On the limit existence principles in elementary arithmetic and Σ^0_n-consequences of theories

Dedicated to Wolfram Pohlers on the occasion of his 60-th birthday

Lev D. Beklemishev [a,1]

[a] *Department of Philosophy, Utrecht University, Heidelberglaan 8, 3584 CS Utrecht, the Netherlands*

Albert Visser [b]

[b] *Department of Philosophy, Utrecht University, Heidelberglaan 8, 3584 CS Utrecht, the Netherlands*

Abstract

We study the arithmetical schema asserting that every eventually decreasing elementary recursive function has a limit. Some other related principles are also formulated. We establish their relationship with restricted parameter-free induction schemata. We also prove that the same principle, formulated as an inference rule, provides an axiomatization of the Σ_2-consequences of $I\Sigma_1$.

Using these results we show that **ILM** is the logic of Π_1-conservativity of any reasonable extension of parameter-free Π_1-induction schema. This result, however, cannot be much improved: by adapting a theorem of D. Zambella and G. Mints we show that the logic of Π_1-conservativity of primitive recursive arithmetic properly extends **ILM**.

In the third part of the paper we give an ordinal classification of Σ^0_n-consequences of the standard fragments of Peano arithmetic in terms of reflection principles. This is interesting in view of the general program of ordinal analysis of theories, which in the most standard cases classifies Π-classes of sentences (usually Π^1_1 or Π^0_2).

Key words: elementary arithmetic, parameter-free induction, inference rule, interpretability logic, conservativity, reflection principles, ordinal analysis

Email addresses: `lev@phil.uu.nl` (Lev D. Beklemishev), `Albert.Visser@phil.uu.nl` (Albert Visser).

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1 Introduction

This paper was motivated by a particular question in the area of interpretability logic. It has been open for some time if a proof of the arithmetical completeness theorem for the logic of Π_1-conservativity **ILM** can be carried through in a theory weaker than $I\Sigma_1$. The essential principle used in this proof is the statement that a certain elementary function h (Solovay function), for which one can prove that it is eventually descending, has a limit. We consider the schema

$$\exists m \forall n \geq m\, h(n+1) \leq h(n) \rightarrow \exists m \forall n \geq m\, h(n) = h(m), \qquad (Lim)$$

for each elementary function h. The schema (Lim) obviously follows from $I\Sigma_1$ (see Lemma 1 below), but is properly weaker, because its arithmetical complexity is $\Sigma_2 \rightarrow \Sigma_2$. ($I\Sigma_1$ does not follow from any set of true Σ_3-sentences.)

We establish that over elementary arithmetic EA this schema is equivalent to the parameter-free Σ_1-induction schema $I\Sigma_1^-$. Also, (Lim) turns out to be equivalent to some other natural limit existence principles in elementary arithmetic. On the other hand, in the proof of completeness theorem for **ILM**, e.g., in the one due to G. Japaridze [10], the principle (Lim) is actually only applied once in the form of a rule

$$\frac{\exists m \forall n \geq m\, h(n+1) \leq h(n)}{\exists m \forall n \geq m\, h(n) = h(m)}, \qquad (LimR)$$

for a certain elementary function h. We show that the closure of EA under unnested applications of $(LimR)$ is equivalent to the fragment $I\Pi_1^-$ of PA axiomatized over EA by the parameter-free induction schema for Π_1-formulas. $I\Pi_1^-$ is known to be much weaker than $I\Sigma_1^-$, it has the same provably total computable functions as EA (elementary functions).

On the other hand, we prove that nested applications of $(LimR)$ provide an axiomatization of the Σ_2-consequences of $I\Sigma_1$. To the best of our knowledge, Σ_2-consequences of $I\Sigma_1$ are characterized here for the first time.

In the second part of this paper we briefly formulate the corollaries of these results for conservativity logics. In particular, **ILM** is the conservativity logic for any sound extension of $I\Pi_1^-$. We also give an example showing that this cannot be much improved. Based on a result of D. Zambella and G. Mints we demonstrate that the conservativity logic of primitive recursive arithmetic properly extends **ILM**.

The limit existence rule and $I\Pi_1^-$ provide concrete examples of mathematically meaningful theorems of PA of complexity Σ_2. In the third part we consider the general question of characterizing Σ_n-consequences of arithmetical theories. For standard fragments of PA we give an ordinal classification of such consequences based on iterated reflection principles of a special kind. This result presents an independent interest in view of the general program of ordinal analysis and is largely based on the ideas in [3]. The third part of the paper can be read independently from the first two.

2 Limit existence schemata

We relate some natural limit existence principles to the standard hierarchies of fragments of PA. As our basic fragment we take elementary arithmetic EA (or $I\Delta_0$+exp) which we formulate in the language containing terms for all Kalmar elementary functions. EA can be axiomatized over basic defining equations for all such functions by the schema of induction for bounded (Δ_0) formulas in its language. PA is defined by the schema of induction for arbitrary formulas in the language. $I\Sigma_n$ is axiomatized over EA by the induction schema for Σ_n-formulas, and $I\Sigma_n^-$ is axiomatized by the parameter-free induction schema for Σ_n-formulas. See [9,11] for more details.

First we study the schema (*Lim*). The most natural proof of (*Lim*) can be carried out in $I\Sigma_1$.

Lemma 1 *$I\Sigma_1$ proves* (*Lim*).

Proof. We reason inside $L\Sigma_1$, the least element principle for Σ_1-formulas, which is equivalent to $I\Sigma_1$.

Let m_0 be such that

$$\forall n \geq m_0\, h(n) \geq h(n+1),$$

where h is the given function. Using $L\Sigma_1$ pick an x such that

$$\exists n \geq m_0\, h(n) = x \wedge \forall y < x\, \neg\exists n \geq m_0\, h(n) = y.$$

We have $\forall n \geq m_0\, h(n) \geq x$. By an auxiliary Δ_0-induction we also obtain

$$\forall n \geq m_0\, h(m_0) \geq h(n).$$

Now if m_1 is such that $h(m_1) = x$, then $\forall n \geq \max(m_0, m_1)\, h(n) = x$. Hence, the limit of h exists. ⊠

As a corollary of a general proof-theoretic result from [11] we obtain the following improvement.

Corollary 2 *The parameter-free Σ_1-induction schema $I\Sigma_1^-$ proves* (*Lim*).

Proof. By [11], $I\Sigma_1$ is conservative over $I\Sigma_1^-$ for Σ_3-sentences, which includes (Lim). $\boxtimes$

The converse to this corollary is also true, which follows from the next theorem.

Theorem 3 *The following schemata are equivalent over* EA*:*

(i) $I\Sigma_1^-$
(ii) $\exists x \forall m\, fm \leq x \to \exists m \forall n\, fn \leq fm$
(iii) $\forall n\, fn \leq f(n+1) \wedge \exists x \forall m\, fm \leq x \to \exists m \forall n > m\, fn = fm$
(iv) $\exists m \forall n \geq m\, f(n+1) \leq f(n) \to \exists m \forall n \geq m\, f(n) = f(m)$.

Here f ranges over all elementary terms in one variable.

Proof. Notice that (iv) is (Lim). By Corollary 2 it is sufficient to prove (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i). We reason in EA.

(iii)$\Rightarrow$(ii): We argue by contraposition. Assume $\forall m \exists n\, f(n) > f(m)$. Consider the elementary function

$$f'(n) := \max_{i \leq n} f(i).$$

The graph of f' provably in EA satisfies

$$f'(n) = y \leftrightarrow (\exists m \leq n\, f(m) = y \wedge \forall m \leq n\, f(m) \leq y).$$

Hence, EA proves $\forall n\, f'(n) \leq f'(n+1)$.

We claim: $\forall m \exists n > m\, f'(n) > f'(m)$.

Indeed, if $f'(m) = a$, then $\exists k \leq m\, f(k) = a$. The assumption about f implies $\exists n\, f(n) > f(k) = a$. Hence, for this n, $f'(n) \geq f(n) > a = f'(m)$, and by the provable monotonicity of f' we also have $n > m$ (for otherwise $f'(n) \leq f'(m)$). This proves the claim.

Applying now (iii) to f' we conclude that $\forall x \exists m\, f'(m) > x$, which implies $\forall x \exists m\, f(m) > x$.

(ii)$\Rightarrow$(i): We have to show, for each Δ_0-formula $\varphi(x, u)$, with only free variables x and u,

$$\exists u \varphi(0, u) \wedge \forall x\, (\exists u \varphi(x, u) \to \exists u \varphi(x+1, u)) \to \forall x \exists u \varphi(x, u).$$

Let IH_φ denote the premise of the above implication. We introduce an elementary function

$$f(u) = \begin{cases} 0, & \text{if } u \text{ does not code a sequence} \\ \max_{x \leq \mathrm{lh}(u)} \forall i < x\, \varphi(i, (u)_i), & \text{otherwise.} \end{cases}$$

Here $\mathrm{lh}(u)$ denotes the length of the sequence u.

We claim: $\mathrm{IH}_\varphi \to \forall u \exists v\, f(v) > f(u)$.

Indeed, assume $f(u) = a$. If $a = 0$, consider a y such that $\varphi(0, y)$ and let $v = \langle y \rangle$. Obviously, $f(v) = 1 > 0$.

If $a > 0$ we have $\forall i < a\, \varphi(i, (u)_i)$. Let u' denote the initial segment of u of length a and let $j = a - 1$. We have $\varphi(j, (u)_j)$, hence by IH_φ there is a z such that $\varphi(a, z)$. Let v be the concatenation of u' and $\langle z \rangle$. We obviously have $\forall i \leq a\, \varphi(i, (v)_i)$, hence $f(v) = a + 1 > a$. This proves the claim.

From this using (ii) we obtain $\forall x \exists u\, f(u) > x$. However, if $f(u) > x$, then $\varphi(x, (u)_x)$ by the definition of f. Hence, $\forall x \exists y\, \varphi(x, y)$.

(iv)⇒(iii) Suppose $\forall x\, (f(x) \leq f(x+1))$ and f is bounded by a. We define elementary functions h and j as follows.

- $j0 = 0$, $h0 = 0$.
- $j(z+1) = \begin{cases} jz, & \text{if } fz \leq jz \\ z, & \text{otherwise,} \end{cases}$

  $h(z+1) = \begin{cases} jz - fz, & \text{if } fz \leq jz \\ 0, & \text{otherwise.} \end{cases}$

We show that h is eventually descending. In case $fy \leq ja$ for all y, we find that, for all $y > a$, $jy = ja$ and hence $hy = ja - fy$. So clearly h will be descending from $a + 1$ on.

Suppose that, for some y, $fy > ja$. Since f is weakly ascending, we may assume that $y \geq a$. Let y^* be the smallest $y \geq a$ such that $fy > ja$ (it exists by the Δ_0-least element principle). It follows that $j(y^* + 1) = y^* \geq a$. Hence, for all z, $fz \leq j(y^* + 1)$. Thus, from $y^* + 2$ on, h will be weakly descending.

By (iv) we find that h is eventually constant. If the limit of h is 0, then, for almost all z, $fz \geq jz$. Suppose for infinitely many z, $fz > jz$. Then, for any such z, $j(z+1) = z$ and hence $f(z+1) \geq j(z+1) = z$. This contradicts the boundedness of f.

Otherwise, $fz = jz$, for almost all z. Then, by the definition of j, $j(z+1) = jz$, for almost all z. Hence, the functions j and f are eventually constant.

If the limit of h is $c \neq 0$, then, similarly to the previous case, for almost all z, $jz = fz + c$. It follows that $j(z+1) = jz$, for almost all z, and the functions j and f are eventually constant. ⊠

If one wants to get a more direct proof of (Lim) in $I\Sigma_1^-$ than the one using Corollary 2, one can prove (i)⇒(ii)⇒(iii)⇒(iv).

(i)⇒(ii): We argue by contraposition. Assume $\forall m \exists n \, fn > fm$ and prove $\forall x \exists m \, f(m) > x$ by induction on x. Basis of induction is easy. For the induction Step assume that, for some m, $f(m) > x$. Then $\exists n \, f(n) > f(m) > x$ by our assumption. Hence, $f(n) > x + 1$, q.e.d.

(ii)⇒(iii): From $\forall n \, fn \leq f(n+1)$ we obtain

$$\forall m \forall n \, (m < n \to fm \leq fn)$$

by Δ_0-induction. Hence, $\forall m \exists n > m \, fm \neq fn$ implies $\forall m \exists n > m \, fn > fm$. Axiom (ii) then yields $\forall x \exists m fm > x$.

Now we derive (iv), i.e., (Lim) from (iii) using an idea of J. Joosten.

Let h be the given elementary function such that $\forall n \geq m \, h(n) \geq h(n+1)$ for some m. Consider the function

$$g(x) = \sum_{i \leq x} |h(i) - h(i+1)|.$$

Obviously, g is provably nondecreasing. We show that g is bounded from above.

We claim: $\forall x \, g(x) \leq g(m) + h(m+1)$.

Indeed, if $x \leq m$, then $g(x) \leq g(m)$ by monotonicity. If $x > m$ we have

$$g(x) = g(m) + \sum_{i=m+1}^{x} |h(i) - h(i+1)|.$$

Now by an auxiliary Δ_0-induction on x with parameter m we show

$$\forall x > m \sum_{i=m+1}^{x} |h(i) - h(i+1)| = h(m+1) - h(x+1).$$

We use the fact that for $x > m$ the function h is weakly descending. If $x = m + 1$, this is obviously true, because $h(x) \geq h(x+1)$. Similarly, for the induction step we have

$$\sum_{i=m+1}^{x+1} |h(i) - h(i+1)| = h(m+1) - h(x+1) + (h(x+1) - h(x+2)) = h(m+1) - h(x+2),$$

as required.

Applying (iii) we may conclude that g has a limit. This implies that $\exists m \forall n \geq m \, g(n) = g(n+1)$, which yields $|h(n) - h(n+1)| = 0$. Hence, h must have a limit, as well.

Remark 4 If in the formulation of Theorem 3 the terms f are allowed to depend on an additional parameter a, then the corresponding schemata will be equivalent to $I\Sigma_1$. The proof remains essentially unchanged.

3 Limit existence rule

Let us now consider the limit existence rule

$$\frac{\exists m \forall n \geq m\, h(n+1) \leq h(n)}{\exists m \forall n \geq m\, h(n) = h(m)}, \qquad (LimR)$$

for each elementary function h. The following result shows that the closure of EA under one application of $(LimR)$ is contained in $I\Pi_1^-$.

Theorem 5 *If* $\mathsf{EA} \vdash \exists m \forall n \geq m\, h(n) \geq h(n+1)$,
then $I\Pi_1^- \vdash \exists m \forall n \geq m\, h(n) = h(m)$.

Proof. We use a version of the Herbrand Theorem for $\exists\forall$-formulas. In our treatment EA is formulated as a purely universal theory with terms for all Kalmar elementary functions. So, the assumption of the theorem yields a sequence of terms $m_0, m_1(x_0), m_2(x_0, x_1), \dots, m_k(x_0, \dots, x_{k-1})$ such that the following disjunction is provable in EA:

$$\begin{aligned}
&(x_0 \geq m_0 \rightarrow h(x_0) \geq h(x_0+1)) \vee && (1)\\
&(x_1 \geq m_1(x_0) \rightarrow h(x_1) \geq h(x_1+1)) \vee && (2)\\
&\dots \\
&(x_k \geq m_k(x_0, \dots, x_{k-1}) \rightarrow h(x_k) \geq h(x_k+1)).
\end{aligned}$$

Notice that the variables $x_0, \dots, x_k$ are free.

We prove $\exists m \forall n \geq m\, h(n) = h(m)$ using the parameter-free least element principle for Σ_1-formulas $L\Sigma_1^-$. The standard argument shows that $L\Sigma_1^-$ is equivalent to $I\Pi_1^-$.

Consider the least y_0 such that $\exists x_0 \geq m_0\, h(x_0) = y_0$ (such a y_0 exists by $I\Pi_1^-$). If $\forall x_0 \geq m_0\, h(x_0) \geq h(x_0+1)$, $x \geq m_0$ and $h(x) = y_0$, then by Δ_0-induction

$$\forall n \geq x\, y_0 \leq h(n) \leq y_0,$$

hence y_0 is the limit of h.

Otherwise, $\exists x_0 \geq m_0\, h(x_0) < h(x_0+1)$, so we can take the least such $x_0 \geq m_0$. For this x_0 the first line (1) of the disjunction is false.

Consider now the least y_1 such that $\exists x_1 \geq m_1(x_0)\, h(x_1) = y_1$. Notice that $I\Pi_1^-$ proves the existence of y_1 because x_0 is Δ_0-definable:

$$y_1 = \mu y.\, \exists x_0, x_1\, (x_0 \geq m_0 \wedge h(x_0) < h(x_0+1) \wedge \\ \forall x < x_0\, \neg(x \geq m_0 \wedge h(x) < h(x+1)) \wedge x_1 \geq m_1(x_0) \wedge h(x_1) = y).$$

Therefore we may proceed with the second line of the disjunction in the same way as with the first one: either $\forall x_1 \geq m_1(x_0)\, h(x_1) \geq h(x_1+1)$ and then y_1 is the limit of h, or (2) is false and using Δ_0-induction we take the least $x_1 \geq m_1(x_0)$ such that $h(x_1) < h(x_1+1)$. The point x_1 is Δ_0-definable, so we may proceed in a similar manner and successively obtain the elements $y_2, \ldots, y_k$ until all the remaining lines of the disjunction are falsified. ⊠

The following theorem shows that unnested applications of $(LimR)$ are, in fact, sufficient to derive $I\Pi_1^-$.

Theorem 6 *Any instance of $I\Pi_1^-$ is provable by one application of $(LimR)$ over* EA.

Proof. We prove any instance of $L\Sigma_1^-$ (and therby of $I\Pi_1^-$) by an application of $(LimR)$. Assume φxu is a Δ_0-formula for which we want to prove

$$\exists x \exists u \varphi xu \to \exists x\, (\exists u \varphi xu \wedge \forall y < x\, \forall v \neg \varphi yv).$$

We construct the following elementary functions using the standard coding of pairs.

$$g_1(n) = \begin{cases} n, & \text{if } \forall v \leq n.\, \neg\varphi((v)_0, (v)_1) \\ (\mu v \leq n.\, \varphi((v)_0, (v)_1))_0, & \text{otherwise.} \end{cases}$$

In other words, $g_1(n) = n$ for all n smaller than the code of the first pair $\langle x, u\rangle$ such that φxu. Thereafter, the value of g_1 does not change and is equal to that particular x.

$$g(n) = \begin{cases} (n)_0, & \text{if } \exists u \leq (n)_1.\, \varphi((n)_0, u) \\ g_1(n), & \text{otherwise.} \end{cases}$$

The function $g(n)$ enumerates the set $\{x : \exists u \varphi xu\} \cup X$, where X is a finite set coming from the values of g_1. (More precisely, X will be finite, if $\exists x \exists u\, \varphi xu$.) Notice that the points from $X \setminus \{x : \exists u \varphi xu\}$ are enumerated only once, whereas the points from $\{x : \exists u \varphi xu\}$ are enumerated infinitely often. Thus, it would be sufficient for us to find the minimum of $\{x : \exists^{>1} n\, g(n) = x\}$.

We define the function h by setting $h(0) = g(0)$. $h(n+1)$ is defined as follows. If $\forall k, m \leq n\, (k \neq m \to g(k) \neq g(m))$, then $h(n+1) = g(n+1)$. If $\exists m \leq$

$n\, g(n+1) = g(m)$ and $g(n+1) < h(n)$, then $h(n+1) = g(n+1)$. Otherwise, $h(n+1) = h(n)$.

$h(n)$ coincides with $g(n)$ until some value of g is repeated. Thereafter, h can only decrease (or not change) its value. So, $\exists x \exists u\, \varphi x u$ implies

$$\exists m \forall n \geq m\, h(n) \geq h(n+1).$$

Define another function h' as follows:

$$h'(x) = \begin{cases} h(x), & \text{if } \exists n \leq x\, \varphi((n)_0, (n)_1) \\ 0, & \text{otherwise.} \end{cases}$$

Thus, h' eventually coincides with h, if $\exists x \exists u \varphi x u$, and is identically 0, otherwise. In any case, it is provable in EA that h' is eventually descending. So, applying $(LimR)$ once, we conclude that h' has a limit y. If $\exists x \exists u \varphi x u$, then y is also the limit of h.

By the definition of h, y will also be the minimum of $\{x : \exists^{>1} n\, g(n) = x\}$, because h decreases every time a repeated value of g smaller than the current value of h appears. Hence, y is also the minimum of $\{x : \exists u \varphi x u\}$. ⊠

From the previous two theorems we conclude that the closure of EA under unnested applications of $(LimR)$ is equivalent to $I\Pi_1^-$. The fragment $I\Pi_1^-$ is properly stronger than EA, in fact, $I\Pi_1^-$ proves consistency of EA (and even finitely many times iterated consistency of EA) [3]. On the other hand, $I\Pi_1^-$, being a set of true Σ_2-sentences, is included in the theory axiomatized by all true Π_1-sentences. Therefore, $I\Pi_1^-$ has the same class of provably total computable functions as EA itself, that is, the elementary functions. So, $I\Pi_1^-$ is much weaker than $I\Sigma_1^-$ and does not contain the primitive recursive arithmetic (see [11,2] for more information on $I\Pi_1^-$).

Now we consider nested applications of the limit existence rule. Our main result below shows that this rule gives an exact axiomatization of the Σ_2-consequences of $I\Sigma_1$. In particular, nested rule is properly stronger than the unnested rule.

Theorem 7 *$I\Sigma_1$ is conservative over $\mathsf{EA} + (LimR)$ for Σ_2-sentences.*

A proof of this theorem consists of three steps. First, we introduce a class of arithmetical sentences that we call *special Σ_2-sentences* and prove that $(LimR)$ is equivalent to a more general version of this rule, where one allows special Σ_2-sentences as side formulas. Secondly, we show that $(LimR)$ provides an axiomatization of *special* Σ_2-consequences of $\mathsf{EA} + (Lim)$ (and thus of $I\Sigma_1$, using the Σ_3-conservativity of $I\Sigma_1$ over $I\Sigma_1^- \equiv \mathsf{EA} + (Lim)$). Thirdly, we prove the theorem by reducing arbitrary Σ_2-consequences of $I\Sigma_1$ to special Σ_2-consequences.

A sentence in the language of EA is called *special* Σ_2, if it has the form $\exists m \forall n \geq m\, \varphi(n)$, where φ is a bounded formula in one variable. We denote this class of sentences $s\Sigma_2$. Particular cases of $s\Sigma_2$-sentences are the assertions that a function f has a limit $\exists m \forall n \geq m\, f(n) = f(n+1)$ and the statement that a function f is eventually decreasing $\exists m \forall n \geq m\, f(n+1) \leq f(n)$.

Lemma 8 *Let A be a Σ_2-sentence of the form $\exists x \forall y\, \varphi(x,y)$ with φ bounded. Then there is an elementary term f such that the sentence $A^* := \exists m \forall n \geq m\, f(n+1) = f(n)$ satisfies*

(i) $\mathsf{EA} + (Lim) \vdash A \to A^*$;
(ii) $\mathsf{EA} \vdash A^* \to A$;
(iii) If A is special Σ_2, then $\mathsf{EA} \vdash A^* \leftrightarrow A$.

Proof. Let f be defined using the standard pairing function as follows:

$$f(n) = \mu x \leq n.\ \forall y \leq n\, (\langle x, y\rangle \leq n \to \varphi(x,y)).$$

We assume that provably in EA the pairing function is increasing in both arguments and $\langle x, y\rangle$ exceeds both x and y. Notice that f is elementary and provably in EA non-decreasing.

(i) Reason in $\mathsf{EA} + (Lim)$. Assume $\exists x \forall y \varphi(x,y)$ and let x_0 be such that $\forall y \varphi(x_0, y)$. We claim that $\forall n\, f(n) \leq x_0$. Indeed, $f(n) > x_0$ implies

$$\exists y \leq n\, (\langle x_0, y\rangle \leq n \land \neg\varphi(x_0, y)),$$

which in particular yields $\exists y \neg\varphi(x_0, y)$, contrary to the choice of x_0.

Now we are in a position to apply Theorem 3 and conclude that f must have a limit.

(ii) Reason in EA. Assume f is constant from m_0 on. Let $x_0 := f(m_0)$. Then, assuming A false we obtain a y_0 such that $\neg\varphi(x_0, y_0)$. Let $n_0 := \langle x_0, y_0\rangle$. Notice that $m_0 < n_0$, for otherwise $y_0 \leq n_0 \leq m_0$ and $\neg\varphi(x_0, y_0)$, hence $f(m_0) = x_0$ would be violated.

By the choice of n_0 we have that $\forall y \leq n_0\, (\langle x_0, y\rangle \leq n_0 \to \varphi(x_0, y))$ is false. Hence $f(n_0) \neq f(m_0)$ contradicting the choice of m_0.

(iii) Suppose $A = \exists x \forall y \geq x\, \varphi(y)$. In this case

$$f(n) = \mu x \leq n.\ \forall y \leq n\, (\langle x, y\rangle \leq n \land y \geq x \to \varphi(y)).$$

To prove $A \to A^*$ reason in EA. Let m_0 be such that

$$\forall y \geq m_0\, \varphi(y). \tag{3}$$

Define $n_0 := \langle m_0, m_0 \rangle$ and $k_0 := f(n_0)$. We claim that f is constant from n_0 on.

Since $f(n_0) = k_0$ we have

$$\forall y \leq n_0\, (\langle k_0, y \rangle \leq n \land y \geq k_0 \to \varphi(y)). \tag{4}$$

Notice that $k_0 \leq m_0$, because (4) is satisfied with m_0 in place of k_0, by the choice of m_0, and k_0 is the minimal such number. Since the coding of pairs is monotonous, from (4) we infer

$$\forall y\, (k_0 \leq y \leq m_0 \to \varphi(y)).$$

Together with (3) this implies $\forall y \geq m_0\, \varphi(y)$. By the definition of f, and since we already know that f is non-decreasing, it follows that $f(n) = k_0$ for all $n \geq n_0$. ⊠

Thus, special Σ_2-sentences are equivalent to sentences asserting that some elementary function has a limit. Next we show that the set of special Σ_2-sentences is closed under disjunction modulo provability in EA.

Lemma 9 *For every $s\Sigma_2$-sentences A_1 and A_2 there is a $s\Sigma_2$-sentence A such that $\mathsf{EA} \vdash A \leftrightarrow (A_1 \lor A_2)$.*

Proof. Assume (modulo EA) that A_1 asserts that a limit of g_1 exists, and A_2 asserts that a limit of g_2 exists. Define by elementary recursion $f(0) = 0$ and

$$f(n+1) = \begin{cases} 1, & \text{if } f(n) = 0 \text{ and } g_0(n+1) \neq g_0(n); \\ 0, & \text{if } f(n) = 1 \text{ and } g_1(n+1) \neq g_1(n); \\ f(n), & \text{otherwise.} \end{cases}$$

Let $A := \exists m \forall n \geq m\, f(n) = f(n+1)$.

Reason in EA. Suppose g_0 has a limit and choose m_0 such that $\forall n \geq m_0 g_0(n) = g_0(m_0)$. If $f(m_0) = 0$, then f will also stay at 0. If $f(m_0) = 1$, then either $\forall n \geq m_0 g_1(n) = g_1(m_0)$, and then f stays at 0, or $\exists m_1 \geq m_0 g_1(m_1) \neq g_1(m_0)$. We can take the smallest such m_1, by Δ_0-least element principle in EA. Then necessarily $f(m_1) = 0$ and hence $\forall n \geq m_1\, f(n) = 0$. This shows that f has a limit, if g_0 has a limit. The reasoning is symmetrical, if g_1 has a limit.

Now assume that f has a limit and $\forall n \geq m\, f(n) = 0$. Then $\forall n \geq m\, g_0(n) = g_0(m)$, otherwise f would jump to 1. Hence, g_0 has a limit. If the limit of f is 1, g_1 will have a limit, by a similar reasoning. ⊠

A limit existence rule with side formulas is defined (in Hilbert-style format) as follows:

$$\frac{\sigma \vee \exists m \forall n \geq m\, h(n+1) \leq h(n)}{\sigma \vee \exists m \forall n \geq m\, h(n) = h(m)}, \qquad (LimR^*)$$

for some elementary function h and a sentence $\sigma \in s\Sigma_2$.

Lemma 10 *$(LimR^*)$ is equivalent to $(LimR)$ over EA.*

Proof. Assume $\mathsf{EA} + (LimR)$ proves a premise of the rule $(LimR^*)$ of the form

$$\sigma \vee \exists m \forall n \geq m\, h(n+1) \leq h(n), \qquad (5)$$

where σ has the form $\exists m \forall n \geq m\ f(n) = f(n+1)$. From the terms f and h define a new function g by elementary recursion: $g(0) = h(0)$ and

$$g(x+1) = \begin{cases} g(x), & \text{if } f(x+1) = f(x); \\ h(x+1), & \text{otherwise.} \end{cases}$$

Reason in $\mathsf{EA} + (LimR)$. Obviously, if f has a limit, then g is eventually constant. If f does not have a limit, then by (5) h will be eventually decreasing. Therefore, g will be eventually decreasing from the same moment on (because the values of g follow those of h). Hence, we can infer by $(LimR)$ that g has a limit.

Notice that either f has a limit or infinitely often $g(x) = h(x)$. In the latter case, if g has a limit, then also h has a limit, because h is eventually decreasing and attains the limit value of g infinitely often. So we infer $\sigma \vee \exists m \forall n \geq m\, h(n) = h(m)$. ⊠

Notice that the given proof shows that the equivalence of the two rules respects the number of nested applications of them. Also, notice that if one drops the requirement of the side formula σ being $s\Sigma_2$, then the resulting rule will be equivalent to the schema (Lim).

Lemma 11 *$\mathsf{EA} + (Lim)$ is conservative over $\mathsf{EA} + (LimR)$ for special Σ_2-sentences.*

Proof. Let T denote the closure of EA under $(LimR)$, and let Φ_n denote a conjunction of the first n instances of (Lim):

$$\Phi_n := \bigwedge_{i=1}^{n} (A_i \to B_i).$$

We prove by induction on n that, for any $\sigma \in s\Sigma_2$,

$$T \vdash \Phi_n \to \sigma \quad \Rightarrow \quad T \vdash \sigma.$$

Basis of induction is clear. For the induction step notice that $\Phi_{n+1} = \Phi_n \wedge (A_{n+1} \to B_{n+1})$.

Assume $T \vdash \Phi_{n+1} \to \sigma$, where σ is a special Σ_2-sentence. Then by propositional logic

$$T \vdash (\Phi_n \wedge B_{n+1}) \to \sigma \tag{6}$$

and

$$T \vdash \Phi_n \to (A_{n+1} \vee \sigma). \tag{7}$$

Notice that $A_{n+1} \vee \sigma$ is equivalent in EA to a special formula, so from (7) by the induction hypothesis we obtain $T \vdash A_{n+1} \vee \sigma$. By (LimR^*) and Lemma 10 we infer $T \vdash B_{n+1} \vee \sigma$. Together with (6) this yields $T \vdash \Phi_n \to \sigma$, essentially by an application of the rule of cut. Hence, the induction hypothesis is applicable once again and we obtain $T \vdash \sigma$. ⊠

Proof of Theorem 7. Assume $I\Sigma_1$ proves a Σ_2-sentence A. By Σ_3-conservativity of $I\Sigma_1$ over $I\Sigma_1^-$ and by Theorem 3, $\mathsf{EA} + (\mathit{Lim}) \vdash A$. By Lemma 8 (i), $\mathsf{EA} + (\mathit{Lim}) \vdash A^*$. By Lemma 11, $\mathsf{EA} + (\mathit{LimR}) \vdash A^*$. Hence, by Lemma 8 (ii) also $\mathsf{EA} + (\mathit{LimR}) \vdash A$. ⊠

4 An application to conservativity logic

Interpretability logic studies the binary modality $\varphi \rhd \psi$ which is translated in the language of arithmetic as the formalized statement

$$\text{“}T + \varphi \text{ interprets } T + \psi\text{”}.$$

Here T is any fixed theory containing enough arithmetic and interpretability is understood as the relative interpretability in the usual sense of Tarski. *The interpretability logic* $\mathbf{IL}(T)$ of a theory T is the set of all propositional modal formulas in the language with $\rhd$ whose arithmetical translations are always provable in T. This notion attracted the attention of researchers from the late 80's, particularly, because it represented a significant strengthening of the more usual *provability logic*. The modality $\Box\varphi$ meaning "φ is provable in T" can easily be expressed in terms of $\rhd$ by $\Box\varphi \leftrightarrow (\neg\varphi) \rhd \bot$.

An important result, obtained by A. Berarducci [4] and V. Shavrukov [13], states that if T is an essentially reflexive theory, in particular, if T is a (sound) extension of Peano Arithmetic PA in the language of PA, then the interpretability logic of T is decidable and axiomatized by a certain axiom system **ILM**.

By the well-known Orey-Hájek characterization (see [9]), for essentially reflexive theories, the notion of interpretability coincides with that of Π_1-conservativity.

In general, however, the two notions diverge, most notably for finitely axiomatizable theories.

This suggests an alternative translation of the modality $\varphi \rhd \psi$ as the arithmetized statement

$$\forall \pi \in \Pi_1 \; (T + \psi \vdash \pi \Rightarrow T + \varphi \vdash \pi)$$

expressing the Π_1-conservativity of $T + \psi$ over $T + \varphi$. Accordingly, the *logic of* Π_1*-conservativity* $\mathbf{CL}(T)$ of T is defined as the set of all modal formulas that are always provable in T under the Π_1-conservativity translation of $\rhd$.

P. Hájek and F. Montagna [7,8] showed that $\mathbf{CL}(T) = \mathbf{ILM}$ not only for extensions of PA, but also for any sound theory T extending $I\Sigma_1$. G. Japaridze [10], among others, found a simplified proof of their result. However, it was not clear from any of these proofs if the bound $I\Sigma_1$ is optimal.

Using the result of the previous section the Berarducci–Shavrukov–Hájek–Montagna theorem can now be improved to the following statement.

Theorem 12 *If T is a sound extension of $I\Pi_1^-$, then* $\mathbf{CL}(T) = \mathbf{ILM}$.

Proof. We analyze the proof given in [5]. Without going into the details, we briefly indicate where the use of $I\Pi_1^-$ is essential. We assume the knowledge of Theorem 14.2 in [5] and the terminology used there.

The proof rests on a diagonal definition of a certain primitive recursive function g, which is associated with an arbitrary finite **ILM**-model. Firstly, it is easy to see that g is Kalmar elementary, so it can be introduced within EA. The essential point in the proof is Lemma 14.3 (a), that is, the statement that g has a limit. The rest of the reasoning is formalizable in EA.

We prove Lemma 14.3 (a) using $(LimR)$. With the function g we associate another elementary function h defined by $h(0) = 0$ and

$$h(n+1) = \begin{cases} 0, & \text{if } n \text{ is an } R'\text{-transfer of } g, \\ \mathrm{rank}(n), & \text{if } n \text{ is an } S'\text{-transfer of } g, \\ h(n), & \text{otherwise.} \end{cases}$$

The notions of S'- and R'-transfer are taken from [5], and $\mathrm{rank}(n)$ indicates the rank of the S'-transfer at n.

The fact that the **ILM**-model at hand is finite and the relation R' is irreflexive implies that only finitely many R'-transfers of g are possible. Moreover, this fact is verifiable purely within EA. One can establish it by external induction on the R'-depth of the model.

So, EA proves that, for almost all n, g only makes S'-transfers or remains unchanged. As in [5], page 526, we see directly from the definition of g that the ranks of consecutive S'-transfers must decrease (strictly, if the last transfer is a move to a different node). This implies within EA that h is eventually descending.

Applying Theorem 5, by $(LimR)$ we conclude that h has a limit. This can only happen in the case that eventually no R'- and S'-moves are made. Hence, g has a limit, too. ⊠

Now we turn to some examples showing that the above bound $I\Pi_1^-$ cannot be much improved. The following theorem has been discovered by D. Zambella [15]. G. Mints (private correspondence) gave a finitary proof of this statement based on the Herbrand Theorem. Below, we reproduce a modified version of that proof.

We write $T \equiv_{\Pi_1} U$ if T and U are mutually Π_1-conservative.

Theorem 13 (Zambella) *Let $T, S \subseteq \Pi_2$ and $T \equiv_{\Pi_1} S$. Then $T + S \equiv_{\Pi_1} T$.*

Proof. Assume $S + T \vdash M(\vec{a})$, where M is a quantifier-free formula with the parameters $\vec{a}$. Then the following disjunction is logically provable:

$$M(\vec{a}) \vee \exists x_1 \forall y\, P(x_1, y) \vee \exists x_2 \forall z\, Q(x_2, z),$$

where the second and the third formula are refutable in T and S, respectively. The Herbrand Theorem for $\exists\forall$-formulas delivers a sequence of terms $t_0(\vec{a}), s_0(\vec{a}), t_1(\vec{a}, y_0, z_0), s_1(\vec{a}, y_0, z_0), \ldots, s_n(\vec{a}, y_0, z_0, \ldots, y_{n-1}, z_{n-1})$ such that

$$\vdash K_n \vee M(\vec{a}),$$

where K_n denotes

$$\bigvee_{i=0}^{n} (P(t_i, y_i) \vee Q(s_i, z_i)).$$

We prove by induction on j that for all $j = 0, \ldots, n$,

$$S \vdash M \vee K_{n-j}.$$

The statement is obvious for $j = 0$. For the induction step assume

$$S \vdash M \vee K_{l+1},$$

where $l = n - j - 1$. We have

$$S \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}) \vee Q(s_{l+1}, z_{l+1}).$$

Hence,

$$S \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}) \vee \forall z_{l+1}\, Q(s_{l+1}, z_{l+1}),$$

because the variable z_{l+1} only occurs in $Q(s_{l+1}, z_{l+1})$. Since

$$S \vdash \neg\exists x_2 \forall z_{l+1} Q(x_2, z_{l+1}),$$

this yields

$$S \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}).$$

This formula is quantifier-free, so by the Π_1-conservativity of S over T we obtain

$$T \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}).$$

Now we can get rid of $P(t_{l+1}, y_{l+1})$ in the same way using the fact that $T \vdash \forall x_1 \exists y_{l+1} \neg P(x_1, y_{l+1})$. So, we obtain $T \vdash M \vee K_l$, whence $S \vdash M \vee K_l$ by the Π_1-conservativity of T over S. This proves the induction step and the theorem. $\boxtimes$

Let EA^+ be axiomatized over EA by the axiom stating that the superexponentiation function is total (see [9]). The Herbrand Theorem is formalizable in EA^+, hence so is the proof of Zambella's theorem.

Let us consider the conservativity logic $\mathbf{CL}(T)$ for a Π_2-axiomatizable theory T containing EA^+. Examples of such T are EA^+ itself or the *primitive recursive arithmetic* PRA.

Notice that any arithmetical translation of a modal formula of the form $\varphi \rhd \psi$ is Π_2, and the translations of the formulas of the form $\Diamond\varphi$ and $\Box\varphi$ are Π_1 and Σ_1, respectively.

Call a modal formula Π_2, if it is obtained from formulas of the forms $\varphi \rhd \psi$, $\Box\varphi$ and $\Diamond\varphi$ by positive boolean connectives $\vee$ and $\wedge$. By the above remark any arithmetical translation of such a formula belongs to the class Π_2 of the arithmetical hierarchy.

Corollary 14 *If $T \subseteq \Pi_2$ and contains EA^+, then the following schema belongs to $\mathbf{CL}(T)$:*

$$(\varphi \rhd \psi \wedge \psi \rhd \varphi) \to \varphi \rhd (\varphi \wedge \psi), \qquad (Zam)$$

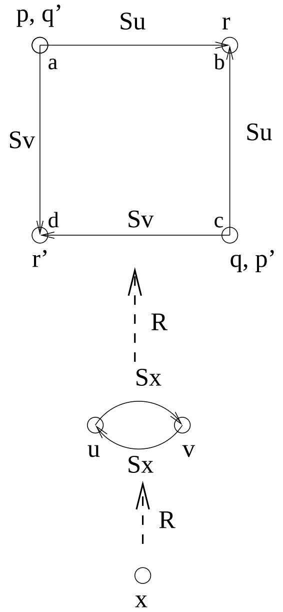
where φ and ψ are Π_2-formulas.

Proof. This is Zambella's theorem applied to the theories $T + \varphi$ and $T + \psi$. Mints' proof shows that this statement is actually provable in EA^+ and hence in T. $\boxtimes$

Now we prove that the principle (Zam) is independent from **ILM**.

Theorem 15 *The instance of (Zam) with $\varphi = (\Diamond p \wedge (q \rhd r))$ and $\psi = (\Diamond p' \wedge (q' \rhd r'))$ is not provable in* **ILM**.

Proof. We exhibit an **ILM**-model that falsifies the given instance of (Zam). For the notion of **ILM**-model see [5].



Here, each of the nodes a, b, c, d is R-accessible from any of the nodes u, v and both u and v are R-accessible from x. The S-arrows are all indicated except for the obvious reflexivity arrows for each of the nodes.

At every node the shown variables are true; all the other variables are assumed to be false. It is not difficult to see that this model falsifies the given instance of (Zam) at the node x. ⊠

By the previous theorem, the principle (Zam) is independent from **ILM**. Thus, the Π_1-conservativity logics of Π_2-axiomatizable theories such as PRA properly extend **ILM**. We know from [2] that the schema $I\Pi_1^-$ is equivalent to the local Σ_2-reflection principle over EA w.r.t. cut-free provability. This implies that $I\Pi_1^-$ is not contained in any consistent r.e. extension of EA of quantifier complexity Π_2, which agrees with Theorem 12 above.

To find a complete axiomatization of **CL**(PRA) and **CL**(EA) remains an intriguing open question.

5 On Σ_n-consequences of $I\Sigma_m$

Results on proof-theoretic (ordinal) analysis of theories T can be viewed as characterizations of consequences of T of specific logical complexity, e.g., of complexity Π^1_1 or Π^0_2. Typically, given a logical complexity class Γ one considers a mapping $P : \alpha \longmapsto P_\alpha$ of ordinals to schemata in the language of T satisfying the following demands:

(1) For each α, $P_\alpha \in \Gamma$.
(2) $\{\varphi \in \Gamma : T \vdash \varphi\}$ can be axiomatized by $\{P_\beta : \beta < \alpha\}$, where

$$\alpha = |T|_P := \min\{\beta : T \nvdash P_\beta\}.$$

The ordinal α is then called the *proof-theoretic ordinal of T w.r.t. the mapping P*. Somewhat loosely, one also calls it the *Γ-ordinal of T*, if there is one distinguished hierarchy P is associated with Γ (see [3] for a more detailed discussion).

Thus, P serves as a measure of proof-theoretic strength of theories. Assuming fixed some natural system of ordinal notation one can define the following typical hierarchies considered in the literature:

(a) $P_\alpha := \mathrm{WF}(\alpha) \in \Pi^1_1$, the transfinite induction axiom up to α;
(b) $Q_\alpha := (\forall x \exists y\, F_\alpha(x) = y) \in \Pi^0_2$, where F_α is the α-th function of the fast growing hierarchy;
(c) $R_\alpha := (\forall x \exists y\, G_\alpha(x) = y) \in \Pi^0_2$, where G_α is the α-th function of the slow growing hierarchy.

For example, it follows from Gentzen's results that the ordinal of Peano arithmetic (conservatively extended by second order variables) w.r.t. the above measure P is ϵ_0. This was improved later by H. Schwichtenberg and S. Wainer to the statement that the ordinal of PA w.r.t. Q is ϵ_0. On the other hand, the ordinal of PA w.r.t. R is much higher and was calculated by J.-Y. Girard.

In [3], developing the ideas in [14,6] and [12], the hierarchies of uniform reflection principles were used to classify Π^0_n-consequences of theories, for any n. Below we shall work in the language of EA. Recall that the *n-consistency assertion* $n\text{-}\mathsf{Con}(T)$ for a theory T formalizes the fact that T together with all true Π_n-sentences is consistent. This can be naturally expressed using a truth-definition for Π_n-formulas in EA.

Iterated consistency assertions, assuming fixed an elementary system of ordinal notation, are defined by the following fixed point equation:

$$\mathsf{EA} \vdash n\text{-}\mathsf{Con}^\alpha(T) \leftrightarrow n\text{-}\mathsf{Con}(T \cup \{n\text{-}\mathsf{Con}^\beta(T) : \beta < \alpha\}).$$

It is proved in [1,3] that the solution of this equation is unique up to provable equivalence in EA. We also denote the theory $T \cup \{n\text{-}\mathsf{Con}^\beta(T) : \beta < \alpha\}$ by $(T)^n_\alpha$. Notice that, provably in EA, $n\text{-}\mathsf{Con}^0(T) \leftrightarrow n\text{-}\mathsf{Con}(T)$ and $n\text{-}\mathsf{Con}^\alpha(T) \leftrightarrow n\text{-}\mathsf{Con}((T)^n_\alpha)$. See [3] for more information on such hierarchies.

As shown in [3], the statements $P^n_\alpha := n\text{-}\mathsf{Con}^\alpha(\mathsf{EA})$ can be used to classify Π_n-consequences of theories, for any n. For $n = 2$ the ordinals obtained in this way agree with the ordinals w.r.t. the traditional Π^0_2-measure Q defined by the fast growing hierarchy. However, to the best of our knowledge, there were no results on ordinal classifications of Γ-consequences of theories, for Σ-type classes of sentences Γ.

In the realm of fragments of arithmetic some suggestive relationships are known. Thus, by a result of [11], $I\Sigma_m$ is Σ_{m+2}-conservative over $I\Sigma^-_m$. Hence, Σ_{m+2}-consequences of $I\Sigma_m$ are axiomatizable by $I\Sigma^-_m$ which has the smaller complexity $\Sigma_{m+1} \to \Sigma_{m+1}$. However, already the question about Σ_2-consequences of $I\Sigma_1$ was open. Among the Σ_2-consequences of $I\Sigma_1$ we find the system $I\Pi^-_1$, but it does not exhaust all its Σ_2-consequences. An exact characterization is given by the limit existence rule and Theorem 7.

Below we give a complete classification of Σ_n-consequences of the fragments $I\Sigma_m$ of PA, for any n and m. We notice some trivial relationships. Firstly, Σ_1-consequences of any sound theory containing EA coincide with the set of true Σ_1-sentences. Secondly, if $n > m + 2$, then $I\Sigma_m$ itself provides a Σ_n-axiomatization of its Σ_n-consequences. The other cases, except for $n = m + 2$, are covered by the following theorem.

Define: $\omega_0 = 1$, $\omega_{k+1} = \omega^{\omega_k}$. We consider the schema

$$S^n_\alpha := \{\pi \to n\text{-}\mathsf{Con}^\alpha(\mathsf{EA} + \pi) : \pi \in \Pi_{n+2}\}.$$

Theorem 16 *Let $0 \le n < m$. Then Σ_{n+2}-consequences of $I\Sigma_m$ are axiomatized over* EA *by the schemata*

$$\{S^n_\alpha : \alpha < \omega_{m-n+1}\}. \qquad (*)$$

Proof. The proof of this theorem is based on the results of [3]. We note that the arithmetical complexity of the schema S^n_α is, indeed, Σ_{n+2}.

First, we derive $(*)$ from $I\Sigma_m$. Consider a Π_{n+2}-sentence π. We know that $I\Sigma_m$ is equivalent to $(m+1)\text{-}\mathsf{Con}(\mathsf{EA})$. Since $m > n$, we have that $I\Sigma_m + \pi$ is equivalent to $(m+1)\text{-}\mathsf{Con}(\mathsf{EA} + \pi)$. Applying now Theorem 4 from [3] we obtain that

$$(\mathsf{EA} + \pi)^{m+1}_1 \equiv_{\Pi_{n+1}} (\mathsf{EA} + \pi)^n_{\omega_{m+1-n}}. \qquad (**)$$

Hence,

$$I\Sigma_m + \pi \vdash n\text{-}\mathsf{Con}^\alpha(\mathsf{EA} + \pi),$$

for all $\alpha < \omega_{m+1-n}$. Deduction theorem implies

$$I\Sigma_m \vdash \pi \to n\text{-}\mathsf{Con}^\alpha(\mathsf{EA} + \pi),$$

for any such α.

For the converse direction, assume $I\Sigma_m \vdash \neg\pi$, where $\pi \in \Pi_{n+2}$. Then $I\Sigma_m + \pi \vdash \bot$. By (∗∗) we have $(\mathsf{EA}+\pi)^n_{\omega_{m+1-n}} \vdash \bot$. Hence, there is $\alpha < \omega_{m+1-n}$ such that $\mathsf{EA} + \pi + n\text{-}\mathsf{Con}^\alpha(\mathsf{EA}+\pi) \vdash \bot$. Therefore,

$$\mathsf{EA} + \pi + (\pi \to n\text{-}\mathsf{Con}^\alpha(\mathsf{EA}+\pi)) \vdash \bot,$$

and by the Deduction Theorem

$$\mathsf{EA} + (\pi \to n\text{-}\mathsf{Con}^\alpha(\mathsf{EA}+\pi)) \vdash \neg\pi.$$

This proves the theorem. ⊠

Remark 17 Σ_{m+2}-consequences of $I\Sigma_m$ axiomatized by $I\Sigma_m^-$ have a characterization in terms of reflection principles which is similar to the one in Theorem 16. By the results of [2], $I\Sigma_m^-$ is equivalent to

$$\mathsf{EA} + \{\pi \to m\text{-}\mathsf{Con}(\mathsf{EA}+\pi) : \pi \in \Pi_{m+1}\}.$$

Therefore, Theorem 16 also holds with $m = n$, but with the complexity of π decreased by 1.

Now we mention some corollaries of Theorem 16.

Corollary 18 *Σ_2-consequences of $I\Sigma_m$ are axiomatized by*

$$\{\pi \to \mathsf{Con}^\alpha(\mathsf{EA}+\pi) : \pi \in \Pi_2,\ \alpha < \omega_{m+1}\}.$$

In particular, for $I\Sigma_1$ one obtains all S^0_α with ordinals $\alpha < \omega^\omega$.

Corollary 19 *Σ_{n+2}-consequences of PA are axiomatized by S^n_α with $\alpha < \epsilon_0$.*

Thus, the hierarchy of schemata S^n_α allows to define proof-theoretic ordinals of theories for logical complexity Σ_{n+2}. According to this measure the Σ_2-ordinals of $I\Sigma_1$ and PA are ω^ω and ϵ_0, respectively.

We show that S^n_α can also be perceived as iterated local reflection schemata. For any theory T, let $S^n(T)$ denote the relativized local reflection schema

$$\{\pi \to n\text{-}\mathsf{Con}(T+\pi) : \pi \in \Pi_{n+2}\}.$$

(Notice that for $n = 0$ it is equivalent to the usual local Σ_2-reflection schema.) Iterated version thereof is a progression of theories $S^n_\alpha(T)$ obtained, similarly to the iterated consistency assertions, by formalizing the fixed point equation

$$S^n_\alpha(T) \equiv T + \{S^n(S^n_\beta(T)) : \beta < \alpha\}.$$

See [3] for more details. We have the following relationship.

Theorem 20 *Verifiably in* EA, *for any* α,

$$S^n_\alpha(\mathsf{EA}) \equiv \mathsf{EA} + \{S^n_\beta : \beta < \omega^\alpha\}.$$

For the proof, first we need the following lemma.

Lemma 21 *For any* α *and* $\pi \in \Pi_{n+2}$, *provably in* EA,

$$S^n_\alpha(\mathsf{EA}) + \pi \equiv S^n_\alpha(\mathsf{EA} + \pi).$$

Proof. The proof goes by the so-called reflexive induction on α in EA (see [3]). We give an informal argument.

Assume the statement holds provably for all $\beta < \alpha$. Consider the axioms of $S^n_\alpha(\mathsf{EA} + \pi)$ that have the form

$$\pi_1 \to n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA} + \pi) + \pi_1),$$

for some $\beta < \alpha$ and $\pi_1 \in \Pi_{n+2}$. By the reflexive induction hypothesis the latter is equivalent to

$$\pi_1 \to n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA}) + \pi + \pi_1). \tag{8}$$

Notice that $S^n_\alpha(\mathsf{EA})$ proves

$$\pi \wedge \pi_1 \to n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA}) + \pi + \pi_1),$$

for both π and π_1 are Π_{n+2}. It follows that $S^n_\alpha(\mathsf{EA}) + \pi$ proves (8).

The proof in the opposite direction is easy. ⊠

Proof of Theorem 20. Again, we use reflexive induction. We also rely on the results of [1,**?**]. Recall that α-times iterated local reflection principle contains (and is Π_1-conservative over) ω^α-times iterated consistency assertion, and similar relationship holds for the relativized versions of these schemata.

To prove ($\supseteq$), consider any $\beta < \omega^\alpha$ and any $\pi \in \Pi_{n+2}$. We have:

$$\begin{aligned} S^n_\alpha(\mathsf{EA}) + \pi &\vdash S^n_\alpha(\mathsf{EA} + \pi) \\ &\vdash (\mathsf{EA} + \pi)^n_{\omega^\alpha}, \quad \text{by Proposition 6.2 of [3]} \\ &\vdash n\text{-}\mathsf{Con}^\beta(\mathsf{EA} + \pi), \quad \text{for } \beta < \omega^\alpha. \end{aligned}$$

Hence, by the Deduction Theorem

$$S^n_\alpha(\mathsf{EA}) \vdash \pi \to n\text{-}\mathsf{Con}^\beta(\mathsf{EA} + \pi),$$

q.e.d.

To prove ($\subseteq$) consider any axiom of $S^n_\alpha(\mathsf{EA})$ of the form

$$\pi \to n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA}) + \pi),$$

with $\beta < \alpha$ and $\pi \in \Pi_{n+2}$. By Proposition 6.2 of [3] we have, provably in EA,

$$S^n_\beta(\mathsf{EA} + \pi) \equiv_{\Pi_{n+1}} (\mathsf{EA} + \pi)^n_{\omega^\beta}.$$

Hence, $n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA} + \pi))$ is equivalent to $n\text{-}\mathsf{Con}^{\omega^\beta}(\mathsf{EA} + \pi)$. Since $\omega^\beta < \omega^\alpha$ we obtain

$$\begin{aligned} \mathsf{EA} + \{S^n_\gamma : \gamma < \omega^\alpha\} &\vdash \pi \to n\text{-}\mathsf{Con}^{\omega^\beta}(\mathsf{EA} + \pi) \\ &\vdash \pi \to n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA} + \pi)) \\ &\vdash \pi \to n\text{-}\mathsf{Con}(S^n_\beta(\mathsf{EA}) + \pi). \end{aligned}$$

Hence, $\mathsf{EA} + \{S^n_\gamma : \gamma < \omega^\alpha\}$ contains $S^n_\alpha(\mathsf{EA})$. ⊠

In connection with the results of Theorems 7 and 5 the following natural question arises: how can we characterize the the closure of EA under k nested applications of the limit rule? Notice that $I\Pi^-_1$ is equivalent to the local Σ_2 reflection principle over EA (for cut-free provability), whereas the same principle iterated ω times axiomatizes all Σ_2-consequences of $I\Sigma_1$, by Corollary 18 and Theorem 20. Hence, it is natural to conjecture that k times iterated local Σ_2-reflection principle would precisely correspond to k nested applications of the limit rule. However, we have not yet verified this claim.

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