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MULTIVALUED GROUPS, THEIR REPRESENTATIONS AND HOPF ALGEBRAS

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Abstract. This paper introduces the concept of n-valued groups and studies their algebraic and topological properties. We explore a number of examples. An important class consists of those that we call n-coset groups; they arise as orbit spaces of groups G modulo a group of automorphisms with n elements. However, there are many examples that do not arise from this construction. We see that the theory of n-valued groups is distinct from that of groups with a given automorphism group. There are natural concepts of the action of an n-valued group on a space and of a representation in an algebra of operators. We introduce the (purely algebraic) notion of an n-Hopf algebra and show that the ring of functions on an n-valued group and, in the topological case, the cohomology has an n-Hopf algebra structure. The cohomology algebra of the classifying space of a compact Lie group admits the structure of an n-Hopf algebra, where n is the order of the Weyl group; the homology with dual structure is also an n-Hopf algebra. In general the group ring of an n-valued group is not an n-Hopf algebra but it is for an n-coset group constructed from an abelian group. Using the properties of n-Hopf algebras we show that certain spaces do not admit the structure of an n-valued group and that certain commutative n-valued groups do not arise by applying the n-coset construction to any commutative group.

1. Introduction

Multivalued mappings arise naturally in many parts of mathematics. A very familiar example is that given by the roots of a polynomial of degree n. This mapping can be regarded either as a multivalued function of the coefficients of the polynomial or as a single valued function $\mathbf{C}^n \to (\mathbf{C})^n$ where $(\mathbf{C})^n$ denotes the symmetric product (and in this form it is a diffeomorphism). From a geometric viewpoint, coverings give rise to natural generalisations of this example and several other classes of multivalued mappings have been intensively studied in algebraic geometry (see, for example

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[H]). A particularly important class consists of those defined by algebraic correspondences and some of them arise naturally in the study of certain dynamical systems ([BP], [V]).

The aim of this paper is the study of the algebraic and topological properties of multivalued groups and we focus almost entirely on the case of n-valued groups. The general concepts associated with multivalued groups have been intensively studied from many viewpoints and in many contexts; a large bibliography is given in the survey article [Li]. The multivalued groups that we study here can be regarded as a special class of hypergroups but our emphasis on the n-valued case enables us to make a much more detailed analysis. An important class of examples are those we call n-coset groups; they arise as orbit spaces of groups G modulo a group A of automorphisms where A has n elements and give particular cases of Delsarte's explicit construction ([De]) in the theory of hypergroups. However, there are many examples that do not arise from this construction and we also give examples of 2-coset groups which arise as orbit spaces of two different groups. Thus we see that the theory of n-valued groups is distinct from that of groups with a given automorphism group. We introduce the concepts of actions and representations of n-valued groups; applications of these ideas to the study of some dynamical systems are given in [BV]. The definitions we introduce are entirely compatible with those that are familiar in the theory of groups and hypergroups [Li].

The topological aspects of multivalued mappings and branched coverings have been studied a great deal (see, for example [Ar], [Do]) and an important aspect is the study of the transfer map [BeG]. In another paper we make a fuller study of the notion of an n-ring homomorphism which can be regarded as a transfer and indeed is a generalisation of the classical trace. Some of the formulae that are satisfied by n-ring homomorphisms are multiplicative in nature and give rise to rather surprising identities (and which are therefore satisfied by the classical trace).

Given a group G and a contravariant functor F, such as a ring of functions on G or cohomology, it is natural to consider the Hopf algebra structure on $F(G)$. If G is an n-valued group, then $F(G)$ inherits an algebraic structure and it is natural to call it an n-Hopf algebra; we define this concept using purely algebraic axioms. Its main property is that the diagonal map is an n-ring homomorphism. The idea of an n-Hopf algebra gives a way of interpolating between a general co-algebra and a Hopf algebra (which is a 1-Hopf algebra). By analogy with the result of H. Hopf ([S] page 269) about the cohomology ring structure of topological groups, one can use information about n-Hopf algebras to show that certain spaces cannot admit an n-valued group structure. In particular, we show that $\mathbf{C}P^2$ does not admit a 2-valued group structure; this result has been extended by T. E. Panov [P].

2-valued formal groups were introduced in [BN] in order to describe the Pontryagin classes of tensor products of vector bundles and were a natural development of the formal group methods used in algebraic topology. The theory of n-valued formal groups, with emphasis on their applications in algebraic topology, has been developed in a series of papers, starting with [B1] and summarised in the survey paper [B2]; the algebraic theory was developed by A. Kholodov [Kh1] and [Kh2]. The global theory was introduced in [B3] and the motivation came from problems arising in constructing integrable systems based on addition theorems and the basic ideas were further developed in [BG]. The present paper introduces new concepts on which further developments of the theory can be based. A preliminary version of some of this work was prepared in 1994 and a summary [BR] has appeared. The paper has the following other sections:

Main definitions.
Examples and basic properties.
Multivalued group structures on euclidean spaces and spheres.
Actions and representations.
Hopf algebras.
Commutative, singly generated 2-coset groups.
Examples of group algebras.

1.1. Acknowledgments

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2. Main definitions

If X is a space, let $(X)^n$ denote its n-fold symmetric product, i.e., $(X)^n = X^n/\Sigma_n$ where the symmetric group Σ_n acts by permuting the coordinates. An n-valued multiplication on X is a map

$$\mu : X \times X \to (X)^n.$$

1. μ is *associative* if the following diagram commutes

$$\begin{tikzcd}
 & (X)^n \times X \arrow[r, "1 \times D"] & (X \times X)^n \arrow[dr, "(\mu)^n"] & \\
X^3 \arrow[ur, "\mu \times 1"] \arrow[dr, "1 \times \mu"'] & & & (X)^{n^2} \\
 & X \times (X)^n \arrow[r, "D \times 1"] & (X \times X)^n \arrow[ur, "(\mu)^n"'] &
\end{tikzcd}$$

where $(1 \times D)([x_1, x_2, \ldots, x_n], x) = [(x_1, x), (x_2, x), \ldots, (x_n, x)]$ and $D \times 1$ has a similar definition.

2. A *strong* left *unit* $e_L \in X$ satisfies the condition that

$$\mu(e_L, x) = [x, x, \ldots, x] =: D(x).$$

A *weak* left *unit* satisfies the condition that $x \in \mu(e_L, x)$,, i. e., there is a map $E_L : X \to (X)^{n-1}$ such that $\mu(e_L, x) = [E_L(x), x]$.

One also has the corresponding concept of a right unit and some obvious variants. We usually assume that there is a strong (two-sided) unit e. If there are both strong left and right units, it is easy to check that they must be equal.

3. When there is a unit, e, one can consider the existence of inverses. One requires a map inv : $X \to X$; it gives a *strong* left inverse if

$$\mu(\mathrm{inv}(x), x) = D(e)$$

and a *weak* left inverse if $e \in \mu(\mathrm{inv}(x), x)$. There are some interesting examples where inv has to be taken to be a multivalued map. Similarly one can consider right inverses and two-sided inverses.

For applications, the following seems the most useful.

Definition 2.1. An n-valued group structure on X is a mapping

$$\mu : X \times X \to (X)^n$$

which is associative together with a *strong* (two sided) unit $e \in X$ and a *weak* (two sided) inverse inv : $X \to X$.

Definition 2.2. A map $f : X \to Y$ is a homomorphism of n-valued groups if $f(e) = e$, $f(\mathrm{inv}(x)) = \mathrm{inv}(f(x))$ for all $x \in X$ and $\mu_Y(f(x), f(y)) = (f)^n \mu_X(x, y)$ for all $x, y \in X$, i.e., the following diagram commutes

$$\begin{tikzcd} X \times X \arrow[r, "\mu_X"] \arrow[d, "f \times f"'] & (X)^n \arrow[d, "(f)^n"] \\ Y \times Y \arrow[r, "\mu_Y"] & (Y)^n \end{tikzcd} .$$

Such a mapping could be called a *strong homomorphism.* There are weaker variants that may be useful in some circumstances.

3. Examples and basic properties

(1) For each $m \in \mathbf{N}$, an n-valued group can be regarded as an example of an mn-valued group by using the composition

$$G \times G \longrightarrow (G)^n \xrightarrow{(D)^m} (G)^{mn}.$$

(2) If $f : G_1 \to G_2$ is a (strong) homomorphism between two n-valued groups, then $\mathrm{Ker}(f) = \{x \in G_1 \mid f(x) = e\}$ is an n-valued group.

(3) *n-coset groups.*

Let G be any group and A a group of automorphisms of G with $\#A = n$. Then one can define an n-valued group structure on $X = G/A$ as follows.

Let $\pi : G \to G/A$ be the quotient map and define $\mu : X \times X \to (X)^n$ by $\mu(x, y) = \pi(\mu_o(\pi^{-1}(x), \pi^{-1}(y)))$ where μ_o denotes the multiplication on G.

An n-valued group of this type will be called an *n-coset group.*

An important special case is when $\#A = 2$, say $A = \{1, \alpha\}$; then the elements of X can be written as $\{g, g^\alpha\}$ and μ as

$$\{g, g^\alpha\} * \{h, h^\alpha\} = [\{gh, g^\alpha h^\alpha\}, \{gh^\alpha, g^\alpha h\}].$$

Another important example is the quotient of $\mathbf{Z}$ by its automorphism group; it arises naturally in the study [BV] of certain dynamical systems. The quotient can be identified with $\mathbf{W} = \{k \mid k \geq 0, k \in \mathbf{Z}\}$ and, with this identification, the product of k and ℓ is $[k + \ell, |k - \ell|]$.

Proposition 3.1. *Let $X = G/A$ be a 2-coset group; then $x * y = [z, z]$ if and only if either x or y is the image of a point fixed under A.*

Proof. Let $x = \{g, g^\alpha\}$ and $y = \{h, h^\alpha\}$. Then, if $x * y = [z, z]$, one has either $gh = gh^\alpha$ or $gh = g^\alpha h$. So either g or h is fixed under α. The other implication also follows from the formula given above for the product $*$. □

When G is a commutative group, we can consider the involution defined by the map $g \to g^{-1}$. In the special case of a cyclic group, it is easy to construct the multiplication table for G/A.

Proposition 3.2. *There is an example of a noncommutative group with an automorphism of order two whose associated 2-coset group is isomorphic to that obtained from the integers with its nontrivial automorphism.*

Proof. Let G be the infinite dihedral group, $G = \langle a, b \mid a^2 = b^2 = e \rangle$ with the automorphism α interchanging a and b. The elements of the 2-valued group G/A are $u_{2n} = \{(ab)^n, (ba)^n\}$ and $u_{2n+1} = \{b(ab)^n, a(ba)^n\}$ for $n \geq 0$. The multiplication is given by $u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell|}]$ and so this 2-valued group is isomorphic with the example $\mathbf{W}$ considered above. □

Another important special case is where G is finite and $A = G$ acts by inner automorphisms. In this case X can be identified with the set of conjugacy classes in G.

(4) Let $H \subset G$ be a subgroup with $\#H = n$. Then the right coset space G/H admits an n-valued multiplication using the formula $Hx.Hy := \{Hxhy \mid h \in H\}$. In this case, H is a strong left unit and a weak right unit. A two sided weak inverse is given by the multivalued map $\mathrm{inv}(Hx) := \{Hx^{-1}y \mid y \in H\}$. This often does not give an n-valued group.

(5) However, $X = H\backslash G/H$ admits an n-valued group structure by the formula $HxH \cdot HyH := \{HxhyH \mid h \in H\}$. In this case, H is a strong two sided unit and $Hx^{-1}H$ is a weak two sided inverse for HxH.

(6) If G is a group and s is an indeterminate, let $\widetilde{G} = G \cup \{s\}$. Then $\widetilde{G}$ has a 2-valued group structure by

$$\begin{aligned}
\mu(s,s) &= e, \\
\mu(s,e) &= \mu(e,s) = s, \\
\mu(s,g) &= \mu(g,s) = g \text{ for all } g \in G \setminus \{e\}, \\
\mu(g_1,g_2) &= \begin{cases} g_1g_2 & \text{if } g_1g_2 \neq e, \\ \{e,s\} & \text{if } g_1g_2 = e \text{ and } g_1 \neq e. \end{cases}
\end{aligned}$$

A straightforward application of Proposition 3.1 shows that this example does not, in general, arise as a 2-coset group; indeed if $G = \mathbf{Z}/m$ and $m > 2$ then this is not a 2-coset group although for $m = 2$, one obtains the 2-coset group arising from $\mathbf{Z}/4$ modulo its nontrivial automorphism.

(7) The construction of the previous example introduces a square root s of the element e and, more generally, one can construct the multivalued extension of two groups G_1 and G_2 giving an exact sequence

$$0 \longrightarrow G_2 \longrightarrow \widetilde{G} \longrightarrow G_1 \longrightarrow 0$$

where the multiplication $*$ in $\widetilde{G}$ is defined as

$$\begin{aligned}
u * v &= u \text{ for } u \in G_1, v \in G_2, \\
v_1 * v_2 &= v_1v_2 \text{ for } v_1, v_2 \in G_2, \\
u_1 * u_2 &= \begin{cases} \{G_2\} & \text{when } u_1u_2 = e \in G_1, \\ u_1u_2 & \text{otherwise.} \end{cases}
\end{aligned}$$

(8) Let G denote the cyclic group of order m generated by the element x,. We define a 2-valued 'deformation' of G. Its elements can be identified with those of G and the multiplication is given by (with $0 \leq r, s < m$)

$$\mu(x^r, x^s) = \begin{cases} x^{r+s} & \text{for } r+s < m, \\ \{x^{r+s-m}, x^{r+s+1-m}\} & \text{for } m \leq r+s. \end{cases}$$

When $m = 2$, it is easily checked that this example is isomorphic to the quotient of the cyclic group with three elements by its nontrivial automorphism. On the set with two elements there is only one other 2-valued group structure, namely, the cyclic group. For $m > 2$, one can use Proposition 3.1 to show that this example does not arise as a 2-coset group.

The following is a classification result for a class of finite 2-valued groups.

Proposition 3.3. *Let $X = \{x_0, x_1, \dots, x_{m-1}\}$ be a 2-valued group on a set with m elements with multiplication $*$ such that $x_r * x_s = x_{r+s}$ for $r+s < m$. Then either $x_r * x_s = x_{r+s-m}$ or $[x_{r+s-m}, x_{r+s-m+1}]$ for $r+s \geq m$.*

Proof. It is clear that x_0 is the strong unit and that $\mathrm{inv}(x_1) = x_{m-1}$ since it cannot be one of the other elements. Let $x_1 * x_{m-1} = [x_0, x_\ell]$. If $\ell = 0$ then X is the cyclic group and if $\ell = 1$ we get the deformed cyclic group of the previous example. Now suppose that $\ell > 1$. Then, by associativity, the following are equal: $x_{m-\ell} * (x_1 * x_{m-1}) = x_{m-\ell} * [x_0, x_\ell] = [x_{m-\ell}, x_{m-\ell}, x_0, x_\ell]$ and $(x_{m-\ell} * x_1) * x_{m-1} = [x_{m-\ell+1}, x_{m-\ell+1}] * x_{m-1} = [x_p, x_p, x_q, x_q]$ for some p, q and this is a contradiction. □

4. Multivalued group structures on euclidean spaces and spheres

The spaces $(\mathbf{C})^n = \mathbf{C}^n/\Sigma_n$ and $\mathbf{C}^n$ are identified using the map

$$\mathcal{S} \colon \mathbf{C}^n \to \mathbf{C}^n$$

whose components are given by $(z_1, z_2, \dots, z_n) \to e_r(z_1, z_2, \dots, z_n)$, $1 \leq r \leq n$, where e_r denotes the rth elementary symmetric polynomial. It is often convenient to write the map $\mathcal{S}$ as the polynomial $z^n - e_1 z^{n-1} + e_2 z^{n-2} - \dots + (-1)^n e_n$ whose roots are $[z_1, z_2, \dots, z_n]$. The projectivisation of the map $\mathcal{S}$ induces a diffeomorphism between $(\mathbf{C}P^1)^n$ and $\mathbf{C}P^n$.

4.1. The additive n-valued group structure on C

Consider $(\mathbf{C}, +)$ with its automorphism $z \to \omega z$ of order n, where ω is a primitive nth root of unity. We obtain an n-valued group structure on $\mathbf{C}/A$ where A denotes a cyclic group of order n. But $\mathbf{C}/A$ is diffeomorphic to $\mathbf{C}$ by the mapping $\mathbf{C}/A \to \mathbf{C}$ defined by $z \to z^n$. The multiplication is then given, for $x, y \in \mathbf{C}$ by

$$x * y = [(x^{\frac{1}{n}} + \omega^r y^{\frac{1}{n}})^n : 1 \leq r \leq n].$$

The unit is 0 and the inverse of z is $(-1)^n z$. Using the map $\mathcal{S}$, one obtains the following polynomials for low values of n (note the interesting numerical coefficients) :

$n = 2$:

$$z^2 - 2(x+y)z + (x-y)^2 = (z+y+x)^2 - 4(xy+yz+zx),$$

$n = 3$:

$$z^3 - 3(x+y)z^2 + 3(x^2 - 7xy + y^2)z - (x+y)^3 = (z-x-y)^3 - 27xyz,$$

$n = 4$:

$$z^4 - 4(x+y)z^3 + 2(3x^2 - 62xy + 3y^2)z^2 - 4(x^3 + 31(x+y)xy + y^3)z + (x-y)^4$$

$$= ((z+y+x)^2 - 4(xy+yz+zx))^2 - 2^7(z+y+x)xyz,$$

$n = 5$:

$$z^5 - 5(x+y)z^4 + 5(2x^2 - 121xy + 2y^2)z^3 - 5(2x^3 + 381(x+y)xy + 2y^3)z^2$$
$$+5(x^4 - 121x^3y + 381x^2y^2 - 121xy^3 + y^4)z - (x-y)^5$$
$$= (z-x-y)^5 + 5^4(5(xy - xz - yz) - (z-x-y)^2)xyz.$$

4.2. The additive n!-valued group structure on $\mathbf{C}^n$

Using the map $\mathcal{S}$, $\mathbf{C}^n$ inherits an $n!$-valued group structure from the usual addition on $\mathbf{C}^n$.

In the case $n = 2$, the multiplication map can be written as the composition

$$\mathbf{C}^2 \times \mathbf{C}^2 \xrightarrow{\tilde{\mu}} \mathbf{C} \times (\mathbf{C})^2 \xrightarrow{D \times 1} (\mathbf{C} \times \mathbf{C})^2.$$

Using the identification induced by the map $\mathcal{S}$, the multiplication $\tilde{\mu}$ can be written as

$$(x_1, x_2) * (y_1, y_2) = (x_1 + x_2, 2(y_1 + y_2) + x_1x_2, (y_1 - y_2)^2 + (x_1 + x_2)(x_1y_2 + y_1x_2)).$$

4.3. The multiplicative 2-valued group structure on C

Consider $\mathbf{C}^* := (\mathbf{C} \setminus \{0\}, \cdot)$ with its automorphism $z \to z^{-1}$ of order 2. The space $\mathbf{C}^*/A$ is identified with $\mathbf{C}$ using the map $\mathbf{C}^* \to \mathbf{C}$ given by $\pi(z) = \frac{1}{2}(z + z^{-1})$ and $\pi^{-1}(x) = x \pm (x^2-1)^{\frac{1}{2}}$. Then

$$x * y = [xy + ((x^2-1)(y^2-1))^{\frac{1}{2}}, xy - ((x^2-1)(y^2-1))^{\frac{1}{2}}].$$

This formula can be rewritten as the quadratic $z^2 - 2xyz + x^2 + y^2 - 1$. The unit is 1 and the inverse of each element is itself. Under the change of variables $x, y, z \to x+1, y+1, z+1$ the polynomial becomes $z^2 - 2(x+y+xy)z + (x-y)^2$.

Remark 4.1. The multiplication arising in Example 4.1 occurs as the n-valued formal group for cohomology and that in Example 4.3 arises similarly in K-theory [BN].

Remark 4.2. It is proved in [B1], [B2] that, up to local change of coordinates, there are only two different 2-valued formal group laws on $\mathbf{C}$, the usual additive group and the one described in Example 4.1 above.

Example 4.3 has a generalisation giving a multiplicative $n!$-group structure on $\mathbf{C}^{n-1}$. It is given by the above coset construction where the symmetric group Σ_n acts on the commutative group

$$M = \{z_1, z_2, \dots, z_n : z_1z_2 \dots z_n = 1\}$$

(with pointwise multiplication) by permuting the coordinates. The group M is isomorphic to $\mathbf{C}^* \times \dots \times \mathbf{C}^*$ (with $n-1$ copies) and the quotient by the symmetric group is diffeomorphic to $\mathbf{C}^{n-1}$.

4.4. 2-valued group structure arising from $\mathbf{C}^m$

The space $\mathbf{C}^m/\pm$ can be identified with the space Sym_m of all $m \times m$ symmetric matrices of rank 1 by the map $u \in \mathbf{C}^m \to X(u) = uu^T$. A calculation shows that the product of $X, Y \in \mathrm{Sym}_m$ is given by the roots of the quadratic

$$Z^{\otimes 2} - (\Theta_1(X,Y) \otimes Z + Z \otimes \Theta_1(X,Y)) + \Theta_2(X,Y)$$

where $\Theta_1(X,Y) = X+Y$, $\Theta_2(X,Y) = (X+Y)^{\otimes 2} - X_{12}^{\otimes 2}$ and X_{12} is $X(u+v)-X-Y$ where u, v are defined (up to sign) by $X = X(u)$, $Y = X(v)$. This is a direct generalisation of Example 4.1. Explicitly, if $\Theta_2(X,Y) = (\theta_{ijk\ell})$ then $\theta_{ijk\ell} = x_{ij}x_{k\ell} - x_{kj}y_{i\ell} - x_{i\ell}y_{kj} + y_{ij}y_{kl}$.

The properties of the symmetric products $(\mathbf{C}^m)^n$ and multidimensional analogues of the algebraic equations are considered in detail in [GKZ].

4.5. 2-valued group structure on $\mathbf{C}P^1$

For a lattice $\Lambda \subset \mathbf{C}$, the corresponding Weierstrass $\wp$ - function defines a holomorphic mapping $\mathbf{C}/\Lambda \to \mathbf{C}P^1$ and identifies $\mathbf{C}P^1$ with the 2-coset group derived from the automorphism $x \to -x$ of $\mathbf{C}/\Lambda$. From the addition formula for $\wp$ (see, for example [WW]) one obtains that the 2-coset group structure on $\mathbf{C}P^1$ is given (in the standard affine chart) by

$$x * y = \left[-x-y+\frac{1}{4}\left(\frac{x_1-y_1}{x-y}\right)^2, -x-y+\frac{1}{4}\left(\frac{x_1+y_1}{x-y}\right)^2\right]$$

where $x_1^2 = 4x^3 - g_2x - g_3$ and y_1, y are similarly related. Hence the 2-valued structure is determined as the roots of the equation

$$(z+x+y)(4xyz-g_3) = \left(xy+yz+zx-\frac{g_2}{4}\right)^2.$$

The referee pointed out that this form of the addition law for the $\wp$ function is given in [HC] page 171.

We can also consider the Jacobi elliptic function $\mathrm{sn}(u)$ related (in the standard notation) to $\wp$ via $\wp(u) = e_1 + 1/\mathrm{sn}^2(u)$; its square defines an identification of $\mathbf{C}P^1$ with the above 2-coset group and yields slightly more general formulae. The addition formula is

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}(u)(p(\mathrm{sn}^2(v))^{\frac{1}{2}} + \mathrm{sn}(v)(p(\mathrm{sn}^2(u))^{\frac{1}{2}}}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)}$$

where $p(t) = 1 - 2\delta t + k^2t^2$. Letting $x = \mathrm{sn}^2(u)$ and $y = \mathrm{sn}^2(v)$ gives that the 2-valued multiplication on $\mathbf{C}P^1$ is

$$x * y = \left[\left(\frac{(xp(y))^{\frac{1}{2}} + (yp(x))^{\frac{1}{2}}}{1-k^2xy}\right)^2, \left(\frac{(xp(y))^{\frac{1}{2}} - (yp(x))^{\frac{1}{2}}}{1-k^2xy}\right)^2\right].$$

Identifying the symmetric square of $\mathbf{C}$ with the space of quadratic polynomials, the right hand side of this formula becomes

$$z^2 - 2\frac{xp(y)+yp(x)}{(1-k^2xy)^2}z + \frac{(x-y)^2}{(1-k^2xy)^2}$$

because $xp(y) - yp(x) = (x-y)(1-k^2xy)$. Also, we have

$$xp(y) + yp(x) = (x+y)(1+k^2xy) - 4\delta xy$$

and hence $(1-k^2xy)^2z^2 - 2[(x+y)(1+k^2xy) - 4\delta xy]z + x^2 - 2xy + y^2 = x^2 + y^2 + z^2 - 2k^2xyz(x+y+z) - 2(xy+yz+zx) + k^4x^2y^2z^2 + 8\delta xyz$.

In the local coordinates $(x:1), (y:1), (z:1)$ one has

$$(x+y+z-k^2xyz)^2 = 4(xy+yz+zx-2\delta xyz).$$

When $k = \delta = 0$ we have the additive 2-valued group structure on $\mathbf{C}$ and when $k = 0, \delta = -1/4$ we have the multiplicative structure. In the general case, we can use the change of coordinates

$$(x,y,z) \to (Ax, Ay, Az)$$

to obtain a family of 2-valued group structures on $\mathbf{C}$ which are all equivalent topologically.

Using the identification given by the map S we obtain

Proposition 4.1. *The map $\mu : \mathbf{C}P^1 \times \mathbf{C}P^1 \to \mathbf{C}P^2$ defined by*

$$\mu((x_0:x_1),(y_0:y_1)) =$$

$$((x_0y_0 - k^2x_1y_1)^2 : -2((x_0y_1+x_1y_0)(x_0y_0+k^2x_1y_1) - 4\delta x_0y_0x_1y_1) : (x_1y_0 - x_0y_1)^2)$$

defines a 2-valued group structure on $\mathbf{C}P^1$ if and only if $k(k^2-\delta^2) \neq 0$, i.e., the elliptic curve for $\operatorname{sn}(u)$ is nondegenerate.

Proof. The map μ is not well defined at the point $(\mathbf{x},\mathbf{y}) = ((x_0:x_1),(y_0:y_1))$ if and only if the values of each of the three coordinates of the product is zero. The vanishing of the first and last coordinates is equivalent to the condition that either $x_0 = x_1 = 0$ or $y_0^2 = k^2y_1^2$ and $\mathbf{x} = \mathbf{y}$ in $\mathbf{C}P^1$. Substituting the second of these into the second coordinate yields the equation $2y_0y_1(y_0^2 + k^2y_1^2) - 4\delta y_0^2y_1^2 = 0$ and we are assuming that we also have $y_0^2 = k^2y_1^2$ without $y_0 = y_1 = 0$. So either $k = 0$ or $y_0^2 + k^2y_1^2 - 2\delta y_0y_1 = 0$ and so $k^2 = \delta^2$. The result follows. $\square$

By using the results of [ST], [C] one can find multivalued group structures on $\mathbf{C}P^n$ and by [BS] also on weighted projective spaces.

4.6. A noncommutative 2-valued group on $\mathbf{R}^3$

The Heisenberg groups admit an automorphism of order 2 and so give rise to examples of 2-valued groups. The simplest example of a Heisenberg group is on $\mathbf{R}^3$ with multiplication

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + yx')$$

and the automorphism changes the sign of the first two coordinates. One obtains a 2-valued group structure on the quotient space, which is easily seen to be homeomorphic to $\mathbf{R}^3$.

Similar constructions apply to Heisenberg groups in higher dimensions (associated to arbitrary bilinear forms) as defined, for example in [Au].

4.7. A 2-valued group on $\mathbf{S}^3$

Let G be a group and $g \in G$ an element whose square is central. Then the quotient of G by the inner automorphism induced by conjugation by g gives an example of a 2-valued group. As a special case, let $G = S^3$, the unit quaternions and let g be a purely imaginary unit quaternion then $g^2 = -1$ and so is central. The quotient under conjugation by g is homeomorphic to S^3 as one sees as follows.

First, by choosing the complex structure appropriately, one can assume (with no loss of generality) that $g = i$. Secondly, every point of S^3 can be written in the form

$$z_1 \cos\theta + jz_2 \sin\theta \quad \text{with} \quad z_1, z_2 \in S^1;$$

this describes S^3 as the join of two circles, the first being $S^1 \subset \mathbf{C}$ and the second being jS^1, i.e., $S^3 = S^1 * jS^1$. The action of conjugation by i preserves the join structure; it is trivial on the first circle and multiplication by -1 on the second. The quotient is therefore homeomorphic to S^3 regarded as the join $S^1 * \bar{S}^1$ where $\bar{S}^1$ denotes the circle $jS^1/-1$. So one obtains a 2-valued group structure on S^3.

5. Actions and representations

Definition 5.1. If X is an n-valued group and Y a set, an action of X on Y is a mapping $\phi : X \times Y \to (Y)^n$ such that the following diagram commutes.

$$\begin{tikzcd}
X\times(X\times Y) \arrow[r, "1\times\phi"] \arrow[d, "="'] & X\times(Y)^n \arrow[r, "D\times 1"] & (X\times Y)^n \arrow[dr, "(\phi)^n"] & \\
 & & & (Y)^{n^2}. \\
(X\times X)\times Y \arrow[r, "\mu\times 1"] & (X)^n\times Y \arrow[r, "1\times D"] & (X\times Y)^n \arrow[ur, "(\phi)^n"'] &
\end{tikzcd}$$

For example, X acts on itself.

Another example is: Let $H \subset G$ be a subgroup of order n then the n-valued group defined on the double coset space $X = H\backslash G/H$ acts on the coset space $Y = H\backslash G$ by

$$HxH \cdot Hy := \{Hxhy \mid h \in H\}.$$

Definition 5.2. A representation of an n-valued group X in an algebra A is a map $\rho : X \to A$ such that $\rho(e) = 1$ and $\mathrm{Av}\rho(\mu(x,y)) = \rho(x)\rho(y)$ where $\mathrm{Av}\rho(\mu(x,y))$ denotes the average value of ρ on the set $\mu(x,y)$, i.e.,

$$\text{if } \mu(x,y) = [z_1, z_2, \ldots, z_n], \quad \text{then } \mathrm{Av}\rho(\mu(x,y)) = \frac{1}{n}\sum_{i=1}^{n} \rho(z_i).$$

An important special case is when A is the algebra of endomorphisms of a vector space V. In this case, one has a *linear representation* of X on the vector space V.

5.1. Examples

(1) When X is a (single-valued) group these definitions agree with the usual definitions of a group action and of a representation.

(2) If X is an n-valued group that acts on the set Y and V denotes the vector space spanned by Y, one obtains a representation of X on V as follows:

If $\phi(x,y) = [y_1, y_2, \ldots, y_n]$, let $\rho(x)$ denote the linear transformation defined by

$$y \to \frac{1}{n}(y_1 + y_2 + \ldots + y_n).$$

It is trivial to verify that ρ defines a representation in the above sense.

(3) As a special case consider the regular representation of the 2-valued deformation of $\mathbf{Z}/m$ (Example 7 above). The generator is mapped to the $m \times m$ matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} \\ 1 & 0 & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

(4) *Group algebras.*

For an n-valued group X, the *group algebra* $C(X)$ is the $\mathbf{C}$ vector space spanned by X with multiplication induced by

$$xy = (z_1 + z_2 + \ldots + z_n)/n$$

where the product of x and y in X is $[z_1, z_2, \dots, z_n]$.

One has two linear representations of X in $C(X)$ by left and right multiplication. For example, left multiplication is defined by

$$xy = Av(x * y)$$

on the elements of X and extended by linearity. The obvious map $X \to C(X)$ is a representation. Any representation $X \to A$ has a unique extension to an algebra homomorphism $C(X) \to A$.

(5) Let $C(S^1)$ denote the algebra of trigonometric polynomials on the circle. The subalgebra A of even polynomials has basis $\{\cos(k\theta) \mid k \geq 0\}$. The map $\rho : \mathbf{W} \to A$ defined by $\rho(k) = \cos(k\theta)$ is a representation of $\mathbf{W}$ because the identity

$$\cos(k\theta)\cos(\ell\theta) = (\cos(k+\ell)\theta + \cos(k-\ell)\theta)/2$$

becomes $\rho(k)\rho(\ell) = \mathrm{Av}\rho(k * \ell)$. Indeed, the linear map $C(\mathbf{W}) \to A$ induced by ρ is an isomorphism.

This representation can also be thought of as a map $\rho' : \mathbf{W} \to \mathbf{C}[t]$ where $\rho'(k) = p_k(t)$, the kth Chebyshev polynomial and so this representation is called the *Chebyshev* representation of $\mathbf{W}$. Note that the elements of $\mathbf{C}[t]$ can be regarded as functions on the 2-coset group corresponding to the involution $e^{i\theta} \to e^{-i\theta}$ on S^1.

This last example can be regarded as a special case of the following result which the referee kindly suggested that we should include.

Proposition 5.1. *Let X be the n-coset group G/A. Then the group algebra $C(X)$ is isomorphic to $C(G)^A$, the subalgebra of A-invariants in the group algebra $C(G)$.*

Proof. The map $C(X) \to C(G)$ is defined as the linear extension of the map

$$x = \{g^\alpha \mid \alpha \in A\} \to \frac{1}{n} \sum_{\alpha \in A} g^\alpha.$$

The image of this map clearly lies in $C(G)^A$ and it is easily checked that it is multiplicative. □

6. Hopf algebras

When X has an n-valued group structure and F is a contravariant functor from a suitable category of spaces to a category of algebras then $F(X)$ can be given a Hopf algebra like structure. We investigate this general situation by first considering the important example where $F(X)$ is the ring $\mathbf{C}[X]$ of all complex valued functions on X.

Definition 6.1. The map $\mathrm{Av} : \mathbf{C}[X] \to \mathbf{C}[(X)^m]$ is defined by

$$\mathrm{Av}(f)[x_1, x_2, \ldots, x_m] = \frac{1}{m} \sum_{i=1}^{m} f(x_i).$$

Analogously, the maps $\sigma_r : \mathbf{C}[X] \to \mathbf{C}[(X)^m]$ are defined by means of $e_r[f(x_1), f(x_2), \ldots, f(x_m)]$, the elementary symmetric functions ([Mac]).

Definition 6.2. If $\mu(x,y) = [z_1, z_2, \ldots, z_n]$ and $f \in \mathbf{C}[X]$, let $\Delta f \in \mathbf{C}[X \times X] \cong \mathbf{C}[X] \otimes \mathbf{C}[X]$ be defined by

$$(\Delta f)(x,y) = \mathrm{Av}(f)(\mu(x,y)) = \frac{1}{n} \sum_{i=1}^{n} f(z_i).$$

Remark 6.1. In general, one takes $\mathbf{C}[X \times X]$ to be $\mathbf{C}[X] \hat{\otimes} \mathbf{C}[X]$.

Lemma 6.1. *Δ gives a coassociative coalgebra structure on $\mathbf{C}[X]$ and so, by duality, an associative algebra structure on $\mathbf{C}[X]^* = \mathrm{Hom}(\mathbf{C}[X], \mathbf{C})$, the dual space of $\mathbf{C}[X]$.*

Proof. Consider the map $\mathrm{Av} : \mathbf{C}[X] \to \mathbf{C}[(X)^{n^2}]$ and compose with the diagram for the associativity of an n-valued multiplication. $\square$

In the case of a single-valued group, this gives a Hopf algebra structure on $\mathbf{C}[X]$ and indeed, a dual pair $(\mathbf{C}[X], \mathbf{C}[X]^*)$ of Hopf algebras.

Definition 6.3. An *n-Hopf algebra* structure on a commutative algebra A over the field $\mathbf{C}$ with multiplication $m : A \otimes A \to A$ and unit $\eta : \mathbf{C} \to A$, a map of algebras, consists also of

- a *counit* $\epsilon : A \to \mathbf{C}$, which is a map of algebras,
- an *antipode* $s : A \to A$, which is a map of algebras,
- a *diagonal* $\Delta : A \to A \otimes A$, a linear map making A into a coassociative coalgebra and
- a map $P : A \to (A \otimes A)[t]$ which assigns to each $a \in A$ a monic *polynomial* of degree n

$$P_a(t) = t^n - \beta_1 t^{n-1} + \ldots + (-1)^n \beta_n$$

with $\beta_r = \beta_r(a) \in A \otimes A$.

These are related in the following way. For each $a \in A$, introduce the series

$$\alpha_a(t) := \sum_{q \geq 0} \frac{a^q}{t^{q+1}} \in A[[t^{-1}]].$$

Axiom 1. The polynomial $P_a(t)$ is such that

$$\Delta\alpha_a(t) = \sum_{q\geq 0} \frac{\Delta(a^q)}{t^{q+1}} = \frac{1}{n}\frac{d}{dt}\ln(P_a(t)).$$

Axiom 2 (*unit*). If $\eta(1)$ is denoted by 1, then $\Delta(1) = 1 \otimes 1$.

Axiom 3 (*counit*). Let $i_1, i_2 : A \otimes A \to A$ be defined by $i_1(a \otimes b) = \epsilon(b)a$ and $i_2(a \otimes b) = \epsilon(a)b$; then the composites $i_1\Delta, i_2\Delta : A \to A \otimes A \to A$ are both equal to the identity.

Axiom 4 (*antipode*). The map $s : A \to A$ satisfies $m(1 \otimes s)P_a(\eta\epsilon(a)) = 0$ and $m(s \otimes 1)P_a(\eta\epsilon(a)) = 0$.

Remark 6.2. An *n-bialgebra* will satisfy all these axioms except that the existence of an antipode will not be assumed.

Remark 6.3. In a future paper we hope to explore the properties of n-Hopf algebras over more general rings.

Lemma 6.2. *A* 1*-Hopf algebra is a Hopf algebra* (*in the usual sense*).

Proof. In this case $P_a(t) = t - \beta$ with $\beta \in A \otimes A$, and by Axiom 1,

$$\Delta\alpha_a(t) = \frac{1}{t-\beta},$$

so $\Delta(a^n) = \beta^n$ for each n and therefore $\Delta(a^n) = \beta^n = (\Delta a)^n$.

For $a_1, a_2 \in A$, consider $a_1a_2 = \frac{1}{2}[(a_1 + a_2)^2 - a_1^2 - a_2^2]$; thus

$$\begin{aligned}\Delta(a_1a_2) &= \frac{1}{2}[\Delta((a_1+a_2)^2) - \Delta(a_1^2) - \Delta(a_2^2)] \\ &= \frac{1}{2}[(\Delta((a_1) + \Delta(a_2))^2) - \Delta(a_1)^2 - \Delta(a_2)^2] = \Delta(a_1)\Delta(a_2).\end{aligned}$$

Hence, and using Axiom 2, one sees that Δ is a ring homomorphism.

The counit axiom for a Hopf algebra is exactly Axiom 3 above.

By Axiom 4, $m(1 \otimes s)\Delta(a) = \eta\epsilon(a) = m(s \otimes 1)\Delta(a)$, and so s gives an antipode in the Hopf algebra sense. □

Definition 6.4. A *morphism of n-Hopf algebras* is a morphism of the underlying algebras and coalgebras which also commutes with the antipodes.

Proposition 6.3. *If X is an n-valued group, then* $\mathbf{C}[X]$ *is an n-Hopf algebra.*

Proof. By Lemma 6.1, $\mathbf{C}[X]$ with comultiplication Δ is a coassociative coalgebra. We now check that Axioms 1 and 2 hold for pointwise multiplication. Using the n-valued multiplication μ, we introduce for each function $f \in \mathbf{C}[X]$ functions $\beta_r(f) \in \mathbf{C}[X \times X]$ by $\beta_r(f)(x,y) = \sigma_r\ f(\mu(x,y))$ where σ_r is the r-th elementary symmetric function. Then let $P_f(t) = t^n - \beta_1 t^{n-1} + \ldots + (-1)^n \beta_n$, the polynomial whose set of roots is $f(\mu(x,y))$. Because $(f^q)(x) = (f(x))^q$, it is straightforward to check that

$$\Delta\alpha_f(t) = \sum_{q \geq 0} \frac{\Delta(f^q)}{t^{q+1}} = \frac{1}{n}\frac{d}{dt}\ln(P_f(t)).$$

When $A = \mathbf{C}[X]$, one can identify $A \otimes A$ with $\mathbf{C}[X \times X]$ and $i_1 : A \otimes A \to A$ is given as follows: For $f : X \times X \to \mathbf{C}$, the map $i_1 f : X \to \mathbf{C}$ is $i_1 f(x) = f(e,x)$. Similarly $i_2 f(x) = f(x,e)$. It is now easy to check that, for $\phi \in A$, $i_1 \Delta\phi(x) = (\phi(x) + \ldots + \phi(x))/n = \phi(x)$. This verifies Axiom 3.

The polynomial $m(1 \otimes s)P_f(t)(x)$ for $f \in A$ is of degree n and its roots are $f(z_1), \ldots, f(z_n)$ where $x * \mathrm{inv}(x) = [z_1, \ldots, z_n]$. But, since $e \in [z_1, \ldots, z_n]$, one has that $m(1 \otimes s)P_f(f(e))(x) = 0$ for all $x \in X$. Similarly, $m(s \otimes 1)P_f(f(e)) = 0$. This verifies Axiom 4. □

In a future paper, we intend to explore the converse of Proposition 6.3 and we will prove it, at least in the case where X is finite.

Proposition 6.4. *If X is a topological n-valued group, then $H^{2*}(X;\mathbf{C})$ the even dimensional part of the cohomology algebra of X is an n-Hopf algebra.*

Proof. The diagonal, $\Delta : H^{2*}(X;\mathbf{C}) \to H^{2*}(X;\mathbf{C}) \otimes H^{2*}(X;\mathbf{C})$, is defined as follows. Let $\pi_1 : X^n \to X$ denote the map induced by projection onto the first factor, $\tau : H^*(X^n;\mathbf{C}) \to H^*((X)^n;\mathbf{C})$ the transfer homomorphism, $\mu : X \times X \to (X)^n$ the map defining the multiplication and $\kappa : H^{2*}(X \times X;\mathbf{C}) \to H^{2*}(X;\mathbf{C}) \otimes H^{2*}(X;\mathbf{C})$ be a splitting induced by the Künneth isomorphism. Then $\Delta = \kappa\mu\tau\pi_1^*$. Using the properties of transfer maps (see e.g. [SE]), the proof follows that of Proposition 6.3. □

As an application we show that $\mathbf{C}P^2$ does not admit a topological 2-valued group structure.

Consider the cohomology algebra H of $\mathbf{C}P^2$ with $\mathbf{C}$ coefficients, so

$$H \cong \mathbf{C}[x]/x^3 = 0 \text{ and } \textstyle\sum_{q \geq 0} \frac{x^q}{t^{q+1}} = \frac{1}{t} + \frac{x}{t^2} + \frac{x^2}{t^3} = \frac{(t^2+tx+x^2)}{t^3}.$$

We regard $H \otimes H$ as the quotient of the polynomial algebra on two generators x, y. If $\mathbf{C}P^2$ admits a 2-valued group structure, there is a diagonal map $\Delta : H \to H \otimes H$ with $\Delta(x) = x + y$ and $\Delta(x^2) = x^2 + \lambda xy + y^2$ where $\lambda \in \mathbf{C}$ is to be determined. Since H is a 2-Hopf algebra, there is a polynomial $P_x(t) \in H \otimes H[t]$ say $P_x(t) = t^2 - \beta_1 t + \beta_2$ such that

$$t^3 \frac{d}{dt}\ln P_x(t) = 2(t^2 + t\Delta(x) + \Delta(x^2))$$

so $t^3(2t-\beta_1) = 2(t^2+t\Delta(x)+\Delta(x^2))(t^2-\beta_1 t+\beta_2)$, that is $t^4 - t^3\beta_1/2 = t^4+t^3(\Delta(x)-\beta_1)+t^2(\Delta(x^2)-\beta_1\Delta(x)+\beta_2)+t(\beta_2\Delta(x)-\beta_1\Delta(x^2))+\beta_2\Delta(x^2)$. Comparing coefficients gives $\beta_1 = 2\Delta(x) = 2(x+y), \beta_2 = \beta_1\Delta(x)-\Delta(x^2) = x^2+(4-\lambda)xy+y^2, \beta_2\Delta(x)-\beta_1\Delta(x^2) = 0$, which gives that $2(1+\lambda)(x^2y+xy^2) = (5-\lambda)(x^2y+xy^2)$ so $\lambda = 1$ and finally, $\beta_2\Delta(x^2) = 0$ which gives that $2+(4-\lambda)\lambda = 0$. This yields the required contradiction when compared with the equation $\lambda = 1$ obtained above.

Proposition 6.5. *Let H be a commutative Hopf algebra over* $\mathbf{C}$ *with diagonal Δ and G a finite group of automorphisms of H. Let A be the subalgebra of elements invariant under G. Then the linear map $\Delta^G : H \to H \otimes H$ defined by $\Delta^G(x) = \frac{1}{n}\sum_g (g\otimes 1)\Delta(x)$ gives a diagonal $\Delta^G : A \to A\otimes A$ which makes A into an n-Hopf algebra.*

For $x \in H$, let $\pi x = \frac{1}{n}\sum_g gx$. The following is easy to check.

Lemma 6.6. *If $\beta \in H\otimes H$, then $\beta \in A\otimes A$ if and only if $(\pi\otimes\pi)\beta = \beta$.*

Lemma 6.7. *For $a \in A$, $\Delta^G(a) = (\pi\otimes\pi)\Delta(a)$.*

Proof. Since $g\in G$ is an automorphism of H one has that if $\Delta(a) = \sum_i a_i'\otimes a_i''$, then $\Delta(ga) = \sum_i ga_i'\otimes ga_i''$ so $\Delta(\pi a) = \frac{1}{n}\sum_{g,i} ga_i'\otimes ga_i''$. Hence

$$\begin{aligned}
\Delta^G(a) &= \frac{1}{n}\sum_g (g\otimes 1)\Delta(a)\\
&= \frac{1}{n}\sum_g (g\otimes 1)\frac{1}{n}\sum_h (h\otimes h)\Delta(a)\\
&= \frac{1}{n^2}\sum_{g,h}(gh\otimes h)\Delta(a)\\
&= \frac{1}{n^2}\sum_{g,h}(g\otimes h)\Delta(a)\\
&= (\pi\otimes\pi)\Delta(a). \quad\square
\end{aligned}$$

Proof of Proposition 6.5. For $a \in A$,

$$\begin{aligned}
\Delta^G(\alpha_a(t)) &= \Delta^G\Big(\sum_{q\geq 0}\frac{a^q}{t^{q+1}}\Big)\\
&= \frac{1}{n}\sum_{q\geq 0}\sum_g (g\otimes 1)\frac{\Delta(a^q)}{t^{q+1}}\\
&= \frac{1}{n}\sum_g (g\otimes 1)\frac{1}{(t-\Delta a)}\\
&= \frac{1}{n}\sum_g \frac{1}{(t-(g\otimes 1)\Delta(a))}\\
&= \frac{1}{n}\sum_g \frac{d}{dt}\ln(t-(g\otimes 1)\Delta(a)\\
&= \frac{1}{n}\frac{d}{dt}\ln\prod_g (t-(g\otimes 1)\Delta(a)).
\end{aligned}$$

Let $P_a(t) = \prod_g (t - (g \otimes 1)\Delta(a)) = t^n - \beta_1 t^{n-1} + \ldots + (-1)^n \beta_n$. Then we need to check that $\beta_i \in A \otimes A$. However, this is easy since each β_i is a symmetric function of the elements $\{(g \otimes 1)\Delta(a) \mid g \in G\}$. The other axioms will be easily verified by the interested reader. $\square$

Definition 6.5. An n-coset group is called *very commutative* if it has the form G/A where G is commutative.

We also remark that if G is a group of automorphisms of the Hopf algebra H, it is also a group of automorphisms of the dual H^*. Consequently, if the diagonal of H is cocommutative then H^* is also an n-Hopf algebra. Hence, we obtain

Proposition 6.8. *Let X be a very commutative n-coset group; then $\mathbf{C}[X]^*$ is an n-Hopf algebra.*

This proposition can be used to show that certain commutative n-valued groups are not very commutative n-coset groups.

Important examples of commutative, cocommutative n-Hopf algebras are given by the cohomology of classifying spaces of connected Lie groups. By Proposition 6.5, $H^*(BG)$ is an n-Hopf algebra where n denotes the cardinality of the Weyl group $W(G)$ of the compact, connected Lie group G. This follows since $H^*(BG)$ can be identified with the $W(G)$-invariant subalgebra of the polynomial Hopf algebra $H^*(BT)$ where T denotes a maximal torus of the group G. Using a proof similar to that of Proposition 6.8, one sees the following

Proposition 6.9. *If G is a Lie group, then $H_*(BG)$ is an n-Hopf algebra.*

A geometric realisation of the diagonal of the n-Hopf algebra $H^*(BG)$ can be obtained by applying the Becker–Gottlieb transfer [BeG] to the fibration $BT \to BG$ with fibre G/T.

There are other examples of polynomial algebras that admit n-Hopf algebra structures; many can be constructed as a consequence of the Shephard–Todd–Chevalley theorem on the invariants of reflection groups acting on polynomial algebras ([ST], [C]).

7. Commutative, singly-generated 2-coset groups

If $A \subset X$ is a subset of an n-valued group, then the subgroup generated by A consists of those elements obtained under any number of successive multiplications of elements of A and their inverses. We will pay particular attention to the case where G is a group, α is an automorphism of G of order 2 and $X = G/\alpha$ is the corresponding 2-coset group. In this case, the subgroup generated by $A \subset X$ is the image of the subgroup generated by the inverse image of A in G. We note that there are noncommutative, one-generator subgroups of 2-coset groups. For example, let G be a free group on two generators and α the automorphism that interchanges the two generators. Then $X = G/\alpha$ is noncommutative but generated by the image of one of the generators of G. If the generators are denoted by a and b and the product in X is $*$, then

$$\{ab, ba\} * \{a^2b^2, b^2a^2\} = [\{aba^2b^2, bab^2a^2\}, \{ba^3b^2, ab^3a^2\}]$$

but

$$\{a^2b^2, b^2a^2\} * \{ab, ba\} = [\{a^2b^2ab, b^2a^2ba\}, \{b^2a^3b, a^2b^3a\}]$$

and these are not equal.

Proposition 7.1. *Let G be a group with involution α and for $g \in G$, let $h = g^\alpha$. Then the element $\{g, h\} \in X$ generates a commutative subgroup of X if and only if either $gh = hg$ or $g^2 = h^2$.*

Proof. We can assume that $gh \neq hg$. The product of $\{g, h\}$ with $\{g^2, h^2\}$ is $[\{g^3, h^3\}, \{gh^2, hg^2\}]$ and in the opposite order it is $[\{g^3, h^3\}, \{h^2g, g^2h\}]$. If these are equal then we have either $gh^2 = h^2g$ or $gh^2 = g^2h$; the latter implies that $h = g$.

Similarly, the product of $\{g, h\}$ with $\{g^3, h^3\}$ is $[\{g^4, h^4\}, \{gh^3, hg^3\}]$ and in the opposite order it is $[\{g^4, h^4\}, \{g^3h, h^3g\}]$. If these are equal, then we have either $gh^3 = g^3h$ and so $g^2 = h^2$ or $gh^3 = hg^3$. In the latter case, if we also have that $gh^2 = h^2g$, then one has that $gh = hg$. Since the relation $g^2 = h^2$ implies $gh^2 = h^2g$, the result is proved in one direction.

Conversely, consider the group

$$K =< a, b \mid a^2 = b^2 >$$

(which is the fundamental group of the Klein bottle) and its automorphism α which interchanges a and b. Let $\mathbf{X}$ denote the corresponding 2-coset group. To complete the proof of the proposition, it is enough to show that $\mathbf{X}$ is commutative. There is an epimorphism of K onto D_∞, the infinite dihedral group; its kernel is the cyclic group generated by $c = a^2 = b^2$ and forms the centre of K. The automorphism α acts trivially on the centre, so one has an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{X} \to \mathbf{W} \to 0.$$

Every element of $\mathbf{X}$ can be written as $x_{n,m}$ where

$$x_{n,m} = \begin{cases} \{(ab)^{n/2}c^m, (ba)^{n/2}c^m\} & \text{for } n \text{ even,} \\ \{b(ab)^{n-1/2}c^m, a(ba)^{n-1/2}c^m\} & \text{for } n \text{ odd.} \end{cases}$$

With this notation, a tedious, direct calculation shows that the multiplication in $\mathbf{X}$ is given by

$$x_{k,\ell} * x_{n,m} = [x_{k+n,\ell+m}, x_{|k-n|,\ell+m+\min(k,n)}]$$

and so is commutative. $\square$

Rather surprisingly, the 2-coset group $\mathbf{X}$ is isomorphic to the 2-coset group $(\mathbf{Z} \times \mathbf{Z})/\tau$ where τ is the involution that interchanges the factors and so is very commutative. Consider the map

$$q : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}_+ \times \mathbf{Z}$$

defined by

$$q(m,n) = (|m-n|, \min(m,n))$$

which induces a bijection between $(\mathbf{Z} \times \mathbf{Z})/\tau$ and $\mathbf{Z}_+ \times \mathbf{Z}$. It is easily checked that when the 2-coset multiplication is rewritten via q, it is given by exactly the same formulae as that on the subscripts described above for $\mathbf{X}$.

The 2-coset group $\mathbf{X}$ has the universal property that it maps onto any commutative singly generated 2-coset group and so plays a role similar to that of the integers in the theory of groups. However, note that in the closely related situation of a pair consisting of a group and an automorphism of order two, there is no universal object since $(\mathbf{Z} \times \mathbf{Z}, \tau)$ and (K, α) do not admit equivariant maps from one onto the other.

8. Examples of group algebras

If X is finite, then the group algebra $C(X)$ can be identified with the dual $\mathbf{C}[X]^*$ of the algebra $\mathbf{C}[X]$ of functions on X. As we have seen, when X is an n-valued group, $\mathbf{C}[X]$ is an n-Hopf algebra and if X is a very commutative n-coset group, the dual $C(X)$ is, by Proposition 6.8, also an n-Hopf algebra.

Definition 8.1. An element $x \in C(X)$ is called a *geometric element* if $\Delta(x) = x \otimes x$.

Elements of $C[X]$ arising as evaluation at points of X are geometric. In a general Hopf algebra such elements are called group like (see [K]).

It will be convenient, in the cases where the algebra $C(X)$ is a quotient of a polynomial algebra $\mathbf{C}[x]$, to identify $C(X \times X) \cong C(X) \otimes C(X)$ with a quotient of $\mathbf{C}[x,y]$ where, by abuse of notation, we denote $x \otimes 1$ by x and $1 \otimes x$ by y.

First we study all the (four) examples of 2-valued group structures on a set with three elements.

(1) If X is the cyclic group of order 3, then the algebra structure on $C(X)$ is given by $\mathbf{C}[x]/(x^3 = 1)$. We have $\Delta(x^k) = x^k y^k$.

(2) When X is the deformed cyclic group of order 3 (Example 3.8), then $C(X) \cong \mathbf{C}[x]/(2x^3 = x+1)$ and $\Delta(x^k) = x^k y^k$ for $k = 0, 1, 2$.

This algebra can be regarded as being deformed in the category of hyperalgebras from the previous example.

(3) Consider $\mathbf{Z}/4/\alpha$ whose group algebra is naturally

$$\mathbf{C}[x_1, x_2]/(2x_1^2 = x_2 + 1, x_1 x_2 = x_1, x_2^2 = 1).$$

Writing $x_1=x$ and $x_2=2x^2-1$, one sees that it is isomorphic to $\mathbf{C}[x]/(x^3-x)$. One has $\Delta(x_k)=x_ky_k$ for $k=1,2$ so $\Delta(x)=xy$ and $\Delta(x^2)=\Delta(1+x_2)/2=(1+x_2y_2)/2=(1+(2x^2-1)(2y^2-1))/2=1-x^2-y^2+2x^2y^2$.

This example can also be regarded as a deformation of Example 1.

(4) The example $\mathbf{Z}/5/\alpha$ has group algebra

$$\mathbf{C}[x_1,x_2]/(2x_1^2=x_2+1, 2x_1x_2=x_1+x_2, 2x_2^2=x_1+1)$$

and $\Delta(x_k)=x_ky_k$ for $k=1,2$. Writing $x_1=x$ and $x_2=2x^2-1$, the group algebra becomes $\mathbf{C}[x]/(4x^3=2x^2+3x-1)$.

(5) The deformed cyclic group of order m is the generalisation of Example 2 above. Its group algebra is $\mathbf{C}[x]/(2x^m-x-1)$ and $\Delta(x^k)=x^ky^k$ for $0\le k<m-1$. Write the series

$$\sum_{q\ge 0}\frac{x^q}{t^{q+1}}$$

using the relation $x^m=(x+1)/2$ as $\sum_{k=0}^{m-1}a_k(t)x^k$. So

$$\begin{aligned} 1 &= (t-x)\sum_{k=0}^{m-1}a_k(t)x^k \\ &= (ta_0(t)-a_m(t)/2)+(ta_1(t)-a_m(t)/2)x+\sum_{k=2}^{m-1}(ta_k(t)-a_{k-1}(t))x^k. \end{aligned}$$

Hence, $ta_0(t)-a_{m-1}(t)/2=1, ta_k(t)-a_{k-1}(t)=0$ for $2<k\le m-1$, and $ta_1(t)-a_0(t)=a_{m-1}(t)/2$.

Solving these equations gives

$$a_0(t)=\frac{2t^{m-1}-1}{2t^m-t-1},$$

$$a_k(t)=\frac{2t^{m-k-1}}{2t^m-t-1}\quad\text{for}\quad 1\le k\le m-1.$$

Hence

$$\sum_{q\ge 0}\frac{x^q}{t^{q+1}}=\frac{2}{2t^m-t-1}\left(\frac{t^m-x^m}{t-x}-\frac{1}{2}\right).$$

We therefore obtain

$$\Delta\left(\sum_{q\ge 0}\frac{x^q}{t^{q+1}}\right)=\frac{2}{2t^m-t-1}\left(\frac{t^m-x^my^m}{t-xy}-\frac{1}{2}\right).$$

This shows that $C(X)$ is not a 2-Hopf algebra for $m>2$ and so X is not a very commutative 2-coset group.

(6) The extension example $\mathbf{Z}/m \cup \{s\}$ has group algebra $\mathbf{C}[x]/(x^{m+1}-x)$ and $\Delta(x^k) = x^k y^k$ for $0 \le k < m$, $\Delta(x^m) = 1 - x^m - y^m + 2x^m y^m$. A calculation gives that

$$\sum_{q\ge 0} \frac{x^q}{t^{q+1}} = \frac{1}{t}\left(1 + \frac{\sum_{k=1}^{m} t^{m-k} x^k}{t^m - 1}\right)$$

and hence

$$\Delta\left(\sum_{q\ge 0} \frac{x^q}{t^{q+1}}\right) = \frac{1}{t}\left(1 + \frac{\sum_{k=1}^{m-1} t^{m-k} x^k y^k + (1 - x^m - y^m + 2x^m y^m}{t^m - 1}\right).$$

This shows that $C(X)$ is not a 2-Hopf algebra for $m > 2$ and so X is not a very commutative 2-coset group.

(7) Consider the universal commutative one generator 2-coset group $\mathbf{X}$; its group algebra $C(\mathbf{X})$ is a polynomial algebra with one generator over the algebra k of Laurent polynomials in the central element c of the group K and which is fixed under the automorphism α. So,

$$C(\mathbf{X}) = k[x], k = \mathbf{C}[c, c^{-1}] \text{ and } x = (a+b)/2.$$

The geometric elements in $C(\mathbf{X})$ are $\{p_n c^m | m \in \mathbf{Z}, n \ge 0\}$ where

$$p_n = \begin{cases} ((ab)^s + (ba)^s)/2 & \text{if } n = 2s, \\ (b(ab)^s + a(ba)^s)/2 & \text{if } n = 2s+1. \end{cases}$$

The diagonal $\Delta : k[x] \to k \otimes k[x,y]$ induces an algebra homomorphism $k \to k \otimes k$ and $\Delta(p_n) = p_n q_n$ with $p_0 = 1, p_1 = x$ and $2xp_n = p_{n+1} + cp_{n-1}$ for $n \ge 1$. To see this one needs to use the facts that $c = a^2 = b^2$ and that $a(ab)^s = cb(ab)^{s-1}$ to show that $(a+b)((ab)^s + (ba)^s) = b(ab)^s + a(ba)^s + cb(ab)^{s-1} + ca(ba)^{s-1}$ and $(a+b)(b(ab)^s + a(ba)^s) = (ab)^{s+1} + (ba)^{s+1} + c(ab)^s + c(ba)^s$. Setting

$$\mathcal{P}(x,u) = \sum_{n\ge 0} p_n(x) u^n$$

one calculates easily that $(1 - 2xu + cu^2)\mathcal{P}(x,u) = 1 - xu$. For $c = 1$ one therefore has that the geometric elements are the Chebyshev polynomials.

Writing $\Delta(c) = c_1 c_2$ and $\Delta(x) = xy$, and using $p_2 = 2x^2 - c$ from above, one gets that $\Delta(x^2) = (p_2 q_2 + c_1 c_2)/2 = 2x^2y^2 - c_1 y^2 - c_2 x^2 + c_1 c_2$. Hence the quadratic polynomial in the definition of a 2-Hopf algebra becomes for this case, $P_x(t) = t^2 - 2xyt + c_2 x^2 + c_1 y^2 - c_1 c_2$.

When $c = 1$ and so $c_1 = c_2 = 1$, this polynomial becomes $t^2 - 2xyt + x^2 + y^2 - 1$ which is the polynomial that defines the multiplicative 2-valued group structure on $\mathbf{C}$ that we considered earlier. This relation is a consequence of the duality theory for n-valued groups that we intend to study in another paper.

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