Provability algebras for theories of Tarskian truth

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- Work in progress.
- Joint work with Evgeny Dashkov.
- Influenced by Henryk Kotlarski's study of inductive satisfaction classes.

Provability algebraic view

- We view consistency assertion (along with higher reflection principles) as a function

$$\varphi \longmapsto \mathrm{Con}(S + \varphi)$$

acting on a suitable algebra of sentences. (In principle, on the whole Lindenbaum–Tarski algebra of S.)
- Minimal substructures closed under this map (and some other operations) can provide suitable ordinal notations.
- Using these notations we classify consequences of theories of a specific logical complexity such as Π^0_1 or Π^0_2.

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Why truthpredicates?

- Truthpredicates are tightly related to reflection principles and are convenient in our framework.
- Theories of iterated truth are mutually interpretable with various standard theories of predicative strength (ramified analysis, iterated Π^0_1-comprehension).
- The study leads to some simplifications of the previous approach (going to weak *positive provability logic*).

Reflection principles

Notation:

$\Box_S(\varphi)$ 'φ *is provable in* S'

$Tr_n(\sigma)$ 'σ *is the Gödel number of a true* Σ_n*-sentence*'

Reflection principles:

$R_0(S)$ $\mathrm{Con}(S)$

$R_n(S)$ $\forall \sigma \in \Sigma_n\, (\Box_S \sigma \to Tr_n(\sigma))$, for $n \geq 1$.

$R_n(S) \iff \mathrm{Con}(S + \text{all true } \Pi_n\text{-sentences})$

Reflection calculus RC

Language: $\alpha ::= \top \mid (\alpha_1 \wedge \alpha_2) \mid n\alpha \qquad n \in \omega$
Example: $\alpha = 3(2\top \wedge 32\top)$, or shortly: $3(2 \wedge 32)$.

Sequents: $\alpha \vdash \beta$.

RC rules:

1. $\alpha \vdash \alpha$; $\quad \alpha \vdash \top$; $\quad$ if $\alpha \vdash \beta$ and $\beta \vdash \gamma$ then $\alpha \vdash \gamma$;
2. $\alpha \wedge \beta \vdash \alpha, \beta$; $\quad$ if $\alpha \vdash \beta$ and $\alpha \vdash \gamma$ then $\alpha \vdash \beta \wedge \gamma$;
3. $nn\alpha \vdash n\alpha$; $\quad$ if $\alpha \vdash \beta$ then $n\alpha \vdash n\beta$;
4. $n\alpha \vdash m\alpha$ for $n > m$;
5. $n\alpha \wedge m\beta \vdash n(\alpha \wedge m\beta)$ for $n > m$.

Ex. $3 \wedge 23 \vdash 3(\top \wedge 23) \vdash 323$.

Arithmetical interpretation of RC

Let S be a reasonable theory. Interpretation α_S of α in S:

- $\top_S = \top$; $(\alpha \wedge \beta)_S = (\alpha_S \wedge \beta_S)$;
- $(n\alpha)_S = R_n(S + \alpha_S)$.

Suppose $\mathbb{N} \vDash S$.
Theorem. $\alpha \vdash \beta$ in RC iff $S \vdash \alpha_S \to \beta_S$.

Interpretation of RC in GLP

- RC can be seen as a variable-free $\{\wedge, \Diamond\}$-fragment of polymodal provability logic GLP.
- Interpretation: $3(2\top \wedge 32\top) \mapsto \langle 3\rangle(\langle 2\rangle\top \wedge \langle 3\rangle\langle 2\rangle\top)$

Theorems (E. Dashkov).

1. GLP is a conservative extension of RC (also with variables);
2. RC with variables is polytime decidable;
3. RC with variables enjoys finite model property.

- Weak positive modal logic systems similar to RC with variables have independently been formulated by Zakharyaschev et al. in their work on description logic.

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RC as an ordinal notation system

Define:

- $\alpha \sim \beta$ if ($\alpha \vdash \beta$ and $\beta \vdash \alpha$);
- $\alpha <_n \beta$ if $\beta \vdash n\alpha$.

Let W denote the set of all RC formulas.

Theorem.

1. Every $\alpha \in W$ is equivalent to a *word* (formula without $\wedge$);
2. $(W/{\sim}, <_0)$ is isomorphic to $(\varepsilon_0, <)$.

The ordinal $o(0^k) = k$. If $\alpha = \alpha_1 0 \alpha_2 0 \cdots 0 \alpha_n$, then

$$o(\alpha) = \omega^{o(\alpha_n^-)} + \cdots + \omega^{o(\alpha_1^-)},$$

where $(132)^- = 021$.

Ex. $o(1012) = \omega^{o(01)} + \omega^{o(0)} = \omega^{\omega^1 + \omega^0} + \omega = \omega^{\omega+1} + \omega$

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Reduction property

$$R_n^1(S) = R_n(S), \quad R_n^{k+1}(S) = R_n(S + R_n^k(S))$$

Suppose $U \subseteq \Pi_{n+2}$ and $S \vdash U$.
Th. $R_{n+1}(S) \equiv_n \{R_n^k(S) : k < \omega\}$ modulo U,
where $\equiv_n$ denotes conservativity for Π_{n+1}-formulas.

Example. Modulo elementary arithmetic EA:

$I\Sigma_1 \equiv R_2(\mathsf{EA}) \equiv_1 \{R_1^k : k < \omega\} \equiv \mathsf{PRA}$ (Parsons–Mints);

Key idea: Suppose $\alpha = (n+1)\beta$.
Define $\alpha[\![0]\!] := n\beta$, $\alpha[\![k+1]\!] := n(\beta \wedge \alpha[\![k]\!])$.
Then $\alpha[\![0]\!] <_0 \alpha[\![1]\!] <_0 \alpha[\![2]\!] \cdots \to \alpha$.

Reduction: $\alpha_S \equiv_n \{\alpha[\![k]\!]_S : k < \omega\}$.

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Consistency proof for PA

Th. Transfinite induction over $(W, <_0)$ proves $\mathrm{Con(PA)}$.

Work in $S = \mathrm{EA}$, $\Diamond$ means Con_S. We prove $\forall\alpha \, \Diamond\alpha_S$. Claim:

$$\mathrm{PRA} \vdash \forall\beta <_0 \alpha \, \Diamond\beta_S \to \Diamond\alpha_S.$$

Assume $\forall\beta <_0 \alpha \, \Diamond\beta_S$.
If $\alpha = 0\beta$, then $\Diamond\beta_S$, hence $\Diamond\Diamond\beta_S$ since $\mathrm{PRA} \vdash R_1(S)$.
If $\alpha = \langle n+1\rangle\beta$, then $\forall k \, \Diamond\alpha[\![k]\!]_S$, because $\alpha[\![k]\!] <_0 \alpha$.
By Reduction (provably in PRA):

$$\alpha_S \equiv_n \{\alpha[\![k]\!]_S : k < \omega\}.$$

Therefore $\forall k \, \Diamond\alpha[\![k]\!]_S$ yields $\Diamond\alpha_S$.

Iterated reflection and analysis of PA

W_n is the set of words in the alphabet $\{k \in \omega : k \geq n\}$.

Let $S^n_\alpha \equiv S + \{R_n(S^n_\beta) : \beta <_n \alpha\}$ over $(W_n, <_n)$.

Let S be a Π_{n+1} extension of PRA.
Theorem. For any $\alpha \in W_n$, $S + \alpha_S \equiv_n S^n_\alpha$.

Cor. $\mathsf{PA} \equiv_n \mathsf{PRA}^n_{\varepsilon_0}$ (U. Schmerl)

1. For $n = 0$: Consistency proof for PA (Gentzen);
2. For $n = 1$: Characterizing provably recursive functions of PA (Schwichtenberg–Wainer).

Tarskian truthpredicates

- $T(x)$ 'x is the Gödel number of a true arithmetical sentence'.
- Let $\mathcal{L}(T)$ be the extension of the language of PA by T.

Tarskian principles for truth:

- $\forall\varphi\,(At[\varphi] \to (T[\varphi] \leftrightarrow T_0[\varphi]))$;
- $\forall\varphi,\psi\,(T[\varphi\wedge\psi] \leftrightarrow (T[\varphi] \wedge T[\psi]))$;
- $\forall\varphi\,(T[\neg\varphi] \leftrightarrow \neg T[\varphi])$;
- $\forall\varphi\,(T[\forall x\,\varphi(x)] \leftrightarrow \forall x\,T[\varphi(\underline{x})])$.

Theories:
BT is just PRA plus Tarskian truth; PA(T) is BT + full induction.

NB: PA$(T) \equiv ACA$, second order arithmetic with arithmetical comprehension and full induction.

Higher reflection principles

Extending arithmetical hierarchy to $\mathcal{L}(T)$:

- Π_ω are arithmetical formulas;
- $\Pi_{\omega+n}$ are $\Pi_n(T)$-formulas, $n \geq 1$.

Higher reflection:

- $R_{\omega+1}(S) := \forall \varphi \in \Pi_\omega \, (\Box_S(\varphi) \to T(\varphi))$ (global arithmetical reflection)
- $R_\omega(S) := \{R_n(S) : n < \omega\}$ (uniform reflection)

Fact: $R_{\omega+1}(S)$ is equivalent to uniform reflection for $\Pi_{\omega+1}$-formulas: $\forall x \, (\Box_S \varphi(\underline{x}) \to \varphi(x))$, where $\varphi \in \Pi_{\omega+1}$.

Warning: $R_\omega(S)$ is an infinite schema, not a sentence!

Induction and reflection

Fact. BT is conservative over PRA (Kotlarski,Krajewski,Lachlan model-theoretically; Halbach syntactically).

Modulo BT:

- $R_\omega(\mathsf{BT}) \equiv \mathsf{PA}$;
- $R_{\omega+1}(\mathsf{BT}) \equiv I\Delta_0(T)$ (Kotlarski);
- full reflection $\equiv$ full induction.

Reduction formulas

Let $\equiv_\alpha$ denote conservativity for $\Pi_{1+\alpha}$-formulas. Let $S \vdash U$.

Th. $R_{\alpha+1}(S) \equiv_\alpha \{R^k_\alpha(S) : k < \omega\}$ modulo U, provided U is an extension of BT of the following complexity:

- $U \subseteq \Pi_{\alpha+2}$ if $\alpha < \omega$;
- $U \subseteq \Pi_\omega$ if $\alpha = \omega$;
- $U \subseteq \Pi_{\alpha+1}$ if $\alpha > \omega$.

This is:

- For $\alpha < \omega$, the standard reduction;
- For $\alpha > \omega$, a relativization of the standard reduction: $\Pi_k \mapsto \Pi_k(T)$;
- Key new case: $\alpha = \omega$. For $S = \mathsf{BT}$ this result is due to H. Kotlarski.

Reflection calculus RC_Ω

Fix some infinite ordinal Ω, for $\mathcal{L}(T)$ we choose $\Omega = \omega 2$.

Language: $\alpha ::= \top \mid (\alpha_1 \wedge \alpha_2) \mid x\alpha \qquad x < \Omega$

We interpret x as R_x and $\wedge$ as union of two schemata.
Warning: Formulas now denote schemata, not individual sentences!

Axioms are as before, with the following modifications:

- $x\alpha \wedge y\beta \vdash x(\alpha \wedge y\beta)$ for $x > y$, $y \notin \mathsf{Lim}$;
- $(x+1)x\alpha \sim (x+1)\alpha$, for $x \in \mathsf{Lim}$;
- $y(x\alpha \wedge x\beta) \vdash y(x\alpha \wedge \beta)$, for $y \leq x \in \mathsf{Lim}$.

RC_Ω *as an ordinal notation system*

Define: $\alpha <_x \beta$ iff $\beta \vdash x\alpha$.
Let W denote the set of all RC_Ω words.

Theorem.

1. $(W/\sim, <_0)$ is a well-ordering;
2. $W/\sim$ is closed under reduction:
   - If $\alpha = (x+1)\beta$ then $\alpha[\![k]\!]$ as before;
   - If $\alpha = x\beta$ with $x \in \mathsf{Lim}$, then $\alpha[\![k]\!] := x_k\beta$.

Iterated reflection

Progressions S^x_α are now defined over $(W_x, <_x)$.

Theorem. For any $\alpha \in W_x$, $S + \alpha_S \equiv_x S^x_\alpha$.

Cor.

- $I\Delta_0(T) \equiv_\omega \mathrm{BT}^\omega_\omega \equiv_n \mathrm{BT}^n_{\varepsilon_\omega}$, for $n < \omega$;
- $\mathrm{PA}(T) \equiv_\omega \mathrm{BT}^\omega_{\varepsilon_0} \equiv_n \mathrm{BT}^n_{\varepsilon_{\varepsilon_0}}$.

Words and ordinals

0	$\ldots$	1	$\ldots$	101	$\ldots$	11	$\ldots$	n	$\ldots$	ω
1	$\ldots$	ω	$\ldots$	$\omega 2$	$\ldots$	ω^2	$\ldots$	ω_n	$\ldots$	ε_0
0ω	$\ldots$	10ω	$\ldots$	$\omega 0 \omega$	$\ldots$	1ω	$\ldots$	$\omega\omega$	$\ldots$	$(\omega+1)$
$\varepsilon_0 + 1$	$\ldots$	$\varepsilon_0 + \omega$	$\ldots$	$\varepsilon_0 \cdot 2$	$\ldots$	$\varepsilon_0 \cdot \omega$	$\ldots$	ε_1	$\ldots$	ε_ω

$\omega(\omega+1)$	$\ldots$	$(\omega+1)(\omega+1)$	$\ldots$
$\varepsilon_{\omega+1}$	$\ldots$	$\varepsilon_{\omega \cdot 2}$	$\ldots$

Let Cr_x be the x-th critical class (enumerated by Veblen φ_x function). Let $o(\alpha)$ be the order type of $\{\beta \in W : \beta <_0 \alpha\}$.

Th. $o(W_{\omega^x}) = Cr_x \cup \{0\}$

Words and ordinals

$$
\begin{array}{ccccccccccc}
0 & \dots & 1 & \dots & 101 & \dots & 11 & \dots & n & \dots & \omega \\
1 & \dots & \omega & \dots & \omega 2 & \dots & \omega^2 & \dots & \omega_n & \dots & \varepsilon_0 \\
 & & & & & & & & & & \\
0\omega & \dots & 10\omega & \dots & \omega 0 \omega & \dots & 1\omega & \dots & \omega\omega & \dots & (\omega+1) \\
\varepsilon_0 + 1 & \dots & \varepsilon_0 + \omega & \dots & \varepsilon_0 \cdot 2 & \dots & \varepsilon_0 \cdot \omega & \dots & \varepsilon_1 & \dots & \varepsilon_\omega
\end{array}
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Iterated Tarskian truth

Theories RT_α are in the language with α-many Tarskian truthpredicates T_β, for $\beta < \alpha$. These theories are tightly related to *ramified analysis* systems (S. Feferman 64), as well as to iterated arithmetical comprehension (see the book by V. Halbach for accurate definitions).

Fact: $RT_{<\alpha} \equiv (\Pi^0_1\text{-CA})_{<\omega\alpha}$.

- We believe that everything done for a single truthpredicate works for the theories RT_α (with the system $\mathrm{RC}_{\omega\alpha}$).