathsf{PA} \vdash \Box^m_{\mathsf{PA}}(\varphi \to \psi) \quad &\Rightarrow \quad \mathsf{PA} \vdash \Box^{\max(n,m)}_{\mathsf{PA}}\psi,\\ \mathsf{PA} \vdash \Box^n_{\mathsf{PA}}(\Box_{\mathsf{PA}}\varphi) \quad &\Rightarrow \quad \mathsf{PA} \vdash \Box^{n+1}_{\mathsf{PA}}\varphi,\end{aligned}$$

[6] An even simpler argument: otherwise one can derive from $\Box_{\mathsf{PA}}\Box_{\mathsf{PA}}\bot$ the formulas $\Box^*_{\mathsf{PA}}\bot$ and $\Box_{\mathsf{PA}}\bot$, which yields $\mathsf{PA} \vdash \Box_{\mathsf{PA}}\Box_{\mathsf{PA}}\bot \to \Box_{\mathsf{PA}}\bot$ contradicting Löb's theorem.

and this is formalizable, q.e.d.

Corollary 10.3 $\mathit{EA} \vdash \mathsf{Con}(\mathit{PA}^*) \leftrightarrow \mathsf{Con}(\mathit{PA}_\omega)$.

Proof. By induction, $\mathsf{Con}(\mathsf{PA}_n)$ is equivalent to $\neg\Box_{\mathsf{PA}}\Box^n_{\mathsf{PA}}\bot$, moreover this equivalence is formalizable in EA (with n a free variable). Hence, $\mathsf{Con}(\mathsf{PA}_\omega)$ is equivalent to $\forall n\ \neg\Box_{\mathsf{PA}}\Box^n_{\mathsf{PA}}\bot$, which yields the claim by the previous lemma, q.e.d.

Applying this to Π^0_1-ordinals defined from above, we observe

Proposition 10.4 *(i)* $|\mathit{PA}|^\vee_{\Pi^0_1} = \epsilon_0$;

(ii) $|\mathit{PA}^*|^\vee_{\Pi^0_1} = \epsilon_0 \cdot \omega$.

Proof. Statement (i) follows from the EA^+-provable Π^0_1-regularity of PA.

To prove (ii) we first obtain

$$\mathsf{PA}_\omega \equiv_{\Pi_1} \mathsf{EA}_{\epsilon_0\cdot\omega}$$

by Proposition 7.10(i). Formalization of this in EA^+ yields

$$\mathsf{EA}^+ + \mathsf{EA}_{\epsilon_0\cdot\omega+1} \vdash \mathsf{Con}(\mathsf{PA}_\omega) \vdash \mathsf{Con}(\mathsf{PA}^*),$$

by Corollary 10.3. By Schmerl's formula it is also clear that

$$\mathsf{EA}^+ + \mathsf{EA}_{\epsilon_0\cdot\omega} \subseteq ((\mathsf{EA})^2_1)_{\epsilon_0\cdot\omega} \nvdash \mathsf{Con}(\mathsf{PA}^*),$$

which proves Statement (ii), q.e.d.

Observe that the Π^0_1-ordinals of PA and PA^* defined from below both equal ϵ_0, because PA^* is deductively equivalent to PA. Hence, for PA^* the Π^0_1-ordinals defined from below and from above are different — this reflects the gap between the power of axioms of this theory and the effectiveness of its proofs.

Despite the fact that, as we have seen, the proof-theoretic ordinals defined from above may have some independent meaning, it seems that those from below are more fundamental and better behaved.

11 Appendix C

Here we discuss the role of the metatheory EA^+ that was taken as basic in this paper. On the one hand, it is the simplest choice, and if one is interested in the analysis of strong systems, there is no reason to worry about it. Yet, if one wants to get meaningful ordinal assignments for theories not containing EA^+, such as $\mathsf{EA} + I\Pi^-_1$ or $\mathsf{EA} + \mathsf{Con}(I\Sigma_1)$, the problem of weakening the metatheory has to be addressed. For example, somewhat contrary to the intuition, it can be shown (see below) that $\mathsf{EA} + \mathsf{Con}(I\Sigma_1)$ is not a Π^0_1-regular theory in the usual sense.

These problems can be handled, if one reformulates the hierarchies of iterated consistency assertions using the notion of *cut-free provability* and formalizes Schmerl's formulas in EA using *cut-free conservativity.* Over EA^+ these notions provably coincide with the usual ones, so they can be considered as reasonable generalizations of the usual notions in the context of the weak arithmetic EA. The idea of using cut-free provability predicates for this kind of problems comes from Wilkie and Paris [30]. Below we briefly sketch this approach and consider some typical examples.

A formula φ is *cut-free provable* in a theory T (denoted $T \vdash^{\mathrm{cf}} \varphi$), if there is a finite set T_0 of (closed) axioms of T such that the sequent $\neg T_0, \varphi$ has a cut-free proof in predicate logic. Similarly, φ is *rank k provable*, if for some finite $T_0 \subseteq T$, the sequent $\neg T_0, \varphi$ has a proof with the ranks of all cut-formulas bounded by k.

If T is elementary presented, its cut-free $\Box^{\mathrm{cf}}_T$ and rank k provability predicates can be naturally formulated in EA. It is known that EA^+ proves the equivalence of the ordinary and the cut-free provability predicates. On the other hand, EA can only prove the equivalence of the cut-free and the rank k provability predicates for any fixed k, but not the equivalence of the cut-free and the ordinary provability predicates.

The behavior of $\Box^{\mathrm{cf}}_T$ in EA is very much similar to that of $\Box_T$, e.g., $\Box^{\mathrm{cf}}_T$ satisfies the EA-provable Σ_1-completeness and has the usual provability logic — this essentially follows from the equivalence of the bounded cut-rank and the cut-free provability predicates in EA.

A. Visser [29], building on the work H. Friedman and P. Pudlák, established a remarkable relationship between the predicates of ordinary and cut-free provability: if T is a finite theory, then[7]

$$\mathsf{EA} \vdash \forall x\, (\Box_T \varphi(\dot{x}) \leftrightarrow \Box^{\mathrm{cf}}_{\mathsf{EA}} \Box^{\mathrm{cf}}_T \varphi(\dot{x})). \tag{13}$$

In particular, for EA itself Visser's formula (13) attains the form

$$\mathsf{EA} \vdash \forall x\, (\Box_{\mathsf{EA}} \varphi(\dot{x}) \leftrightarrow \Box^{\mathrm{cf}}_{\mathsf{EA}} \Box^{\mathrm{cf}}_{\mathsf{EA}} \varphi(\dot{x})).$$

This can be immediately generalized (by reflexive induction) to progressions of iterated cut-free consistency assertions. Let $\mathsf{Con}^{\mathrm{cf}}(T)$ denote $\neg\Box^{\mathrm{cf}}_T \bot$ and let a nice well-ordering be fixed.

Proposition 11.1 *The following holds provably in EA:*

(i) If $\alpha \prec \omega$, then $\mathsf{EA}_\alpha \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{2\cdot\alpha}$;

(ii) If α is a limit ordinal, then $\mathsf{EA}_\alpha \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_\alpha$;

(iii) If $\alpha = \omega \cdot \beta + n + 1$, where $\beta \succ 0$ and $n \prec \omega$, then

$$\mathsf{EA}_\alpha \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega\cdot\beta+2n+1}.$$

[7] A. Visser works in a relational language and uses efficient numerals, but this does not seem to be essential for the general result over EA.

We omit a straightforward proof by Visser's formula. We call a theory T Π^0_1-*cf-regular*, if for some α,

$$T \equiv_{\Pi_1} \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_\alpha. \tag{14}$$

The situation with the cut-free reflection principles of higher arithmetical complexity is even easier.

Proposition 11.2 *Let T be a finite extension of EA. Then for any $n > 1$,*

$$\mathsf{EA} \vdash \mathsf{RFN}^{\mathrm{cf}}_{\Pi_n}(T) \leftrightarrow \mathsf{RFN}_{\Pi_n}(T).$$

Proof. We only show the implication $(\rightarrow)$, the opposite one is obvious. For any $\varphi(x) \in \Pi_n$ the formula $\Box_T \varphi(\dot{x})$ implies $\Box^{\mathrm{cf}}_T \Box^{\mathrm{cf}}_T \varphi(\dot{x})$. Applying $\mathsf{RFN}^{\mathrm{cf}}_{\Pi_n}(T)$ twice yields $\varphi(x)$, q.e.d.

This equivalence carries over to the iterated principles. Let $(T)^{n,\mathrm{cf}}_\alpha$ denote $\mathsf{RFN}^{\mathrm{cf}}_{\Pi_n}(T)_\alpha$. For successor ordinals α the theories $(T)^{n,\mathrm{cf}}_\alpha$ are finitely axiomatizable, so we obtain by reflexive induction using Proposition 11.2 for the induction step:

Proposition 11.3 *For any $n > 1$, provably in EA,*

$$\forall \alpha \ \ (T)^{n,\mathrm{cf}}_\alpha \equiv (T)^n_\alpha.$$

We say that T is *cut-free Π_n-conservative* over U, if for every $\varphi \in \Pi_n$, $T \vdash^{\mathrm{cf}} \varphi$ implies $U \vdash^{\mathrm{cf}} \varphi$. Let $T \equiv^{\mathrm{cf}}_{\Pi_n} U$ denote a natural formalization in EA of the mutual cut-free Π_n-conservativity of T and U. Externally $\equiv^{\mathrm{cf}}_{\Pi_n}$ is the same as $\equiv_{\Pi_n}$, so the difference between the two notions only makes sense in formalized contexts.

Analysis of the proof of Schmerl's formula reveals that we deal with an elementary transformation of a cut-free derivation into a derivation of a bounded cut-rank. To see this, the reader is invited to check the ingredient proofs of Theorem 2 and Propositions 4.6 and 5.4. All these elementary proof transformations are verifiable in EA, which yields the following formalized variant of Schmerl's formula (we leave out all the details).

Proposition 11.4 *For all $n \geq 1$, if T is an elementary presented Π_{n+1}-axiomatized extension of EA, the following holds provably in EA:*

$$\forall \alpha \succeq 1 \ (T)^{n+m}_\alpha \equiv^{\mathrm{cf}}_{\Pi_n} (T)^n_{\omega_m(\alpha)}$$

We notice that this relationship holds for the ordinary as well as for the cut-free reflection principles, because the ordinal on the right hand side of the equivalence is a limit (if $m > 0$).

Now we consider a few examples. The following proposition shows that the theory $\mathsf{EA} + I\Pi^-_1$ is both Π^0_1-cf-regular and Π^0_1-regular with the ordinal ω.

Proposition 11.5 *Provably in EA,*

$$\mathsf{EA} + I\Pi_1^- \equiv_{\Pi_1} \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_\omega \equiv_{\Pi_1} \mathsf{EA}_\omega.$$

Proof. The logics of the ordinary and the cut-free provability for EA coincide, so by the usual proof the schema of local reflection w.r.t. the cut-free provability is Π_1-conservative over $\mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_\omega$. But the former contains $\mathsf{EA} + I\Pi_1^-$ by the comment following (E3)(c), moreover this inclusion is easily formalizable in EA (both in the usual and in the cut-free version). This proves the first equivalence. The second one follows from Proposition 11.1(ii), q.e.d.

Consider the theories $\mathsf{EA}+\mathsf{Con}^{\mathrm{cf}}(I\Sigma_1)$ and $\mathsf{EA}+\mathsf{Con}(I\Sigma_1)$. We show that the first one is Π^0_1-cf-regular with the ordinal $\omega^\omega+1$, and the second is Π^0_1-cf-regular with the ordinal $\omega^\omega+2$. Notice, however, that only the *first* theory is Π^0_1-regular in the usual sense: by Proposition 11.1(iii), $\mathsf{EA}_{\omega^\omega+1} \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega^\omega+1}$, whereas

$$\mathsf{EA}_{\omega^\omega+2} \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega^\omega+3} \not\equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega^\omega+2}.$$

Proposition 11.6 *Provably in EA,*

(i) $\mathsf{EA} + \mathsf{Con}^{\mathrm{cf}}(I\Sigma_1) \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega^\omega+1}$;

(ii) $\mathsf{EA} + \mathsf{Con}(I\Sigma_1) \equiv \mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega^\omega+2}$.

Proof. Part (i) follows from the (obvious) formalizability of the equivalence of $I\Sigma_1$ and $(\mathsf{EA})^3_1$ in EA and Proposition 11.4. One can show that the two systems are also cut-free equivalent, provably in EA.

To prove Part (ii) recall that by Visser's formula (13), provably in EA, $\mathsf{Con}(I\Sigma_1)$ is equivalent to $\mathsf{Con}^{\mathrm{cf}}(\mathsf{EA} + \mathsf{Con}^{\mathrm{cf}}(I\Sigma_1))$, and hence to $\mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega^\omega+2}$ by Part (i), q.e.d.

The following facts are also worth noticing. Statement (i) below implies that $I\Sigma_n$ is not EA-provably Π^0_1-regular (and thus the original Schmerl's formula is not formalizable in EA). In contrast, Statement (ii) implies that incidentally PA itself *is* EA-provably Π^0_1-regular.

Proposition 11.7 (i) $\mathsf{EA} \nvdash \mathsf{Con}(\mathsf{EA}_{\omega_{n+1}}) \to \mathsf{Con}(I\Sigma_n)$;

(ii) $\mathsf{PA} \equiv_{\Pi_1} \mathsf{EA}_{\epsilon_0}$, provably in EA, hence $\mathsf{EA} \vdash \mathsf{Con}(\mathsf{EA}_{\epsilon_0}) \to \mathsf{Con}(\mathsf{PA})$.

Proof. Fact (i) has just been proved for $I\Sigma_1$: $\mathsf{Con}(I\Sigma_n)$ is Π^0_1-conservative over $\mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega_{n+1}+2}$, whereas $\mathsf{Con}(\mathsf{EA}_{\omega_{n+1}})$ is equivalent to $\mathsf{Con}^{\mathrm{cf}}(\mathsf{EA})_{\omega_{n+1}+1}$ by Proposition 11.1(iii). Hence, (i) follows by Löb's principle for the cut-free provability.

To prove (ii) we formalize the following reasoning in EA: Assume $\pi \in \Pi_1$ and $\mathsf{PA} \vdash \pi$. Then for some n, $I\Sigma_n \vdash \pi$. Since $I\Sigma_n$ is finitely axiomatized, by (13) we obtain that $\exists n\ \mathsf{EA} \vdash^{\mathrm{cf}} \Box^{\mathrm{cf}}_{I\Sigma_n}\pi$, therefore by Proposition 11.4

$$\exists n\ \mathsf{EA} \vdash^{\mathrm{cf}} \Box^{\mathrm{cf}}_{\mathsf{EA}_{\omega_{n+1}}}\pi,$$

which can be weakened to

$$\exists n\ \mathsf{EA}_{\epsilon_0} \vdash \Box^{\mathrm{cf}}_{\mathsf{EA}_{\omega_{n+1}}} \pi. \tag{15}$$

On the other hand, we notice that (provably in EA) for every fixed $\beta \prec \epsilon_0$ and $\pi \in \Pi_1$,

$$\mathsf{EA}_{\epsilon_0} \vdash \Box^{\mathrm{cf}}_{\mathsf{EA}_\beta} \pi \to \pi,$$

by the cut-free version of the Σ_1-completeness principle, and applying this to (15) yields $\mathsf{EA}_{\epsilon_0} \vdash \pi$, q.e.d.

Corollary 11.8 $|\mathit{EA} + \mathsf{Con}(\mathit{PA})|_{\Pi_1^0} = \epsilon_0 + 1$.

References

[1] L.D. Beklemishev. Remarks on Magari algebras of PA and $I\Delta_0 + \mathrm{EXP}$. In P. Agliano and A. Ursini, editors, *Logic and Algebra*, pages 317–325. Marcel Dekker, New York, 1996.

[2] L.D. Beklemishev. Induction rules, reflection principles, and provably recursive functions. *Annals of Pure and Applied Logic*, 85:193–242, 1997.

[3] L.D. Beklemishev. Notes on local reflection principles. *Theoria*, 63(3):139–146, 1997.

[4] L.D. Beklemishev. A proof-theoretic analysis of collection. *Archive for Mathematical Logic*, 37:275–296, 1998.

[5] L.D. Beklemishev. Parameter free induction and provably total computable functions. *Theoretical Computer Science*, 224(1–2):13–33, 1999.

[6] L.D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. Logic Group Preprint Series 208, University of Utrecht, 2001. `http://www.phil.uu.nl/preprints.html`. To appear in *Annals of Pure and Applied Logic.*

[7] S. Feferman. Arithmetization of metamathematics in a general setting. *Fundamenta Mathematicae*, 49:35–92, 1960.

[8] S. Feferman. Transfinite recursive progressions of axiomatic theories. *Journal of Symbolic Logic*, 27:259–316, 1962.

[9] H. Gaifman and C. Dimitracopoulos. Fragments of Peano's arithmetic and the MDRP theorem. In *Logic and algorithmic (Zurich, 1980), (Monograph. Enseign. Math., 30)*, pages 187–206. Genève, University of Genève, 1982.

[10] S. Goryachev. On interpretability of some extensions of arithmetic. *Mat. Zametki*, 40:561–572, 1986. In Russian. English translation in *Math. Notes*, 40.

[11] P. Hájek and P. Pudlák. *Metamathematics of First Order Arithmetic.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[12] R. Kaye, J. Paris, and C. Dimitracopoulos. On parameter free induction schemas. *Journal of Symbolic Logic*, 53(4):1082–1097, 1988.

[13] G. Kreisel. Wie die Beweistheorie zu ihren Ordinalzahlen kam und kommt. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 78(4):177–223, 1977.

[14] G. Kreisel and A. Lévy. Reflection principles and their use for establishing the complexity of axiomatic systems. *Zeitschrift f. math. Logik und Grundlagen d. Math.*, 14:97–142, 1968.

[15] D. Leivant. The optimality of induction as an axiomatization of arithmetic. *Journal of Symbolic Logic*, 48:182–184, 1983.

[16] P. Lindström. *The modal logic of Parikh provability.* Tech. Rep. Filosofiska Meddelanden, Gröna Serien 5, Univ. Göteborg, 1994.

[17] M. Möllerfeld. *Zur Rekursion längs fundierten Relationen und Hauptfolgen.* Diplomarbeit, Institut für Mathematische Logik, Westf. Wilhelms-Universität, Münster, 1996.

[18] H. Ono. Reflection principles in fragments of Peano Arithmetic. *Zeitschrift f. math. Logik und Grundlagen d. Math.*, 33(4):317–333, 1987.

[19] R. Parikh. Existence and feasibility in arithmetic. *Journal of Symbolic Logic*, 36:494–508, 1971.

[20] C. Parsons. On a number-theoretic choice schema and its relation to induction. In A. Kino, J. Myhill, and R.E. Vessley, editors, *Intuitionism and Proof Theory*, pages 459–473. North Holland, Amsterdam, 1970.

[21] C. Parsons. On n-quantifier induction. *Journal of Symbolic Logic*, 37(3):466–482, 1972.

[22] W. Pohlers. A short course in ordinal analysis. In A. Axcel and S. Wainer, editors, *Proof Theory, Complexity, Logic*, pages 867–896. Oxford University Press, Oxford, 1993.

[23] W. Pohlers. Subsystems of set theory and second order number theory. In S.R. Buss, editor, *Handbook of Proof Theory*, pages 210–335. Elsevier, North-Holland, Amsterdam, 1998.

[24] H.E. Rose. *Subrecursion: Functions and Hierarchies.* Clarendon Press, Oxford, 1984.

[25] U.R. Schmerl. A fine structure generated by reflection formulas over Primitive Recursive Arithmetic. In M. Boffa, D. van Dalen, and K. McAloon, editors, *Logic Colloquium'78*, pages 335–350. North Holland, Amsterdam, 1979.

[26] C. Smoryński. The incompleteness theorems. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 821–865. North Holland, Amsterdam, 1977.

[27] R. Sommer. Transfinite induction within Peano arithmetic. *Annals of Pure and Applied Logic*, 76(3):231–289, 1995.

[28] A.M. Turing. System of logics based on ordinals. *Proc. London Math. Soc.*, ser. 2, 45:161–228, 1939.

[29] A. Visser. Interpretability logic. In P.P. Petkov, editor, *Mathematical Logic*, pages 175–208. Plenum Press, New York, 1990.

[30] A. Wilkie and J. Paris. On the scheme of induction for bounded arithmetic formulas. *Annals of Pure and Applied Logic*, 35:261–302, 1987.