EA and PRA denote the first order versions of the (Kalmar) elementary and the primitive recursive arithmetic.

Let $\prec$ be an elementary well-ordering. Elementary *transfinite induction rule* is as follows:

$$\frac{\forall\alpha(\forall\beta\prec\alpha\varphi(\beta)\to\varphi(\alpha))}{\forall\alpha\varphi(\alpha)},$$

where φ is an elementary (quantifier-free in the language of EA) formula. If $\prec$ has order type λ, then the above rule is denoted $IR(\lambda,\text{elementary})$. In general, $IR(\lambda,\Gamma)$ denotes this rule for formulas φ of a class Γ. We only consider canonical well-orderings of type smaller than ϵ_0.

Theorem 1 *(i) PRA is closed under $IR(\alpha, prim.rec.)$ for all $\alpha\prec\omega^\omega$.*

(ii) PRA $+\ IR(\omega^\omega, prim.rec.)$ proves the consistency of PRA (and even ω times iterated consistency of PRA).

Theorem 2 *(i) EA is closed under $IR(\alpha, elementary)$ for all $\alpha\prec\omega^2$.*

(ii) EA$+IR(\omega^2, elementary)$ proves the consistency of EA (and even ω times iterated consistency of EA).

Proof of Theorem 1. Instead of giving a direct argument we prefer to use a well-known fact, due to Parsons [2], that PRA is closed under $IR(\omega,\Pi_2)$.

By induction on k we shall show that PRA is closed under $IR(\omega^k,\Pi_1)$. For $k=1$ this is obvious, consider now the case $k+1$.

Assume PRA $\vdash\forall\alpha\prec\omega^{k+1}(\forall\beta\prec\alpha\varphi(\beta)\to\varphi(\alpha))$, where $\varphi(\alpha)$ has the form $\forall u\varphi_0(\alpha,u)$ with φ_0 primitive recursive. Define

$$\psi(\alpha,n):=\alpha\prec\omega^k\wedge n\prec\omega\wedge\varphi(\omega\cdot\alpha+n).$$

Notice that ψ is Π_1 if φ is. Every ordinal $\prec\omega\cdot\alpha+n$ either has the form $\omega\cdot\alpha+n'$ for $n'\prec n$, or the form $\omega\cdot\alpha'+n''$ for $\alpha'\prec\alpha$ and some $n''\prec\omega$. So, by the assumption on φ we have

$$\begin{aligned}\mathsf{PRA}\vdash\forall\alpha\prec\omega^k\forall n\prec\omega\,[\neg\psi(\alpha,n) &\to (\exists\alpha'\prec\alpha\exists n'\neg\psi(\alpha',n')\\ &\vee \exists n'<n\neg\psi(\alpha,n'))].\end{aligned}$$

Hence, by contraposition

$$\begin{aligned}\mathsf{PRA}\vdash\alpha\prec\omega^k\wedge\forall\alpha'\prec\alpha\forall n'\psi(\alpha',n') &\to \forall n\,[\forall n'<n\psi(\alpha,n')\to\psi(\alpha,n)]\\ &\to \forall n\psi(\alpha,n),\end{aligned}$$

by $IR(\omega, \Pi_2)$. More precisely: the rule $IR(\omega, \Pi_2)$ can be formulated equivalently in the following form:

$$\frac{A \to \forall \alpha(\forall \beta \prec \alpha \varphi(\beta) \to \varphi(\alpha))}{A \to \forall \alpha \varphi(\alpha)},$$

where we allow side formulas A (possibly containing parameters different from the induction parameter) of logical complexity Π_1. The version with a side formula, which is applied here, is equivalent to the usual one because the side formula can be merged with the induction formula.

So, now $IR(\omega^k, \Pi_1)$ is applicable and the induction hypothesis yields

$$\begin{aligned} \mathsf{PRA} &\vdash \forall \alpha \prec \omega^k \forall n \psi(\alpha, n) \\ &\vdash \forall \alpha \prec \omega^k \forall n \varphi(\omega \cdot \alpha + n) \\ &\vdash \forall \alpha \prec \omega^{k+1} \varphi(\alpha), \end{aligned}$$

as required. This proves the first part of Theorem 1. The second part is essentially known by adapting Gentzen's consistency proof for PRA.[1]

Proof of Theorem 2. It is easy to show that the set of all α such that EA is closed under $IR(\alpha, \text{elementary})$ is closed under addition. This yields the first claim of the theorem. More interesting is the second claim.

We show that the rule $IR(\omega^2, \text{elementary})$ derives $IR(\omega, \Pi_1)$. It is known by [1] that the closure of EA under the latter rule is equivalent to ω-times iterated consistency of EA.

Assume $\varphi(x, v)$ is an elementary (quantifier-free) formula and

$$T \vdash \forall v \varphi(0, v) \land \forall x(\forall v \varphi(x, v) \to \forall v \varphi(x+1, v)).$$

(T is any extension of EA, e.g., EA itself.) Using the natural coding of ordinals up to ω^2 we define the elementary formula

$$\psi(\omega \cdot n + m) := \varphi(n, m).$$

($\omega \cdot n + m$ is the general form of an ordinal below ω^2.) Then we have

$$\begin{aligned} T \vdash \exists m \neg \psi(\omega \cdot (n+1) + m) &\to \exists m \neg \varphi(n+1, m) \\ &\to \exists m \neg \varphi(n, m) \\ &\to \exists m \neg \psi(\omega \cdot n + m). \end{aligned}$$

[1] There is an even simpler argument to infer that from the results my recent paper "Proof-theoretic analysis...".

But, for any m we have $\omega \cdot n + m \prec \omega \cdot (n+1)$, so

$$\begin{aligned} T \vdash \forall \beta \prec \omega \cdot (n+1) + m \ \psi(\beta) &\rightarrow \forall m' \psi(\omega \cdot n + m') \\ &\rightarrow \forall m' \psi(\omega \cdot (n+1) + m') \\ &\rightarrow \psi(\omega \cdot (n+1) + m). \end{aligned}$$

This implies

$$T \vdash \forall \alpha \prec \omega^2 \ (\forall \beta \prec \alpha \psi(\beta) \rightarrow \psi(\alpha)), \tag{1}$$

because provably in EA any $\alpha \prec \omega^2$ is either of the form $\omega \cdot (n+1) + m$ or $\alpha = m$ (that is, $\alpha \prec \omega$). In the latter case obviously

$$T \vdash \alpha \prec \omega \rightarrow \psi(\alpha),$$

because by the assumption $T \vdash \forall v \varphi(0, v)$. So, by $IR(\omega^2, \text{elementary})$ applied to (1) we obtain

$$\begin{aligned} T + IR(\omega^2, \text{elementary}) &\vdash \forall \alpha \prec \omega^2 \psi(\alpha) \\ &\vdash \forall x, v \varphi(x, v), \end{aligned}$$

and we have derived the conclusion of $IR(\omega, \Pi_1)$. Therefore, $IR(\omega^2, \text{elementary})$ subsumes $IR(\omega, \Pi_1)$ and proves (ω-times) iterated consistency of EA.

References

[1] L.D. Beklemishev. Induction rules, reflection principles, and provably recursive functions. *Annals of Pure and Applied Logic*, 85:193–242, 1997.

[2] C. Parsons. On n-quantifier induction. *Journal of Symbolic Logic*, 37(3):466–482, 1972.