Review of
Aspects of Incompleteness
by Per Lindström

As opposed to the structural proof theory, which is concerned with refined structural or combinatorial properties of formal proofs and whose methods, in general, depend on the choice of a particular proof system, there is a family of results in logic that can be formulated and proved for all 'sufficiently strong' effectively presented formal theories, independently of the details of their syntax. Gödel incompleteness theorems are probably the most important results of this type.

Unfortunately, this branch of logic lacks an adequate name, although since the 30's it has acquired all necessary features of a relatively independent subdiscipline: in AMS mathematics subject classification such results usually go under the heading "Gödel numberings in proof theory" (clearly the Gödel numberings themselves are by no means the point of main interest in this case). Other attempts at finding a name have also been not completely successful: "Arithmetization of metamathematics", "Self-reference" (point at the methods used, but not at the subject of study), "Theory of formal systems" (hints at a too wide interpretation of the term 'formal system').

"Aspects of incompleteness" is an excellent textbook on this old but not easy to define subject. Despite its modest size (about 130 pages) it contains a variety of results that cover the most interesting advances in the area since Gödel theorems and are rarely or not at all presented in monographs. Some keywords: strong forms of Gödel incompleteness theorems, including the construction of Mostowski and Scott formulas, Feferman's nonstandard formalized consistency assertions, Gödel and Parikh speed-up of proofs results, Ehrenfeucht–Feferman and Putnam–Smullyan theorems on numerations of r.e. sets and their refinements, a chapter on reflection principles and axiomatizability, including Pour-El theorem on independent axiomatizability, chapters on interpretability, partial conservativity and lattices of the corresponding degrees.

The three latter chapters (Chapters 5–7) are central to the book and lead the reader practically up to the frontiers of current research. Here a theory is developed to which the author himself has made significant contributions. Main results presented in Chapters 5–6 are: Π_2-completeness of the Σ_n- and Π_n-conservativity relations on finite extensions of Peano arithmetic, equivalence of relative interpretability and Π_2-conservativity (Feferman–Orey–Hájek theorem), Lindström's characterization of *faithful interpretability.* In Chapter 7 the lattice of degrees of interpretability is carefully studied. Besides the elementary algebraic properties of this structure the questions of arithmetical complexity classification of the degrees are considered.

In many parts of the book the author has done a real job of improving and shortening the proofs of the existing results. More important still, he has added a few theorems that, to the best of my knowledge, are new. Most notably, Theorem 5 in Chapter 3 gives a solution to Problem 32 of Friedman (H. Friedman. *One hundred and two problems in mathematical logic.* JSL 40 (1975), 113–129.) This theorem characterizes the so-called *types of independence*, that is, propositional theories defining the relations between arbitrary chosen Σ_1-sentences in the Lindenbaum boolean algebra of Peano arithmetic. For that reason alone the book would already be of interest not only for a student, but also for a specialist.

The book demonstrates a remarkable unity of methods. The main technical tools involved, apart from the elementary recursion theory, are the method of arithmetization (in the broad sense), partial truth definitions, and Gödel's fixed point lemma. Practically all the results in the book are obtained by a combination of these three fundamental ideas. This makes the book conceptually clear, elegant, and easy to follow.

Obviously, it would be impossible to squeeze such a lot of material in 130 pages, had the author not made some choices.

First of all, the book is written in a strictly mathematical style, without many comments and philosophical discussions. The author lets the results 'speak for themselves', rather than expounds on his or other's philosophical attitudes.

Second, the book presupposes a certain amount of background knowledge. Familiarity with Gödel incompleteness theorems and some elementary recursion theory is essentially what is required. This particularly concerns the questions of arithmetization. Technically difficult but easy to understand facts, such as provable Σ_1-completeness, are all presented without proofs. They are collected in introductory Chapter 1 and provided with explanations, comments, and references in the form of an overview. In my opinion, this is a good choice, for applications of the fixed-point arguments seem to be a kind of art that is perhaps more important for a logician to learn, than the particular details of coding. Besides, many well-known textbooks do present such technical details but at the same time do not go far beyond Gödel theorems.

Another, less obvious, choice made in the book is the restriction to formal theories containing Peano arithmetic PA, rather than to some weak fragment of it. This restriction simplifies some constructions, makes it easier for the reader to believe in the facts presented without proofs and/or allows in some cases to supply better references for such facts. On the other hand, the natural question, which of the results presented in the book hold for weaker systems of arithmetic — that play a significant role in modern research — is completely ignored.

Notably in the questions of *relative interpretability* the theories containing PA in the language of PA cannot be considered as representative (although they do present an important particular case). Thus, on the class of finitely axiomatizable theories admitting a coding of finite sequences of their objects, like e.g. Gödel-Bernays set theory, the interpretability relation is complete r.e. rather than Π^0_2-complete, and in many other respects the situation is quite different from the one in the case of essentially reflexive theories. These aspects are only occasionally hinted at in the book.

The author is well aware of and explicit about these restrictions, which may be one of the reasons he chose the unambitious title “Aspects” for his book. In general, as remarked above, the whole subject and the book in particular seem to deserve a better name.

The author's style is laconic but remarkably clear and with a good taste for details. The book is simply pleasant to read, at every moment the reader is kept fully aware of what he or she is doing. Each chapter contains a section of exercises that accumulates related folklore (and not only folklore) results and really is an interesting and important part of the chapter, and a section devoted to historical and bibliographical notes. The selection of exercises in the book would satisfy the gourmet's tastes, although for that very reason some of the exercises may appear too complicated for a beginner. The historical notes are a detailed and and reliable source of bibliographical information.

All this together creates for the reader the beautiful feeling of the subject as a whole. The textbook can be highly recommended to all: students, teachers and researchers in logic.