Review of the paper

On the arithmetical content of restricted forms of comprehension, choice and general unifrom boundedness

by U. Kohlenbach

The paper under review can be considered as part of the larger project of *extracting additional information from proofs.* The project originated with the famous question of G. Kreisel: "what can be learned, if we know a proof of a theorem, compared to the mere fact of its truth." Possible answers to this question usually take the form of extracting effective bounds from (ineffective) proofs, highlights being, e.g., the pioneering result of G. Kreisel on Littlewood's Theorem [6] and H. Luckhardt's analysis of Roth's Theorem [7]. More recently, U. Kohlenbach used this methodology to extract effective bounds in Chebycheff approximation theory (see [2]). See also S. Feferman [1] for a critical overall assessment of this line of research.

Technically, analysis of proofs often has the form of a reduction of an *expressively strong* second order or finite type system S to a *weak* first order or quantifier-free system of arithmetic T, such as PRA or its fragments. The requirements are that 1) the strong system naturally formalizes the mathematical proof under consideration, whereas 2) the weak system allows for the extraction of reasonably effective bounds.

In proof theory one is traditionally interested in the reductions which yield conservation results. These concerns are, in particular, motivated by modern versions of Hilbert Program. E.g., the conservativity of a fragment of analysis S over PRA can be interpreted as grounding the part of mathematics formalizable in S on purely finitistic means. Notice that a reduction of this type is the more informative, the stronger the system S (and the weaker the system T) is. This draws, in particular, attention to questions of the following type: *Determine possibly stronger natural systems S which are conservative over PRA.* [1]

[1] This question does not have a unique answer: there are natural examples of systems S_0 and S_1 (even in the language of first order arithmetic) either of which is conservative over PRA such that $S_0 + S_1$ proves the totality of the Ackermann function. Thus, in principle, incomparable systems can be used to extract effective bounds from proofs.

In the project of extracting computational information from proofs one is driven by different concerns. In practice, to simplify a proof analysis, one often deals with more general kind of reductions, which preserve the rate of growth bounds, but need not yield conservation results. (E.g., one often simplifies the proof analysis by freely using arbitrary true universal lemmas, or more complicated *analytical principles*, in the terminology of U. Kohlenbach, which do not contribute to the rate of growth.)

The paper under review is a certain compromise between these two types of concerns. On the one hand, one is interested in the traditional type of conservation results between fragments of analysis in all finite types and fragments of arithmetic w.r.t. the classes of the prenex arithmetical hierarchy. On the other hand, one tries to isolate the fragments S which are closer to the systems occurring in actual 'unwinding' of mathematical proofs.

One of the main results of the paper is an improvement of a well-known result of J. Paris [8] and H. Friedman (unpublished) on Π_{k+2} conservativity of Π_k-collection principle over Σ_k-induction schema in fragments of PA. Corollary 4.8 of the paper states that over some basic system of analysis in all finite types corresponding to the n-th class of the Grzegorczyk hierarchy (denoted $E\text{-}G_nA^\omega$ in the paper) the following combination of schemata is conservative for Π^0_{k+2}-sentences over Σ^0_k-induction principle $\Sigma^0_k\text{-}IA^-$ [2] (where $^-$ denotes the absence of function parameters, type 0 number parameters are allowed):

0) $AC^{1,0}\text{-}qf$ (quantifier-free choice in types $\{1,0\}$);

1) $\Delta^0_{k+1}\text{-CA}^-$ (comprehension);

2) $\Pi^0_k\text{-AC}^-$ (choice);

3) WKL (weak König's lemma).

Since $\Pi^0_k\text{-AC}^-$ contains arithmetical Π_k-collection, this result improves the quoted theorem of J. Paris and H. Friedman. Corollary 4.11 also shows that the above combination of schemata is, in fact, Π^0_{k+3}-conservative over Π^0_k-collection principle without function parameters. This means that we deal here with an essentially second order improvement of Paris–Friedman Theorem. The system given by 0)–3) for $k = 1$ seems to be one of the strongest currently known natural conservative extensions of PRA.

This theory is natural and the result is interesting, because the schema $\Pi^0_k\text{-AC}^-$ often appears in formalizing mathematical arguments. E.g., Π^0_1-

[2]This system is a conservative extension of first order $I\Sigma_k$.

AC$^-$ proves (a weak schematic version of) the principle of convergence of bounded monotone sequences of reals, whereas Π^0_2-AC$^-$ is needed to prove a suitable version of the existence of limit superior for bounded sequences (a proof of this fact appeared in the later publication [5]).

At this point the reader may well ask him/herself, why do we need to consider the rather awkward restrictions of the schemata to the function parameter-free ones (and some other special families of formulas and systems that the reader encounters in the paper). The answer is that from the point of view of the 'practical' task of extracting effective bounds only the principles are of significance, which contribute as little as possible to the rate of growth of provable functions. Unrestricted use of function parameters is well-known to be bad in this respect. Thus, the more common schemata, although they allow sometimes for more compact formulations, are of little use for the purposes of the project of extracting numerical bounds from proofs, as well as for the reductionistic aims of grounding much of mathematics on PRA.

Methods of the paper are based on two special techniques: *monotone functional interpretation* and the elimination of Skolem functions for a class of so-called *monotone formulas*, for which the main conservation results are proved. This class of formulas is not restricted in any of the usual classes Σ_n or Π_n, which allows to state some results in greater generality. Another advantageous feature of the techniques is that it not only allows to provide a characterization of provably recursive functions, but also of the provable type 2 functionals of a system. Details of the techniques, however, can only be found in the preceeding works of the author [3, 4].

The paper is difficult to read for some objective and some subjective reasons. So, it may require some patience on the reader's part. The objective reason is that one deals here with a great variety of schemas and principles, which are often not so nicely formulated. The absence of sufficiently attractive and memorable *abstract* formulations is a general malaise of this subdiscipline, for which nobody has found a good cure so far. In a sense, this is the price one has to pay for the compromise between the traditional proof-theoretic concerns and the needs of practical proof-unwinding.

A subjective reason is that the author has chosen the style of the presentation, where the more general (but difficult to formulate) results have the status of "theorems" and "propositions", whereas some of the interesting and more memorable formulations are degraded to "corollaries".

Despite these criticisms, I believe that the direction of research is motivated by the vital needs of proof theory, the search for new applications, which are especially important at present. The author is doing a very good and useful job on it. The results obtained so far are interesting and have a potential for further developement of the subject. The techniques used can also be of value for the more traditional areas of proof-theory.

Finally, I would like to make a small bibliographycal correction to the claims made on p. 260: J. Paris published his conservation result in [8], not in the paper [9], which actually does not contain this statement. It seems to be fair to call this theorem simply "Paris–Friedman" or "Friedman–Paris", not "Friedman–Paris–Kirby".

References

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