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Hyperelliptic Kleinian Functions and Applications

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Abstract. In this paper the theory of Kleinian hyperelliptic functions is developed. New collections of basis functions for the field of meromorphic functions on hyperelliptic Jacobians are introduced and the algebraic relations between them are explicitly described. As applications, the following results are obtained: the matrix realization of hyperelliptic Jacobians and their Kummer varieties; the construction of a family of 2×2 matrix operators satisfying the conditions of zero curvature and of a generalized shift that takes this family to a parametric family satisfying the same condition; the construction of systems of linear differential equations whose common spectral manifold is a hyperelliptic curve; the construction of solutions to the KdV system and to the Sine-Gordon equation in Kleinian functions.

§0. Introduction

In this paper we develop the Kleinian construction of hyperelliptic Abelian functions, which is a natural generalization of the Weierstrass approach in elliptic functions theory to the case of a hyperelliptic curve of genus $g>1$. Kleinian ζ and $\wp$-functions are defined as

$$\zeta_i(\boldsymbol{u})=\frac{\partial}{\partial u_i}\ln\sigma(\boldsymbol{u}),\quad \wp_{ij}(\boldsymbol{u})=-\frac{\partial^2}{\partial u_i\partial u_j}\ln\sigma(\boldsymbol{u}),\qquad i,j=1,\dots,g,$$

where the vector $\boldsymbol{u}$ belongs to Jacobian $\operatorname{Jac}(V)$ of the hyperelliptic curve

$$V=\Big\{(y,x)\in\mathbb{C}^2 : y^2-\sum_{i=0}^{2g+2}\lambda_i x^i=0\Big\}$$

and $\sigma(\boldsymbol{u})$ is the Kleinian σ-function.

The systematic study of σ-functions, which originated in the paper **[1]** by F. Klein, was an alternative to the developments of Weierstrass (**[2, 3]**) (the hyperelliptic generalization of the Jacobi elliptic functions sn, cn, dn) and the purely θ-functional theory of Göppel (**[4]**) and Rosenhain (**[5]**) for genus 2, generalized further by Riemann. The σ approach was furthered by Burkhardt (**[6]**), Wiltheiss

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([**7**]), Bolza ([**8**]), Baker ([**9**]) and others; a detailed bibliography may be found in [**10**]. We would like to mention Baker's monographs ([**11, 12**]), which merit special attention.

The article is organized as follows. In §1 we present the facts from the theory of hyperelliptic curves and θ-functions needed in this paper. In §2 the Kleinian hyperelliptic functions are defined. Following Baker, the realization of σ-functions via θ-functions is described. The fundamental differential 2-form of the second kind is expressed explicitly, and the Jacobi inversion problem is solved in terms of hyperelliptic functions. Theorems 2.1 and 2.2 in this section are classical (see [**11**]). In §3, using the classical Theorem 3.1, we derive the main relations between the $\wp$-functions, analyze them, and construct new collections of basis functions of the field of meromorphic functions on hyperelliptic Jacobians. Algebraic relations between them are described explicitly and, as the result, meromorphic embeddings of the Jacobians and their Kummer varieties as intersections of cubics and quartics respectively are obtained. In §4 we demonstrate the potential of Kleinian functions in solving contemporary problems. The matrix realization of hyperelliptic Jacobians and their Kummer varieties is presented. A family of 2×2 matrix operators (not depending on a parameter) satisfying the zero curvature condition is described and a generalized shift and gauge transformation taking this family to a parametric one still satisfying the zero curvature condition is constructed. Systems of linear second order differential equations whose joint spectral manifold is a hyperelliptic curve are obtained. It is shown that the natural identification of the independent variables of Korteweg–de Vries (KdV) systems and of the Sine-Gordon equation with the canonical coordinates on the Jacobians yields their algebro-geometric solutions in Kleinian functions. In §5 we discuss some contemporary aspects of the further development of the theory of Kleinian hyperelliptic functions.

The paper is based on recent results partially announced in [**13–16**]. The presented results have already been applied to describe Abelian Bloch solutions of the 2-dimensional Schrödinger equation ([**17**]).

§1. Preliminaries

We recall some basic definitions from the theory of hyperelliptic curves and θ-functions; see, e.g., [**11, 12, 18–21**] for a detailed exposition.

1.1. **Hyperelliptic curves.** The set of points $V(y,x)$ satisfying the equation

$$y^2 = \sum_{i=0}^{2g+2} \lambda_i x^i = \lambda_{2g+2} \prod_{k=1}^{2g+2} (x - e_k) = f(x) \tag{1.1}$$

is a model of a plane *hyperelliptic curve* of genus g, realized as a 2-sheeted covering over the Riemann sphere with the *branching points* $e_1, \dots, e_{2g+2}$. Any pair (y,x) in $V(y,x)$ is called an *analytic point*; an analytic point that is not a branching point is called a *regular point.* The *hyperelliptic involution* ϕ (the swap of the sheets of the covering) acts as $(y,x) \mapsto (-y,x)$, leaving the branching points fixed.

To make y a single-valued function of x, it suffices to draw $g+1$ cuts, connecting pairs of branching points e_i—$e_{i'}$ for some partition of $\{1,\dots,2g+2\}$ into a set of $g+1$ disjoint pairs i, i'. Those of the branching points at which the cuts start will be denoted by a_i, the end points of the cuts by b_i, respectively; an exception is one of the cuts, whose starting point is denoted by a and the end point by b. In the case $\lambda_{2g+2} \mapsto 0$ this point a goes to ∞. The equation of the curve in the case $\lambda_{2g+2} = 0$ and $\lambda_{2g+1} = 4$ can be rewritten as

$$y^2 = 4P(x)\,Q(x), \tag{1.2}$$

$$P(x) = \prod_{i=1}^{g}(x - a_i), \qquad Q(x) = (x - b)\prod_{i=1}^{g}(x - b_i)\,.$$

The local parametrization of the point (y, x) in the vicinity of a point (w, z),

$$x = z + \begin{cases} \xi & \text{near regular point } (\pm w, z), \\ \xi^2 & \text{near branching point } (0, e_i), \\ \xi^{-1} & \text{near regular point } (\pm\infty, \infty), \\ \xi^{-2} & \text{near branching point } (\infty, \infty), \end{cases}$$

provides the structure of the *hyperelliptic Riemann surface*, a one-dimensional compact complex manifold. We shall use the same notation for the plane curve and the Riemann surface, $V(y, x)$ or V. All curves and Riemann surfaces throughout the paper are assumed to be hyperelliptic if the converse is not stated.

A *marking* on $V(y, x)$ is given by the base point x_0 and the canonical basis of cycles $(A_1, \ldots, A_g; B_1, \ldots, B_g)$, i.e., the basis in the one-dimensional homology group $H_1(V(y, x), \mathbb{Z})$ of the surface $V(y, x)$ with the symplectic intersection matrix $I = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}$, where 1_g is the unit $g \times g$-matrix.

1.2. **Differentials.** Traditionally, three kinds of differential 1-forms are distinguished on a Riemann surface.

1.2.1. *Holomorphic differentials* or differentials of the *first kind*, are the differential 1-forms $\mathrm{d}u$ that can be locally expressed as $\mathrm{d}u = (\sum_{i=0}^{\infty} \alpha_i \xi^i)\, d\xi$ in the vicinity of any point (y, x) with some constants $\alpha_i \in \mathbb{C}$. It can be checked directly that forms satisfying this condition can all be expressed as

$$\sum_{i=0}^{g-1} \beta_i x^i \frac{\mathrm{d}x}{y}\,.$$

The forms $\{\mathrm{d}u_i\}_{i=1}^{g}$ given by

$$\mathrm{d}u_i = \frac{x^{i-1}\mathrm{d}x}{y}, \qquad i = 1, \ldots, g,$$

constitute the set of *canonical holomorphic differentials* in $H^1(V, \mathbb{C})$. The $g \times g$-matrices of their A- and B-periods,

$$2\omega = \left(\oint_{A_k} \mathrm{d}u_l\right), \qquad 2\omega' = \left(\oint_{B_k} \mathrm{d}u_l\right),$$

are nondegenerate. Under the map $(2\omega)^{-1}$, the vector $\mathbf{du} = (\mathrm{d}u_1, \ldots, \mathrm{d}u_g)^T$ is taken to the vector of normalized holomorphic differentials $\mathrm{d}\mathbf{v} = (\mathrm{d}v_1, \ldots, \mathrm{d}v_g)^T$, i.e., to the vector in $H^1(V,\mathbb{C})$ satisfying the conditions

$$\oint_{A_k} \mathrm{d}v_k = \delta_{kl}, \qquad k,l = 1,\dots,g\,.$$

It is known that the $g\times g$ matrix,

$$\tau = \left(\oint_{B_k} \mathrm{d}v_l\right) = \omega^{-1}\omega',$$

belongs to the *upper Siegel halfspace* $\mathcal{S}_g$ of degree g, i.e., is symmetric and has a positive definite imaginary part.

Let us denote by $\mathrm{Jac}(V)$ the *Jacobian* of the curve V, i.e., the factor $\mathbb{C}^g/\Gamma$, where $\Gamma = 2\omega\oplus 2\omega'$ is the lattice generated by the periods of canonical holomorphic differentials.

Divisors $\mathcal{D}$ on Riemann surfaces are given by formal sums of analytic points $\mathcal{D} = \sum_i^n m_i(y_i,x_i)$, and the *degree* of $\mathcal{D}$ is $\deg\mathcal{D} = \sum_i^n m_i$. A divisor is *effective* if $m_i > 0$ for all i.

Let $\mathcal{D}$ be a divisor of degree 0, $\mathcal{D} = \mathcal{X} - \mathcal{Z}$, where $\mathcal{X}$ and $\mathcal{Z}$ are the effective divisors, $\deg\mathcal{X} = \deg\mathcal{Z} = n$, presented by

$$\mathcal{X} = \{(y_1,x_1),\dots,(y_n,x_n)\} \quad \text{and} \quad \mathcal{Z} = \{(w_1,z_1),\dots,(w_n,z_n)\} \in (V)^n;$$

here $(V)^n$ is the nth symmetric power of V.

The *Abel map* has the form

$$\mathfrak{A}(\mathcal{X}-\mathcal{Z})\colon (V)^n \to \mathrm{Jac}(V), \qquad \mathfrak{A}(\mathcal{X}-\mathcal{Z}) = (u_1,\dots,u_g),$$

where

$$u_i = \sum_{k=1}^{n} \int_{(w_k,z_k)}^{(y_k,x_k)} \mathrm{d}u_i, \qquad i = 1,\dots,g,$$

and $\mathcal{Z} = \{(w_1,z_1),\dots,(w_n,z_n)\} \in (V)^n$ is a fixed divisor, the path of integration being the same for all $i = 1,\dots,g$.[1]

Abel's theorem says that the points of the divisors $\mathcal{Z}$ and $\mathcal{X}$ are the poles and zeros, respectively, of a meromorphic function on $V(y,x)$ if and only if $\mathfrak{A}(\mathcal{X}-\mathcal{Z}) = 0 \mod \Gamma$. In the case when $n = g$, the Abel map is onto and one-to-one, except for the so-called *special divisors.* The *Jacobi inversion problem* can be stated as the problem of inversion for such a map $\mathfrak{A}$. Note that a special divisor of degree g is a set of points such that for at least one pair j and $k = 1,\dots,g$ the point (y_j,x_j) is the image of the point (y_k,x_k) under the hyperelliptic involution.

1.2.2. *Meromorphic differentials* or differentials of the *second kind* are the differential 1-forms $\mathrm{d}r$ that can be locally expressed as $\mathrm{d}r = (\sum_{i=-k}^{\infty}\alpha_i\xi^i)\,d\xi$ in the vicinity of any point (y,x) with some constants α_i, and $\alpha_{(-1)} = 0$. It can also be verified directly that forms satisfying this condition can all be written as

$$\sum_{i=0}^{g-1} \beta_i x^{i+g}\frac{\mathrm{d}x}{y}$$

(mod holomorphic differentials).

[1] To shorten the notation, from this point forward we shall denote the integration limits only by the coordinate x of the point (y,x) of the curve.

Let us introduce the following *canonical Abelian differentials of the second kind*

$$\mathrm{d}r_j = \sum_{k=j}^{2g+1-j} (k+1-j)\,\lambda_{k+1+j}\,\frac{x^k\mathrm{d}x}{4y}, \qquad j=1,\dots,g. \tag{1.3}$$

We denote their matrices of A- and B-periods by

$$2\eta = \left(-\oint_{A_k}\mathrm{d}r_l\right), \qquad 2\eta' = \left(-\oint_{B_k}\mathrm{d}r_l\right).$$

The *Riemann bilinear identity* for the period matrices of the differentials of the first and second kind implies the following statement.

LEMMA 1.1. *The* $(2g\times 2g)$*-matrix* $\mathcal{G} = \begin{pmatrix}\omega & \omega'\\ \eta & \eta'\end{pmatrix}$ *belongs to* PSp_{2g}:

$$\mathcal{G}\begin{pmatrix}0 & -1_g\\ 1_g & 0\end{pmatrix}\mathcal{G}^T = -\frac{\pi i}{2}\begin{pmatrix}0 & -1_g\\ 1_g & 0\end{pmatrix}.$$

1.2.3. *Differentials of the third kind* are the differential 1-forms $\mathrm{d}\Omega$ that have only poles of order 1 and have total residue 0, and so are locally given in the vicinity of any of the poles as $\mathrm{d}\Omega = (\sum_{i=-1}^{\infty}\alpha_i\xi^i)\,d\xi$ with some nonzero constants α_i, α_{-1}. Such forms (mod holomorphic differential) may be represented as

$$\sum_{i=0}^{n}\beta_i\left(\frac{y+y_i^+}{x-x_i^+} - \frac{y+y_i^-}{x-x_i^-}\right)\frac{\mathrm{d}x}{y},$$

where $(y_i^{\pm}, x_i^{\pm})$ are the analytic points of the poles of positive (respectively, negative) residue.

Let us introduce the *canonical differential of the third kind*

$$\mathrm{d}\Omega(x_1,x_2) = \left(\frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2}\right)\frac{\mathrm{d}x}{2y}; \tag{1.4}$$

for this differential we have

$$\int_{x_3}^{x_4}\mathrm{d}\Omega(x_1,x_2) = \int_{x_1}^{x_2}\mathrm{d}\Omega(x_3,x_4).$$

1.2.4. *The fundamental* 2*-differential of the second kind.* For an arbitrary pair $\{(y_1,x_1),(y_2,x_2)\}\in(V)^2$, we introduce the function $F(x_1,x_2)$ defined by the conditions

$$\begin{aligned} &\text{(i)} \quad F(x_1,x_2) = F(x_2,x_1),\\ &\text{(ii)} \quad F(x_1,x_1) = 2f(x_1),\\ &\text{(iii)} \quad \left.\frac{\partial F(x_1,x_2)}{\partial x_2}\right|_{x_2=x_1} = \frac{\mathrm{d}f(x_1)}{\mathrm{d}x_1}. \end{aligned} \tag{1.5}$$

This function $F(x_1,x_2)$ can be represented in the following equivalent forms

$$F(x_1,x_2) = 2y_2^2 + 2(x_1-x_2)y_2\frac{\mathrm{d}y_2}{\mathrm{d}x_2} + (x_1-x_2)^2\sum_{j=1}^{g} x_1^{j-1}\sum_{k=j}^{2g+1-j}(k-j+1)\lambda_{k+j+1}x_2^k, \tag{1.6}$$

$$F(x_1,x_2) = 2\lambda_{2g+2}x_1^{g+1}x_2^{g+1} + \sum_{i=0}^{g} x_1^i x_2^i (2\lambda_{2i} + \lambda_{2i+1}(x_1+x_2)) . \tag{1.7}$$

Properties (1.5) of $F(x_1,x_2)$ allow us to construct the *global Abelian* 2*-differential of the second kind* with unique pole of order 2 along $x_1 = x_2$:

$$\omega(x_1,x_2) = \frac{2y_1y_2 + F(x_1,x_2)}{4(x_1-x_2)^2}\,\frac{\mathrm{d}x_1}{y_1}\,\frac{\mathrm{d}x_2}{y_2}, \tag{1.8}$$

which expands in the vicinity of the pole as

$$\omega(x_1,x_2) = \left(\frac{1}{2(\xi-\zeta)^2} + O(1)\right)\mathrm{d}\xi\,\mathrm{d}\zeta,$$

where ξ and ζ are the local coordinates at the points x_1 and x_2, respectively.

Using (1.6), we rewrite (1.8) in the form

$$\omega(x_1,x_2) = \frac{\partial}{\partial x_2}\left(\frac{y_1+y_2}{2y_1(x_1-x_2)}\right)\mathrm{d}x_1\,\mathrm{d}x_2 + \mathrm{d}\boldsymbol{u}^T(x_1)\,\mathrm{d}\boldsymbol{r}(x_2), \tag{1.9}$$

where the differentials $\mathrm{d}\boldsymbol{u}$, $\mathrm{d}\boldsymbol{r}$ are as above. So, the periods of this 2-form (the double integrals $\oint\oint\omega(x_1,x_2)$) can be expressed in terms of $(2\omega,2\omega')$ and $(-2\eta,-2\eta')$, e.g., for the A-periods we have:

$$\left\{\oint_{A_i}\oint_{A_k}\omega(x_1,x_2)\right\}_{i,k=1,\dots,g} = -4\omega^T\eta .$$

1.3. **The Riemann θ-function.** The standard θ-function $\theta(\boldsymbol{v}\,|\,\tau)$ on $\mathbb{C}^g\times\mathcal{S}_g$ is defined by its Fourier series,

$$\theta(\boldsymbol{v}\,|\,\tau) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g}\exp\pi i\{\boldsymbol{n}^T\tau\boldsymbol{n} + 2\boldsymbol{v}^T\boldsymbol{n}\} .$$

The θ-function possesses the following periodicity properties:

$$\theta(\boldsymbol{v}+\boldsymbol{m}+\tau\boldsymbol{m}'\,|\,\tau) = \exp\{-2\pi i\boldsymbol{m}'^T(\boldsymbol{v}+\tfrac{1}{2}\tau\boldsymbol{m}')\}\theta(\boldsymbol{v}\,|\,\tau)$$

for all $\boldsymbol{m},\boldsymbol{m}'\in\mathbb{Z}^g$. The θ-functions with characteristics $[\varepsilon] = \begin{bmatrix}\varepsilon'^T\\ \varepsilon^T\end{bmatrix} = \begin{bmatrix}\varepsilon_1' & \dots & \varepsilon_g'\\ \varepsilon_1 & \dots & \varepsilon_g\end{bmatrix}\in\mathbb{C}^{2g}$ satisfy

$$\begin{aligned}\theta[\varepsilon](\boldsymbol{v}\,|\,\tau) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^g}\exp\pi i\{(\boldsymbol{n}+\varepsilon')^T\tau(\boldsymbol{n}+\varepsilon') + 2(\boldsymbol{v}+\varepsilon)^T(\boldsymbol{n}+\varepsilon')\}\\ &= \exp\{2\pi i\varepsilon'^T(\boldsymbol{v}+\tfrac{1}{2}\tau\varepsilon') + 2\pi i\varepsilon^T\varepsilon'\}\,\theta(\boldsymbol{v}+\varepsilon+\tau\varepsilon'\,|\,\tau);\end{aligned}$$

their periodicity properties are

$$\theta[\varepsilon](\boldsymbol{v}+\boldsymbol{m}+\tau\boldsymbol{m}'\,|\,\tau)=\exp\{-2\pi i\boldsymbol{m}'^T(\boldsymbol{v}+\tfrac{1}{2}\tau\boldsymbol{m}')\}$$
$$\times\exp\{2\pi i(\boldsymbol{m}^T\varepsilon'-\boldsymbol{m}'^T\varepsilon)\}\theta(\boldsymbol{v}\,|\,\tau)$$

for all $\boldsymbol{m},\boldsymbol{m}'\in\mathbb{Z}^g$.

Further, consider the half-integer characteristics $[\varepsilon]$: the θ-function $\theta[\varepsilon](\boldsymbol{v}\,|\,\tau)$ is even or odd whenever $4\boldsymbol{\varepsilon}'^T\boldsymbol{\varepsilon}=0$ or 1 modulo 2. There are $\frac{1}{2}(4^g+2^g)$ even characteristics and $\frac{1}{2}(4^g-2^g)$ odd ones.

Let $\boldsymbol{w}^T=(w_1,\dots,w_g)\in\mathrm{Jac}(V)$ be a point; then the function

$$\mathcal{R}(x)=\theta\left(\int_{x_0}^{x}\mathrm{d}\mathbf{v}-\boldsymbol{w}\,\middle|\,\tau\right),\qquad x\in V,$$

is said to be a *Riemann θ-function.*

The Riemann θ-function $\mathcal{R}(x)$ is either identically 0 or has exactly g zeros $x_1,\dots,x_g\in V$, for which the *Riemann vanishing theorem* says that

$$\sum_{k=1}^{g}\int_{x_0}^{x_i}\mathrm{d}\mathbf{v}=\boldsymbol{w}+\mathbf{K}_{x_0},$$

where $\mathbf{K}_{x_0}^T=(K_1,\dots,K_g)$ is the vector of Riemann constants with respect to the base point x_0; it is defined by the formula

$$K_j=\frac{1+\tau_{jj}}{2}-\sum_{l\neq j}\oint_{A_l}\mathrm{d}v_l(x)\int_{x_0}^{x}\mathrm{d}v_j,\qquad j=1,\dots,g. \tag{1.10}$$

§2. Kleinian functions

Let $\boldsymbol{m},\boldsymbol{m}'\in\mathbb{Z}^g$ be two arbitrary vectors; introduce the following vectors

$$\mathbf{E}(\boldsymbol{m},\boldsymbol{m}')=2\eta\boldsymbol{m}+2\eta'\boldsymbol{m}',\qquad\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}')=2\omega\boldsymbol{m}+2\omega'\boldsymbol{m}'.$$

2.1. **The σ-function.** In [**1, 9**] it was shown that properties (2.1) and (2.2) define a function that plays the central role in the theory of Kleinian functions.

DEFINITION 1. An entire function $\sigma(\boldsymbol{u})$ is the Kleinian *fundamental σ-function* if and only if the following holds.

1. For any vector $\boldsymbol{u}\in\mathbb{C}^g$ we have

$$\sigma(\boldsymbol{u}+\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}'))=\exp\{\mathbf{E}^T(\boldsymbol{m},\boldsymbol{m}')(\boldsymbol{u}+\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}'))\}$$
$$\times\exp\{-\pi i\boldsymbol{m}^T\boldsymbol{m}+2\pi i(\boldsymbol{m}^T\boldsymbol{q}'-\boldsymbol{m}'^T\boldsymbol{q})\}\sigma(\boldsymbol{u}), \tag{2.1}$$

where $\left[{\boldsymbol{q}' \atop \boldsymbol{q}}\right]$ is the half-integer characteristic of the vector of Riemann constants $\mathbf{K}_a$.

2. In the vicinity of $\boldsymbol{u}=0$, the decomposition

$$\sigma(\boldsymbol{u})=\delta(\boldsymbol{u})+\text{higher order terms} \tag{2.2}$$

holds, where $\delta(\boldsymbol{u})=\det(\{u_{i+j-1}\}_{i,j=1,\dots,[(g+1)/2]})$ is a homogeneous polynomial of degree $[(g+1)/2]$.

Note that for small genera we have

$$\begin{aligned} \delta(\boldsymbol{u}) &= u_1 && \text{for } g=1 \text{ and } 2,\\ \delta(\boldsymbol{u}) &= u_1u_3-u_2^2 && \text{for } g=3 \text{ and } 4,\\ \delta(\boldsymbol{u}) &= -u_3^3+2u_2u_3u_4-u_1u_4^2-u_2^2u_5+u_1u_3u_5 && \text{for } g=5 \text{ and } 6, \text{ etc.} \end{aligned}$$

In the case $g=1$, the above definition becomes that of the standard Weierstrass σ-function; for all $m, m' \in \mathbb{Z}$ the latter possesses the following periodicity property

$$\sigma(u+2m\omega+2m'\omega') = (-1)^{mm'+m+m'}\sigma(u)\, e^{(2\eta m+2\eta' m)(u+m\omega+m'\omega')}.$$

For all vectors $\boldsymbol{r}^T, \boldsymbol{r}' \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$, we introduce the *$\sigma$-functions $\sigma_{\boldsymbol{r},\boldsymbol{r}'}$ with characteristic* by the formula

$$\sigma_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{u}) = e^{-\mathbf{E}^T(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{u}}\,\frac{\sigma(\boldsymbol{u}+\boldsymbol{\Omega}(\boldsymbol{r},\boldsymbol{r}'))}{\sigma(\boldsymbol{\Omega}(\boldsymbol{r},\boldsymbol{r}'))}.$$

These functions are completely analogous to Weierstrass' σ_α appearing in the theory of elliptic functions ([**22**]).

2.1.1. *σ-functions as θ-functions.* The fundamental hyperelliptic Kleinian σ-function belongs to the class of modified θ-functions. We can explicitly express σ in terms of the standard θ-function as follows:

$$\sigma(\boldsymbol{u}) = C[e^{\boldsymbol{u}^T\varkappa\boldsymbol{u}}\theta((2\omega)^{-1}\boldsymbol{u}-\mathbf{K}_a\,|\,\tau)]\,e^{2i\pi\boldsymbol{q}'^T\{-(2\omega)^{-1}\boldsymbol{u}+\frac{1}{2}\tau\boldsymbol{q}'-\boldsymbol{q}\}}, \tag{2.3}$$

where $\varkappa = (2\omega)^{-1}\eta$ and $\mathbf{K}_a$ is the vector of Riemann constants (1.10) with base point a, which is equal to

$$\mathbf{K}_a = \sum_{k=1}^{g}\int_a^{a_i}\mathrm{d}\mathrm{v}, \tag{2.4}$$

because a is a branching point. Hence $\mathbf{K}_a$ is a half period, so that it can be written as $\mathbf{K}_a = \boldsymbol{q}+\tau\boldsymbol{q}'$ with half-integers $\boldsymbol{q}$, $\boldsymbol{q}'$, which are uniquely determined by the fixed basis of cycles $(A_1,\dots,A_g,B_1,\dots,B_g)$ (see e.g. [**18**]).

In (2.3) the constant C equals

$$\frac{\epsilon_4}{\theta(0\,|\,\tau)}\prod_{r=1}^{g}\frac{\sqrt{P'(a_r)}}{\sqrt[4]{f'(a_r)}}\,\frac{1}{\prod_{k<l}\sqrt{e_k-e_l}},$$

where $(\epsilon_4)^4 = 1$.

Direct calculations show that the function defined by (2.3) satisfies (2.1) and (2.2).

Putting $g=1$ and fixing the elliptic curve $y^2 = f(x) = 4x^3 - g_2x - g_3$ in (2.3), we see that the function

$$\sigma(u) = \frac{1}{\vartheta_3(0\,|\,\tau)\sqrt[4]{(e_1-e_2)(e_2-e_3)}}\,e^{\eta u^2/(2\omega)}\vartheta_1\Big(\frac{u}{2\omega}\,\Big|\,\tau\Big)$$

is the standard Weierstrass σ-function, where we have used the standard notation for Jacobi ϑ-functions (see e.g. [**22**]): $\vartheta_1(v\,|\,\tau) = -\theta\big[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\big](v\,|\,\tau)$.

2.2. **The functions ζ and $\wp$.** The Kleinian ζ and $\wp$-functions are defined as the logarithmic derivatives of the fundamental function σ, namely

$$\zeta_i(\boldsymbol{u}) = \frac{\partial \ln \sigma(\boldsymbol{u})}{\partial u_i}, \qquad i = 1, \dots, g,$$

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2 \ln \sigma(\boldsymbol{u})}{\partial u_i \partial u_j}, \quad \wp_{ijk}(\boldsymbol{u}) = -\frac{\partial^3 \ln \sigma(\boldsymbol{u})}{\partial u_i \partial u_i \partial u_k}, \quad \dots, \qquad i, j, k, \dots = 1, \dots, g.$$

The functions $\zeta_i(\boldsymbol{u})$ and $\wp_{ij}(\boldsymbol{u})$ have the following periodicity properties

$$\begin{aligned} \zeta_i(\boldsymbol{u} + \boldsymbol{\Omega}(\boldsymbol{m}, \boldsymbol{m}')) &= \zeta_i(\boldsymbol{u}) + E_i(\boldsymbol{m}, \boldsymbol{m}'), & i &= 1, \dots, g, \\ \wp_{ij}(\boldsymbol{u} + \boldsymbol{\Omega}(\boldsymbol{m}, \boldsymbol{m}')) &= \wp_{ij}(\boldsymbol{u}), & i, j &= 1, \dots, g. \end{aligned}$$

2.2.1. *Realization of the fundamental 2-differential of the second kind by Kleinian functions.* The construction is based on the following

THEOREM 2.1. *Let $(y(a_0), a_0)$, (y, x) and (ν, μ) be arbitrary distinct points on V and let $\{(y_1, x_1), \dots, (y_g, x_g)\}$, $\{(\nu_1, \mu_1), \dots, (\nu_g, \mu_g)\} \in (V)^g$ be arbitrary sets of distinct points. Then the following relation is valid:*

$$\begin{aligned} &\int_\mu^x \sum_{i=1}^g \int_{\mu_i}^{x_i} \frac{2yy_i + F(x, x_i)}{4(x - x_i)^2} \frac{\mathrm{d}x}{y} \frac{\mathrm{d}x_i}{y_i} \\ &\quad = \ln \left\{ \frac{\sigma\left(\int_{a_0}^x \mathbf{du} - \sum_{i=1}^g \int_{a_i}^{x_i} \mathbf{du}\right)}{\sigma\left(\int_{a_0}^x \mathbf{du} - \sum_{i=1}^g \int_{a_i}^{\mu_i} \mathbf{du}\right)} \right\} - \ln \left\{ \frac{\sigma\left(\int_{a_0}^\mu \mathbf{du} - \sum_{i=1}^g \int_{a_i}^{x_i} \mathbf{du}\right)}{\sigma\left(\int_{a_0}^\mu \mathbf{du} - \sum_{i=1}^g \int_{a_i}^{\mu_i} \mathbf{du}\right)} \right\}, \end{aligned} \tag{2.5}$$

where the function $F(x, z)$ is given by (1.7).

PROOF. Let us consider the sum

$$\sum_{i=1}^g \int_\mu^x \int_{\mu_i}^{x_i} \left[\omega(x, x_i) + \mathbf{du}^T(x) \varkappa \mathbf{du}(x_i)\right], \tag{2.6}$$

with $\omega(\cdot, \cdot)$ given by (1.9). It is the normalized Abelian integral of the third kind with logarithmic residues at the points x_i and μ_i. By the Riemann vanishing theorem, we can express (2.6) in terms of Riemann θ-functions as

$$\begin{aligned} &\ln \left\{ \frac{\theta\left(\int_{a_0}^x \mathbf{dv} - \left(\sum_{i=1}^g \int_{a_0}^{x_i} \mathbf{dv} - \mathbf{K}_{a_0}\right)\right)}{\theta\left(\int_{a_0}^x \mathbf{dv} - \left(\sum_{i=1}^g \int_{a_0}^{\mu_i} \mathbf{dv} - \mathbf{K}_{a_0}\right)\right)} \right\} \\ &\qquad - \ln \left\{ \frac{\theta\left(\int_{a_0}^\mu \mathbf{dv} - \left(\sum_{i=1}^g \int_{a_0}^{x_i} \mathbf{dv} - \mathbf{K}_{a_0}\right)\right)}{\theta\left(\int_{a_0}^\mu \mathbf{dv} - \left(\sum_{i=1}^g \int_{a_0}^{\mu_i} \mathbf{dv} - \mathbf{K}_{a_0}\right)\right)} \right\}; \end{aligned} \tag{2.7}$$

to obtain the right-hand side of (2.5), we must combine (2.3), the expression of the vector $\mathbf{K}_{a_0}$ (2.4), the matrix $\varkappa = (2\omega)^{-1}\eta$, and Lemma 1.1. The left-hand side of (2.5) is obtained by using (1.8). □

The fact that the right-hand side of (2.5) is independent of the arbitrary point a_0 (that will appear later) has its origin in the properties of the vector of Riemann constants. Consider the difference $\mathbf{K}_{a_0}-\mathbf{K}_{a_0'}$ of vectors of Riemann constants with arbitrary base points a_0 and a_0'; it follows from (1.10) that

$$\mathbf{K}_{a_0}-\mathbf{K}_{a_0'}=(g-1)\int_{a_0'}^{a_0}\mathrm{d}\mathbf{v}\,.$$

This property implies that

$$\int_{a_0}^{x_0}\mathrm{d}\mathbf{v}-\Big(\sum_{i=1}^{g}\int_{a_0}^{x_i}\mathrm{d}\mathbf{v}-\mathbf{K}_{a_0}\Big)=\int_{a_0'}^{x_0}\mathrm{d}\mathbf{v}-\Big(\sum_{i=1}^{g}\int_{a_0'}^{x_i}\mathrm{d}\mathbf{v}-\mathbf{K}_{a_0'}\Big)$$

for arbitrary x_i, with $i=0,\dots,g$ on V, so the arguments of the σ's in (2.5), which are linear transformations by 2ω of the arguments of θ's in (2.7), do not depend on a_0.

COROLLARY 2.1.1. *For arbitrary distinct points $(y(a_0),a_0)$ and (y,x) on V and an arbitrary set of distinct points $\{(y_1,x_1),\dots,(y_g,x_g)\}\in(V)^g$ we have*
(2.8)

$$\sum_{i,j=1}^{g}\wp_{ij}\Big(\int_{a_0}^{x}\mathrm{d}\mathbf{u}+\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}\mathbf{u}\Big)x^{i-1}x_r^{j-1}=\frac{F(x,x_r)-2yy_r}{4(x-x_r)^2},\qquad r=1,\dots,g\,.$$

PROOF. Taking the partial derivative $\partial^2/\partial x_r\partial x$ of both sides of (2.5) and using the hyperelliptic involution $\phi(y,x)=(-y,x)$ and $\phi(y(a_0),a_0)=(-y(a_0),a_0)$, we obtain (2.8). □

In the case $g=1$, formula (2.8) is actually the addition theorem for the Weierstrass elliptic functions,

$$\wp(u+v)=-\wp(u)-\wp(v)+\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2$$

on the elliptic curve $y^2=f(x)=4x^3-g_2x-g_3$.

Now we can give the expression for $\omega(x,x_r)$ in terms of Kleinian functions. We send the base point a_0 to the branch place a; for $r=1,\dots,g$ the fundamental 2-differential of the second kind is given by

$$\omega(x,x_r)=\sum_{i,j=1}^{g}\wp_{ij}\Big(\int_{a}^{x}\mathrm{d}\mathbf{u}-\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}\mathbf{u}\Big)\frac{x^{i-1}\mathrm{d}x}{y}\,\frac{x_r^{j-1}\mathrm{d}x_r}{y_r}\,.$$

COROLLARY 2.1.2. *For all $r\neq s=1,\dots,g$*

$$\sum_{i,j=1}^{g}\wp_{ij}\Big(\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}\mathbf{u}\Big)x_s^{i-1}x_r^{j-1}=\frac{F(x_s,x_r)-2y_sy_r}{4(x_s-x_r)^2}\,.\tag{2.9}$$

PROOF. For $s \neq r$ in (2.8) we have

$$\int_{\phi(a_0)}^{x} \mathrm{d}\mathbf{u} + \sum_{k=1}^{g} \int_{a_k}^{x_k} \mathrm{d}\mathbf{u} = -2\omega\Big(\int_{a_0}^{\phi(x)} \mathrm{d}\mathbf{v} - \Big(\sum_{i=1}^{g} \int_{a_0}^{x_i} \mathrm{d}\mathbf{v} - \mathbf{K}_{a_0} \Big)\Big)$$

$$= -2\omega\Big(\int_{x_s}^{\phi(x)} \mathrm{d}\mathbf{v} - \Big(\sum_{\substack{i=1\\ i\neq s}}^{g} \int_{x_s}^{x_i} \mathrm{d}\mathbf{v} - \mathbf{K}_{x_s} \Big)\Big)$$

$$= \int_{a_s}^{x} \mathrm{d}\mathbf{u} + \sum_{\substack{i=1\\ i\neq s}}^{g} \int_{a_i}^{x_i} \mathrm{d}\mathbf{u}$$

and the change of notation $x \to x_s$ gives (2.9). □

2.2.2. *Solution of the Jacobi inversion problem.* The equations of the Abel map under the conditions of the Jacobi inversion problem,

$$u_i = \sum_{k=1}^{g} \int_{a_k}^{x_k} \frac{x^{i-1}\mathrm{d}x}{y}, \tag{2.10}$$

are invertible for nonspecial divisors for which the points (y_k, x_k), $k = 1, \ldots, g$, are distinct and for all $j, k = 1, \ldots, g$, $\phi(y_k, x_k) \neq (y_j, x_j)$, where ϕ is the hyperelliptic involution. Using (2.8), we find the solution of Jacobi inversion problem on the curves with $a = \infty$ in a very effective form.

THEOREM 2.2. *Let the preimage of the point* $\boldsymbol{u} \in \mathrm{Jac}(V)$ *of the Abel map* $\{(y_1, x_1), \ldots, (y_g, x_g)\} \in (V)^g$ *be a nonspecial divisor. Then* $\{x_1, \ldots, x_g\}$ *are the zeros of the polynomial*

$$\mathcal{P}(x; \boldsymbol{u}) = 0, \tag{2.11}$$

where

$$\mathcal{P}(x; \boldsymbol{u}) = x^g - x^{g-1}\wp_{g,g}(\boldsymbol{u}) - x^{g-2}\wp_{g,g-1}(\boldsymbol{u}) - \cdots - \wp_{g,1}(\boldsymbol{u}), \tag{2.12}$$

and the points $\{y_1, \ldots, y_g\}$ *are given by*

$$y_k = x_k^{g-1}\wp_{g,g,g}(\boldsymbol{u}) + \cdots + \wp_{g,g,1}(\boldsymbol{u}) = -\left.\frac{\partial \mathcal{P}(x; \boldsymbol{u})}{\partial u_g}\right|_{x = x_k}. \tag{2.13}$$

PROOF. In (2.8) let $a_0 \to a = \infty$. Then

$$\lim_{x\to\infty} \frac{F(x, x_r)}{4x^{g-1}(x - x_r)^2} = \sum_{i=1}^{g} \wp_{gi}(\boldsymbol{u}) x_r^{i-1}. \tag{2.14}$$

The limit in the left-hand side of (2.14) is equal to x_r^g, and we obtain (2.11).

From (2.10) we find

$$\sum_{i=1}^{g} \frac{x_i^{k-1}}{y_i} \frac{\partial x_i}{\partial u_j} = \delta_{jk}, \qquad \frac{\partial x_k}{\partial u_g} = \frac{y_k}{\prod_{i\neq k}(x_k - x_i)}.$$

On the other hand,

$$\left.\frac{\partial\mathcal{P}}{\partial u_g}\right|_{x=x_k} = -\frac{\partial x_k}{\partial u_g}\prod_{i\neq k}(x_i - x_k),$$

and we obtain (2.13). □

Let us denote by $\wp$, $\wp'$ the following g-dimensional vectors,

$$\wp = (\wp_{g1}, \dots, \wp_{gg})^T, \qquad \wp' = \frac{\partial\wp}{\partial u_g}.$$

The companion matrix $\mathcal{C}$ ([**25**]) of the polynomial $\mathcal{P}(z;\boldsymbol{u})$ given by (2.12), is

$$\mathcal{C} = \mathcal{B} + \wp\,\mathbf{e}_g^T, \quad \text{where } \mathcal{B} = \sum_{k=1}^{g}\mathbf{e}_k\mathbf{e}_{k-1}^T\,.$$

The companion matrix $\mathcal{C}$ has the property

$$x_k^n = \mathbf{X}_k^T\mathcal{C}^{n-g+1}\mathbf{e}_g = \mathbf{X}_k^T\mathcal{C}^{n-g}\wp, \qquad \forall n\in\mathbb{Z}, \tag{2.15}$$

with the vector $\mathbf{X}_k^T = (1, x_k, \dots, x_k^{g-1})$, where x_k is one of the roots of (2.11). From (2.9) we find

$$-2y_ry_s = 4(x_r - x_s)^2\sum_{i=1}^{g}\wp_{ij}(\boldsymbol{u})x_r^{i-1}x_s^{j-1} - F(x_r, x_s)\,.$$

Introducing the matrices

$$\Pi = (\wp_{ij}), \quad \Lambda_0 = \operatorname{diag}(\lambda_{2g-2}, \dots, \lambda_0), \quad \Lambda_1 = \operatorname{diag}(\lambda_{2g-1}, \dots, \lambda_1),$$

we have, taking into account (2.15),

$$\begin{aligned}\mathbf{X}_r^T\wp'\wp'^T\mathbf{X}_s = -4\mathbf{X}_r^T(\mathcal{C}^2\Pi - 2\mathcal{C}\Pi\mathcal{C}^T + \Pi\mathcal{C}^{T^2})\mathbf{X}_s + 4\mathbf{X}_r^T(\mathcal{C}\wp\wp^T + \wp\wp^T\mathcal{C}^T)\mathbf{X}_s\\ + 2\mathbf{X}_r^T\Lambda_0\mathbf{X}_s + \mathbf{X}_r^T(\mathcal{C}\Lambda_1 + \Lambda_1\mathcal{C}^T)\mathbf{X}_s\,.\end{aligned}$$

Hence (see [**16**]), we obtain the following statement.

COROLLARY 2.2.1. *The relation*

$$2\wp'\wp'^T = -4(\mathcal{C}^2\Pi - 2\mathcal{C}\Pi\mathcal{C}^T + \Pi\mathcal{C}^{T^2}) + 4(\mathcal{C}\wp\wp^T + \wp\wp^T\mathcal{C}^T) + \mathcal{C}\Lambda_1 + \Lambda_1\mathcal{C}^T + 2\Lambda_0 \tag{2.16}$$

connects odd functions $\wp_{ggi}$ *with poles of order* 3 *and even functions* $\wp_{jk}$ *with poles of order* 2 *in the field of meromorphic functions on* $\operatorname{Jac}(V)$.

DEFINITION 2. The *umbral derivative* ([**23**]) $D_s(p(z))$ of a polynomial $p(z) = \sum_{k=0}^{n} p_kz^k$ is given by

$$D_sp(z) = \left(\frac{p(z)}{z^s}\right)_+ = \sum_{k=s}^{n} p_kx^{k-s},$$

where $(\,\cdot\,)_+$ means taking the purely polynomial part.

Considering the polynomials $p=\prod_{k=1}^{n}(z-z_k)$ and $\tilde{p}=(z-z_0)p$, we immediately deduce the elementary properties of D_s:

$$\begin{aligned}D_s(p)&=zD_{s+1}(p)+p_s=zD_{s+1}(p)+S_{n-s}(z_1,\dots,z_n),\\ D_s(\tilde{p})&=(z-z_0)\,D_s(p)+p_{s-1}=(z-z_0)\,D_s(p)+S_{n+1-s}(z_1,\dots,z_n),\end{aligned}\tag{2.17}$$

where $S_l(\cdots)$ is $(-1)^l$ times the lth order elementary symmetric function of its variables (we assume $S_0(\cdots)=1$).

From (2.17) we see that $S_{n-s}(z_0,\dots,\hat{z}_l,\dots,z_n)=(D_{s+1}(\tilde{p})|_{z=z_l})$. It is particularly useful to write down the inversion of (2.13), namely

$$\wp_{ggk}(\boldsymbol{u})=\sum_{l=1}^{g}y_l\left(\frac{D_k(P(z))}{(\partial P/\partial z)\,(z)}\bigg|_{z=x_l}\right),\tag{2.18}$$

where $P(z)=\prod_{k=1}^{g}(z-x_k)$.

It is important to describe the set of common zeros of the functions $\wp_{ggk}(\boldsymbol{u})$.

COROLLARY 2.2.2. *The vector function $\wp'(\boldsymbol{u})$ vanishes if and only if $\boldsymbol{u}$ is a half-period.*

PROOF. The equations $\wp_{ggk}(\boldsymbol{u})=0$, $k=1,\dots,g$ imply (2.18), the condition $y_i=0$ for all $i=1,\dots,g$. This is possible if and only if the points $x_1,\dots,x_g$ coincide with any g points $e_{i_1},\dots,e_{i_g}$ from the set of branching points $e_1,\dots,e_{2g+2}$. So the point

$$\boldsymbol{u}=\sum_{l=1}^{g}\int_{a_l}^{e_{i_l}}\mathrm{d}\mathbf{u}\in\operatorname{Jac}(V)$$

is of the second order in the Jacobian and hence is a half-period. □

§3. Basic relations

In this section we shall derive explicit algebraic relations between the generating functions in the field of meromorphic functions on $\operatorname{Jac}(V)$. After some immediate preparations, in 3.1 we shall find the explicit cubic relations between $\wp_{ggi}$ and $\wp_{ij}$. These, in turn, lead to very special corollaries: the variety $\operatorname{Kum}(V)=\operatorname{Jac}(V)/\pm$ is mapped into the space of symmetric matrices of rank not greater than 3.

The conditions $\lambda_{2g+2}=0$, $\lambda_{2g+1}=4$ being imposed, we start with the following theorem, which is the starting point for the derivation of the basic relations.

THEOREM 3.1. *Let $(y_0,x_0)\in V$ be any point and $\{(y_1,x_1),\dots,(y_g,x_g)\}\in(V)^g$ be the Abel preimage of the point $\boldsymbol{u}\in\operatorname{Jac}(V)$. Then*

$$-\zeta_j\left(\int_a^{x_0}\mathrm{d}\mathbf{u}+\boldsymbol{u}\right)=\int_a^{x_0}\mathrm{d}r_j+\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}r_j-\frac{1}{2}\sum_{k=0}^{g}y_k\left(\frac{D_j(R'(z))}{R'(z)}\bigg|_{z=x_k}\right),\tag{3.1}$$

where $R(z)=\prod_0^g(z-x_j)$ and $R'(z)=(\partial R/\partial z)(z)$. Further,

$$-\zeta_j(\boldsymbol{u})=\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}r_j-\frac{1}{2}\,\wp_{gg,j+1}(\boldsymbol{u}).\tag{3.2}$$

PROOF. Putting in (2.5) $\mu_i = a_i$, we obtain

$$\ln\left\{\frac{\sigma\left(\int_{a_0}^{x}\mathrm{d}\mathbf{u}-\boldsymbol{u}\right)}{\sigma\left(\int_{a_0}^{x}\mathrm{d}\mathbf{u}\right)}\right\}-\left\{\frac{\sigma\left(\int_{a_0}^{\mu}\mathrm{d}\mathbf{u}-\boldsymbol{u}\right)}{\sigma\left(\int_{a_0}^{\mu}\mathrm{d}\mathbf{u}\right)}\right\}=\int_{\mu}^{x}\mathrm{d}\mathbf{r}^T\boldsymbol{u}+\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}\Omega(x,\mu), \tag{3.3}$$

where $\mathrm{d}\Omega$ is as in (1.4). Taking derivative in u_j from both sides of (3.3), letting a_0 tend to μ, applying $\phi(y,x)=(-y,x)$ and $\phi(\nu,\mu)=(-\nu,\mu)$, we get

$$\zeta_j\left(\int_{\mu}^{x}\mathrm{d}\mathbf{u}+\boldsymbol{u}\right)+\int_{\mu}^{x}\mathrm{d}r_j-\frac{1}{2}\sum_{k=1}^{g}\frac{1}{y_k}\frac{\partial x_k}{\partial u_j}\frac{y_k-y}{x_k-x}=\zeta_j(\boldsymbol{u})-\frac{1}{2}\sum_{k=1}^{g}\frac{1}{y_k}\frac{\partial x_k}{\partial u_j}\frac{y_k-\nu}{x_k-\mu}.$$

Put $x=x_0$. Denoting $P(z)=\prod_1^g(z-x_j)$, we find

$$\begin{aligned}\sum_{k=1}^{g}\frac{1}{y_k}\frac{\partial x_k}{\partial u_j}\frac{y_k-y}{x_k-x}&=\sum_{k=1}^{g}\left(\frac{D_j(P(z))}{P'(z)}\bigg|_{z=x_k}\right)\frac{y_k-y}{x_k-x}\\&=\sum_{k=0}^{g}y_k\left(\frac{D_j(R'(z))}{R'(z)}\bigg|_{z=x_k}\right)-\sum_{k=1}^{g}y_k\left(\frac{D_{j+1}(P(z))}{P'(z)}\bigg|_{z=x_k}\right).\end{aligned}$$

Hence, using (2.18) and adding $\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}r_j$ to both sides, we deduce

$$\begin{aligned}&\zeta_j\left(\int_{\mu}^{x_0}\mathrm{d}\mathbf{u}+\boldsymbol{u}\right)+\int_{\mu}^{x_0}\mathrm{d}r_j+\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}r_j-\frac{1}{2}\sum_{k=0}^{g}y_k\left(\frac{D_j(R'(z))}{R'(z)}\bigg|_{z=x_k}\right)\\&\quad=\zeta_j(\boldsymbol{u})+\sum_{k=1}^{g}\int_{a_k}^{x_k}\mathrm{d}r_j-\frac{1}{2}\sum_{k=1}^{g}\frac{1}{y_k}\frac{\partial x_k}{\partial u_j}\frac{y_k-\nu}{x_k-\mu}-\frac{1}{2}\wp_{gg,j+1}.\end{aligned} \tag{3.4}$$

Now we see that the left-hand side of the (3.4) is symmetric in $x_0, x_1, \dots, x_g$, while the right-hand side does not depend on x_0. So, it does not depend on any of the x_i. We conclude that it is a constant depending only on μ. Letting $\mu \to a$ and applying the hyperelliptic involution to the whole aggregate, we find this constant to be 0. □

COROLLARY 3.1.1. *For* $(y,x)\in V$ *and* $\boldsymbol{\alpha}=\int_a^x\mathrm{d}\mathbf{u}$*:*

$$\zeta_j(\boldsymbol{u}+\boldsymbol{\alpha})-\zeta_j(\boldsymbol{u})-\zeta_j(\boldsymbol{\alpha})=\frac{(-yD_j+\partial_j)\mathcal{P}(x;\boldsymbol{u})}{2\mathcal{P}(x;\boldsymbol{u})}, \tag{3.5}$$

where $\partial_j=\partial/\partial u_j$.

PROOF. To find $\zeta_j(\boldsymbol{\alpha})$, take the limit as $\{x_1,\dots,x_g\}\to\{a_1,\dots,a_g\}$ in (3.1). The right-hand side of (3.5) is obtained by rearranging

$$\frac{1}{2}\sum_{k=1}^{g}\frac{1}{y_k}\frac{\partial x_k}{\partial u_j}\frac{y_k-y}{x_k-x}.\quad\square$$

COROLLARY 3.1.2. *For* $k=1,\dots,g$, *the functions* $\wp_{gggk}$ *are given by*

$$\wp_{gggi}=(6\wp_{gg}+\lambda_{2g})\wp_{gi}+6\wp_{g,i-1}-2\wp_{g-1,i}+\tfrac{1}{2}\delta_{gi}\lambda_{2g-1}. \tag{3.6}$$

PROOF. Consider relation (3.2). The differentials $\mathrm{d}\zeta_i$, $i = 1, \ldots, g$, can be presented in the following forms

$$-\mathrm{d}\zeta_i = \sum_{k=1}^{g} \wp_{ik}\,\mathrm{d}u_k = \sum_{k=1}^{g} \mathrm{d}r_i(x_k) - \frac{1}{2}\sum_{k=1}^{g} \wp_{gg,i+1,k}\,\mathrm{d}u_k\,.$$

Put $i = g-1$. For each of the x_k, $k = 1, \ldots, g$, we have

$$\left(12x_k^{g+1} + 2\lambda_{2g}x_k^g + \lambda_{2g-1}x_k^{g-1} - 4\sum_{j=1}^{g} \wp_{g-1,j}x_k^{j-1}\right)\frac{\mathrm{d}x_k}{y_k} = 2\sum_{j=1}^{g} \wp_{gggj}x_k^{j-1}\frac{\mathrm{d}x_k}{y_k}\,.$$

Applying formula (2.12) to eliminate the powers of x_k greater than $g-1$, and taking into account the fact that the differentials $\mathrm{d}x_k$ are independent, we come to

$$\sum_{i=1}^{g}\left[(6\wp_{gg} + \lambda_{2g})\wp_{gi} + 6\wp_{g,i-1} - 2\wp_{g-1,i} + \frac{1}{2}\,\delta_{gi}\lambda_{2g-1}\right]x_k^{i-1} = \sum_{j=1}^{g} \wp_{gggj}x_k^{j-1}. \quad \square$$

Calculating the difference $\partial\wp_{gggk}/\partial u_i - \partial\wp_{gggi}/\partial u_k$ according to (3.6), we obtain

COROLLARY 3.1.3.

$$\wp_{ggk}\wp_{gi} - \wp_{ggi}\wp_{gk} + \wp_{g,i-1,k} - \wp_{gi,k-1} = 0\,. \tag{3.7}$$

This means that the 1-form $\sum_{i=1}^{g}(\wp_{gg}\wp_{gi} + \wp_{g,i-1})\,\mathrm{d}u_i$ is closed. We can rewrite this form as $\mathrm{d}\boldsymbol{u}^T\mathcal{C}\boldsymbol{\wp}$.

Differentiation of (3.7) with respect to u_g yields

$$\wp_{gggk}\wp_{gi} - \wp_{gggi}\wp_{gk} + \wp_{gg,i-1,k} - \wp_{ggi,k-1} = 0\,. \tag{3.8}$$

The corresponding closed 1-form is $\mathrm{d}\boldsymbol{u}^T\mathcal{C}\boldsymbol{\wp}'$.

3.1. **Fundamental cubic and quartic relations.** Now we shall find relations connecting the odd functions $\wp_{ggi}$ and the even functions $\wp_{ij}$. In hyperelliptic theory these relations replace the Weierstrass cubic relation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

for elliptic functions, which defines a meromorphic map between the elliptic Jacobian $\mathbb{C}/(2\omega, 2\omega')$ and the plane cubic.

The theorem below is based on the property of an Abelian function to be constant if any gradient of it is identically 0. Equivalently, if for the Abelian functions $G(\boldsymbol{u})$ and $F(\boldsymbol{u})$ there exists a nonzero vector $\boldsymbol{\alpha} \in \mathbb{C}^g$ for which

$$\sum_{i=1}^{g} \alpha_i(\partial/\partial u_i)(G(\boldsymbol{u}) - F(\boldsymbol{u}))$$

vanishes, then $G(\boldsymbol{u}) - F(\boldsymbol{u})$ is a constant.

THEOREM 3.2. *The functions* $\wp_{ggi}$ *and* $\wp_{ik}$ *are related by*

$$\begin{aligned}\wp_{ggi}\wp_{ggk} &= 4\wp_{gg}\wp_{gi}\wp_{gk} - 2(\wp_{gi}\wp_{g-1,k} + \wp_{g,k}\wp_{g-1,i}) \\ &\quad + 4(\wp_{gk}\wp_{g,i-1} + \wp_{gi}\wp_{g,k-1}) + 4\wp_{k-1,i-1} - 2(\wp_{k,i-2} + \wp_{i,k-2}) \\ &\quad + \lambda_{2g}\wp_{gk}\wp_{gi} + \tfrac{1}{2}\lambda_{2g-1}(\delta_{ig}\wp_{kg} + \delta_{kg}\wp_{ig}) + c_{(i,k)},\end{aligned} \tag{3.9}$$

where

$$c_{(i,k)} = \lambda_{2i-2}\delta_{ik} + \tfrac{1}{2}(\lambda_{2i-1}\delta_{k,i+1} + \lambda_{2k-1}\delta_{i,k+1}). \tag{3.10}$$

PROOF. We are looking for a function $G(\boldsymbol{u})$ such that $(\partial/\partial u_g)(\wp_{ggi}\wp_{ggk} - G) = 0$. A direct verification using (3.7) shows that

$$\begin{aligned}\frac{\partial}{\partial u_g}\big(\wp_{ggi}\wp_{ggk} - \big(&4\wp_{gg}\wp_{gi}\wp_{gk} - 2(\wp_{gi}\wp_{g-1,k} + \wp_{g,k}\wp_{g-1,i}) \\ &+ 4(\wp_{gk}\wp_{g,i-1} + \wp_{gi}\wp_{g,k-1}) \\ &+ 4\wp_{k-1,i-1} - 2(\wp_{k,i-2} + \wp_{i,k-2}) \\ &+ \lambda_{2g}\wp_{gk}\wp_{gi} + \tfrac{1}{2}\lambda_{2g-1}(\delta_{ig}\wp_{kg} + \delta_{kg}\wp_{ig})\big)\big) = 0.\end{aligned}$$

It remains to determine c_{ij}. From (2.16) we conclude that $c_{(i,k)}$ is equal to λ_{2i-2} for $k = i$, to $\lambda_{2i-1}/2$ for $k = i + 1$, and vanishes otherwise. So c_{ij} is given by (3.10). □

Consider $\mathbb{C}^{g+g(g+1)/2}$ with the coordinates $(\boldsymbol{z}, p = \{p_{i,j}\}_{i,j=1,\ldots,g})$ with $\boldsymbol{z}^T = (z_1, \ldots, z_g)$ and $p_{ij} = p_{ji}$.

COROLLARY 3.2.1. *The map*

$$\varphi\colon \operatorname{Jac}(V) \setminus (\sigma) \to \mathbb{C}^{g+g(g+1)/2}, \qquad \varphi(\boldsymbol{u}) = (\wp'(\boldsymbol{u}), \Pi(\boldsymbol{u})),$$

where $\Pi = \{\wp_{ij}\}_{i,j=1,\ldots,g}$, *is a meromorphic embedding.*

The image $\varphi(\operatorname{Jac}(V) \setminus (\sigma)) \subset \mathbb{C}^{g+g(g+1)/2}$ *is the intersection of* $g(g+1)/2$ *cubics induced by* (3.9).

Here (σ) denotes the divisor of zeros of σ.

Consider the projection

$$\pi\colon \mathbb{C}^{(g+g(g+1))/2} \to \mathbb{C}^{g(g+1)/2}, \qquad \pi(\boldsymbol{z}, p) = p.$$

COROLLARY 3.2.2. *The restriction* $\pi \circ \varphi$ *is a meromorphic embedding of the Kummer variety* $\operatorname{Kum}(V) = (\operatorname{Jac}(V) \setminus (\sigma))/\pm$ *into* $\mathbb{C}^{g(g+1)/2}$. *The image*

$$\pi(\varphi(\operatorname{Jac}(V) \setminus (\sigma))) \subset \mathbb{C}^{g(g+1)/2}$$

is the intersection of quartics induced by

$$(\wp_{ggi}\wp_{ggj})(\wp_{ggk}\wp_{ggl}) - (\wp_{ggi}\wp_{ggk})(\wp_{ggj}\wp_{ggl}) = 0, \tag{3.11}$$

where the parentheses mean that the substitutions from (3.9) *are made before expanding.*

The quartics (3.11) have no analog in elliptic theory. The first example is given by genus 2, where the celebrated Kummer surface ([**24**]) appears.

3.2. **Analysis of the fundamental relations.** Let us take a second look at the fundamental cubics (3.9) and quartics (3.11).

3.2.1. *Sylvester's identity.* For any matrix K with entries k_{ij}, $i,j=1,\dots,N$, we introduce the symbol $K\left[\begin{smallmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_n\end{smallmatrix}\right]$ to denote the $m\times n$ submatrix:

$$K\left[\begin{smallmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_n\end{smallmatrix}\right] = \{k_{i_k,j_l}\}_{k=1,\dots,m;\, l=1,\dots,n}$$

for subsets of rows i_k and columns j_l.

Here we shall need the *Sylvester's identity* (see, for instance [**25**]). Let us fix a subset of indices $\boldsymbol{\alpha}=\{i_1,\dots,i_k\}$ and consider the $(N-k)\times(N-k)$ matrix $S(K,\boldsymbol{\alpha})$ assuming that

$$S(K,\boldsymbol{\alpha})_{\mu,\nu} = \det K\left[\begin{smallmatrix}\mu;\boldsymbol{\alpha}\\ \nu,\boldsymbol{\alpha}\end{smallmatrix}\right]$$

and μ, ν are not in $\boldsymbol{\alpha}$; then

$$\det S(K,\boldsymbol{\alpha}) = \det K\left[\begin{smallmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}\end{smallmatrix}\right]^{(N-k-1)} \det K\,. \tag{3.12}$$

3.2.2. *Determinantal form.* We introduce (cf. [**13**]) new functions h_{ik} defined by the formula

$$\begin{aligned} h_{ik} &= 4\wp_{i-1,k-1} - 2\wp_{k,i-2} - 2\wp_{i,k-2} \\ &\quad + \tfrac{1}{2}\big(\delta_{ik}(\lambda_{2i-2}+\lambda_{2k-2}) + \delta_{k,i+1}\lambda_{2i-1} + \delta_{i,k+1}\lambda_{2k-1}\big), \end{aligned} \tag{3.13}$$

where $i,k=1,\dots,g+2$. We assume that $\wp_{nm}=0$ if n or m is less than 1 and $\wp_{nm}=0$ if n or m is greater than g. It is evident that $h_{ij}=h_{ji}$. We shall denote the matrix with entries h_{ik} by H.

The map (3.13) from the $\wp$'s and λ's to the h's respects the grading

$$\deg h_{ij} = i+j, \qquad \deg \wp_{ij} = i+j+2, \qquad \deg\lambda_i = i+2,$$

and on a fixed level L (3.13) is linear and invertible. The definition implies

$$\sum_{i=1}^{L-1} h_{i,L-i} = \lambda_{L-2} \implies \boldsymbol{X}^T H \boldsymbol{X} = \sum_{i=0}^{2g+2}\lambda_i x^i$$

for $\boldsymbol{X}^T=(1,x,\dots,x^{g+1})$ with arbitrary $x\in\mathbb{C}$. Moreover, for any roots x_r and x_s of the equation $\sum_{j=1}^{g+2} h_{g+2,j}x^{j-1}=0$, we have (cf. (2.9)) $y_r y_s = \boldsymbol{X}_r^T H \boldsymbol{X}_s$.

From (3.13) we deduce

$$\begin{aligned} -2\wp_{ggi} &= \frac{\partial}{\partial u_g}h_{g+2,i} = \frac{\partial}{\partial u_i}h_{g+2,g} = -\frac{1}{2}\frac{\partial}{\partial u_i}h_{g+1,g+1}, \\ 2\left(\wp_{gi,k-1} - \wp_{g,i-1,k}\right) &= \frac{\partial}{\partial u_k}h_{g+2,i-1} - \frac{\partial}{\partial u_i}h_{g+2,k-1} \\ &= \frac{1}{2}\frac{\partial}{\partial u_i}h_{g+1,k} - \frac{1}{2}\frac{\partial}{\partial u_k}h_{g+1,i}, \end{aligned}$$

etc., and (see (3.6))

$$2\wp_{gggi} = -\frac{\partial^2}{\partial u_g^2}h_{g+2,i} = \det H\left[\begin{smallmatrix} i, & g+1\\ g+1, & g+2\end{smallmatrix}\right] - \det H\left[\begin{smallmatrix} i-1, & g+2\\ g+1, & g+2\end{smallmatrix}\right] + \det H\left[\begin{smallmatrix} i, & g+2\\ g, & g+2\end{smallmatrix}\right]. \tag{3.14}$$

Using (3.13), we rewrite (3.9) in a more effective form:

$$4\wp_{ggi}\wp_{ggk} = \frac{\partial}{\partial u_g} h_{g+2,i} \frac{\partial}{\partial u_g} h_{g+2,k} = -\det H\left[\begin{smallmatrix} i,\, g+1,g+2 \\ k,g+1,g+2 \end{smallmatrix}\right]. \tag{3.15}$$

As an example, consider the case of genus 1. On the Jacobian of the curve

$$y^2 = \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0,$$

we define the following Kleinian functions: $\sigma_K(u_1)$ with expansion $u_1 + \dots$ and its second and third logarithmic derivatives $-\wp_{11}$ and $-\wp_{111}$. By (3.15), and following the definition (3.13), we obtain

$$-4\wp_{111}^2 = \det H\left[\begin{smallmatrix} 1,2,3 \\ 1,2,3 \end{smallmatrix}\right] = \det \begin{pmatrix} \lambda_0 & \lambda_1/2 & -2\wp_{11} \\ \lambda_1/2 & 4\wp_{11} + \lambda_2 & \lambda_3/2 \\ -2\wp_{11} & \lambda_3/2 & \lambda_4 \end{pmatrix};$$

the determinant expands as

$$\wp_{111}^2 = 4\wp_{11}^3 + \lambda_2\wp_{11}^2 + \wp_{11}\frac{\lambda_1\lambda_3 - 4\lambda_4\lambda_0}{4} + \frac{\lambda_0\lambda_3^2 + \lambda_4(\lambda_1^2 - 4\lambda_2\lambda_0)}{16},$$

and (3.14) gives

$$\wp_{1111} = 6\wp_{11}^2 + \lambda_2\wp_{11} + \frac{\lambda_1\lambda_3 - 4\lambda_4\lambda_0}{8}.$$

These equations show that σ_K differs by $\exp(-\lambda_2 u_1^2/12)$ from the standard Weierstrass σ_W built from the invariants

$$g_2 = \lambda_4\lambda_0 + \frac{1}{12}\lambda_2^2 - \frac{1}{4}\lambda_3\lambda_1 \quad \text{and} \quad g_3 = \det \begin{pmatrix} \lambda_0 & \lambda_1/4 & \lambda_2/6 \\ \lambda_1/4 & \lambda_2/6 & \lambda_3/4 \\ \lambda_2/6 & \lambda_3/4 & \lambda_4 \end{pmatrix}$$

(see, e.g. [**22, 26**]).

Further, we find that $\operatorname{rank} H = 3$ at generic points of the Jacobian and $\operatorname{rank} H = 2$ at half-periods. At $u_1 = 0$, where σ_K has is a zero of order 1, we have $\operatorname{rank} \sigma_K^2 H = 3$.

In the general case, on the basis of (3.15), we prove the following result.

THEOREM 3.3. *We have* $\operatorname{rank} H = 3$ *at generic points of* $\operatorname{Jac}(V)$ *and* $\operatorname{rank} H = 2$ *at the half-periods.* $\operatorname{rank} \sigma^2(\boldsymbol{u})H = 3$ *at generic points* (σ) *and* $\operatorname{rank} \sigma^2(\boldsymbol{u})H = 0$ *at the points of* $(\sigma)_{\text{sing}}$.

Here $(\sigma) \subset \operatorname{Jac}(V)$ denotes the divisor of zeros of $\sigma(\boldsymbol{u})$. Further, $(\sigma)_{\text{sing}} \subset (\sigma)$ is the so-called *singular set* of (σ), i.e., $(\sigma)_{\text{sing}}$ is the set of points where σ vanishes and all its first partial derivatives vanish. The set $(\sigma)_{\text{sing}}$ is known (see [**18**] and references therein) to be a subset of dimension $g-3$ in hyperelliptic Jacobians of $g > 3$, to be empty for $g = 2$, and to consist of one point for $g = 3$. Generally, the points of $(\sigma)_{\text{sing}}$ are presented by $\{(y_1, x_1), \dots, (y_{g-3}, x_{g-3})\} \in (V)^{g-3}$.

PROOF. Consider Sylvester's matrix $S = S\big(H\left[\begin{smallmatrix} i,j,g+1,g+2 \\ k,l,g+1,g+2 \end{smallmatrix}\right], \{g+1, g+2\}\big)$. By (3.15) we have

$$S = -4 \begin{pmatrix} \wp_{ggi}\wp_{ggk} & \wp_{ggi}\wp_{ggl} \\ \wp_{ggj}\wp_{ggk} & \wp_{ggj}\wp_{ggl} \end{pmatrix} \quad \text{and} \quad \det S = 0,$$

so by (3.12) we see that

$$\det H\left[\begin{smallmatrix}i,j,g+1,g+2\\k,l,g+1,g+2\end{smallmatrix}\right]\det H\left[\begin{smallmatrix}g+1,g+2\\g+1,g+2\end{smallmatrix}\right]$$

vanishes identically. Since

$$\det H\left[\begin{smallmatrix}g+1,g+2\\g+1,g+2\end{smallmatrix}\right]=\lambda_{2g+2}(4\wp_{gg}+\lambda_{2g})-\lambda_{2g+1}^2/4$$

is not identically 0, we infer that

$$\det H\left[\begin{smallmatrix}i,j,g+1,g+2\\k,l,g+1,g+2\end{smallmatrix}\right]=0. \tag{3.16}$$

Note that this equation is actually (3.11) rewritten in terms of h's. Now, putting $j=l=g$ in (3.16), we see that unless $\boldsymbol{u}$ is such that $H\left[\begin{smallmatrix}g,g+1,g+2\\g,g+1,g+2\end{smallmatrix}\right]$ becomes degenerate, or the entries become singular, i.e., $\boldsymbol{u}\in(\sigma)$, we have, for any i,k,

$$h_{ik}=(h_{i,g},h_{i,g+1},h_{i,g+2})\left(H\left[\begin{smallmatrix}g,g+1,g+2\\g,g+1,g+2\end{smallmatrix}\right]\right)^{-1}\begin{pmatrix}h_{k,g}\\h_{k,g+1}\\h_{k,g+2}\end{pmatrix}. \tag{3.17}$$

This leads to the skeleton decomposition of the matrix H

$$H=H\left[\begin{smallmatrix}1,\dots,\ \ g+2\\g,g+1,g+2\end{smallmatrix}\right]\left(H\left[\begin{smallmatrix}g,g+1,g+2\\g,g+1,g+2\end{smallmatrix}\right]\right)^{-1}H\left[\begin{smallmatrix}g,g+1,g+2\\1,\dots,\ \ g+2\end{smallmatrix}\right], \tag{3.18}$$

which shows that at a generic point of $\operatorname{Jac}(V)$ the rank of H equals 3.

Consider the case $\det H\left[\begin{smallmatrix}g,g+1,g+2\\g,g+1,g+2\end{smallmatrix}\right]=0$. Since by (3.15) we have

$$\det H\left[\begin{smallmatrix}g,g+1,g+2\\g,g+1,g+2\end{smallmatrix}\right]=4\wp_{ggg}^2,$$

this happens if and only if $\boldsymbol{u}$ is a half-period. Therefore, instead of (3.16) we have the relations $H\left[\begin{smallmatrix}i,\ g+1,g+2\\k,g+1,g+2\end{smallmatrix}\right]=0$ and consequently at a half-period matrix H is decomposed as

$$H=H\left[\begin{smallmatrix}1,\dots,g+2\\g+1,g+2\end{smallmatrix}\right]\left(H\left[\begin{smallmatrix}g+1,g+2\\g+1,g+2\end{smallmatrix}\right]\right)^{-1}H\left[\begin{smallmatrix}g+1,g+2\\1,\dots,g+2\end{smallmatrix}\right],$$

and has rank 2.

Next, consider $\sigma(\boldsymbol{u})^2H$ at $\boldsymbol{u}\in(\sigma)$. We have

$$\sigma(\boldsymbol{u})^2h_{i,k}=4\sigma_{i-1}\sigma_{k-1}-2\sigma_i\sigma_{k-2}-2\sigma_{i-2}\sigma_k,$$

where $\sigma_i=(\partial/\partial u_i)\sigma(\boldsymbol{u})$, and, consequently, we obtain the decomposition

$$\sigma(\boldsymbol{u})^2H|_{\boldsymbol{u}\in(\sigma)}=2(\boldsymbol{s}_1,\boldsymbol{s}_2,\boldsymbol{s}_3)\begin{pmatrix}0&0&-1\\0&2&0\\-1&0&0\end{pmatrix}\begin{pmatrix}\boldsymbol{s}_1^T\\\boldsymbol{s}_2^T\\\boldsymbol{s}_3^T\end{pmatrix},$$

where $\boldsymbol{s}_1=(\sigma_1,\dots,\sigma_g,0,0)^T$, $\boldsymbol{s}_2=(0,\sigma_1,\dots,\sigma_g,0)^T$, $\boldsymbol{s}_3=(0,0,\sigma_1,\dots,\sigma_g)^T$. We infer that $\operatorname{rank}(\sigma(\boldsymbol{u})^2H)$ is 3 at generic points of (σ), and vanishes only when $\sigma_1=\dots=\sigma_g=0$, i.e., at the points $\in(\sigma)_{\text{sing}}$, while no other values are possible. □

Conclusion. The correspondence

$$\begin{aligned} h\colon \boldsymbol{u} \mapsto \big\{ & 4\sigma_{i-1}\sigma_{k-1} - 2\sigma_i\sigma_{k-2} - 2\sigma_{i-2}\sigma_k \\ & - \sigma(4\sigma_{i-1,k-1} - 2\sigma_{i,k-2} - 2\sigma_{i-2,k}) \\ & + \tfrac{1}{2}\sigma^2(\delta_{ik}(\lambda_{2i-2} + \lambda_{2k-2}) \\ & \qquad + \delta_{k,i+1}\lambda_{2i-1} + \delta_{i,k+1}\lambda_{2k-1})\big\}_{i,k=1,\dots,g+2}, \end{aligned}$$

induced by (3.13) establishes a meromorphic map of $(\mathrm{Jac}(V) \setminus (\sigma)_{\mathrm{sing}})/\pm$ into the space Q_3 of complex symmetric $(g+2) \times (g+2)$ matrices of rank not greater than 3.

We give an example of genus 2 with $\lambda_6 = 0$ and $\lambda_5 = 4$:

$$H = \begin{pmatrix} \lambda_0 & \frac{1}{2}\lambda_1 & -2\wp_{11} & -2\wp_{12} \\ \frac{1}{2}\lambda_1 & \lambda_2 + 4\wp_{11} & \frac{1}{2}\lambda_3 - 2\wp_{12} & -2\wp_{22} \\ -2\wp_{11} & \frac{1}{2}\lambda_3 - 2\wp_{12} & \lambda_4 + 4\wp_{22} & 2 \\ -2\wp_{12} & -2\wp_{22} & 2 & 0 \end{pmatrix}. \tag{3.19}$$

In this case, $(\sigma)_{\mathrm{sing}} = \{\varnothing\}$, so the Kummer surface in $\mathbb{CP}^3$ with coordinates

$$(X_0, X_1, X_2, X_3) = (\sigma^2, \sigma^2\wp_{11}, \sigma^2\wp_{12}, \sigma^2\wp_{22})$$

is defined by the equation $\det \sigma^2 H = 0$.

3.2.3. *Extended cubic relation.* A generalization ([**13**]) of (3.15) is given by the following theorem.

THEOREM 3.4.

$$\mathbf{R}^T \boldsymbol{\pi}_{jl}\boldsymbol{\pi}_{ik}^T \mathbf{S} = \frac{1}{4}\det\begin{pmatrix} H\left[\begin{smallmatrix} i,\,k.\,g+1,\,g+2 \\ j,\,l,\;g+1,\,g+2 \end{smallmatrix}\right] & \mathbf{S} \\ \mathbf{R}^T & 0 \end{pmatrix}, \tag{3.20}$$

where $\mathbf{R}, \mathbf{S} \in \mathbb{C}^4$ *are arbitrary vectors and*

$$\boldsymbol{\pi}_{ik} = \begin{pmatrix} -\wp_{ggk} \\ \wp_{ggi} \\ \wp_{g,i,k-1} - \wp_{g,i-1,k} \\ \wp_{g-1,i,k-1} - \wp_{g-1,k,i-1} + \wp_{g,k,i-2} - \wp_{g,i,k-2} \end{pmatrix}.$$

PROOF. The vectors $\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi}_{ik}$ and $\boldsymbol{\pi} = \boldsymbol{\pi}_{jl}$ solve the equations

$$H\left[\begin{smallmatrix} i,\,k,\,g+1,\,g+2 \\ j,\,l,\;g+1,\,g+2 \end{smallmatrix}\right]\boldsymbol{\pi} = 0, \qquad \tilde{\boldsymbol{\pi}}^T H\left[\begin{smallmatrix} i,\,k,\,g+1,\,g+2 \\ j,\,l,\;g+1,\,g+2 \end{smallmatrix}\right] = 0.$$

The theorem follows. □

The case of genus 2 was thoroughly studied by Baker ([**12**]). In this case $\boldsymbol{\pi}_{21} = (-\wp_{222}, \wp_{221}, -\wp_{211}, \wp_{111})^T$, exhausts all the possible $\wp_{ijk}$-functions.

§4. Applications

4.1. **Matrix realization of hyperelliptic Kummer varieties.** Here we present the explicit matrix realization (see [**14**]) of hyperelliptic Jacobians $\mathrm{Jac}(V)$ and Kummer varieties $\mathrm{Kum}(V)$ of the curves V with the fixed branching point $e_{2g+2} = a = \infty$. Our approach is based on the results of §3.2.

Let us consider the space $\mathcal{H}$ of complex symmetric $(g+2)\times(g+2)$-matrices $\mathrm{H} = \{\mathrm{h}_{k,s}\}$, with $\mathrm{h}_{g+2,g+2} = 0$ and $\mathrm{h}_{g+1,g+2} = 2$. Let us assign to $\mathrm{H} \in \mathcal{H}$ a symmetric $(g\times g)$-matrix $\mathrm{A(H)}$ with entries $a_{k,s} = \det \mathrm{H}\left[\begin{smallmatrix} k,g+1,g+2 \\ s,g+1,g+2\end{smallmatrix}\right]$.

It follows from Sylvester's identity (3.12) that the rank of the matrix $\mathrm{H} \in \mathcal{H}$ does not exceed 3 if and only if the rank of the matrix $\mathrm{A(H)}$ does not exceed 1.

Let us put $K\mathcal{H} = \{\mathrm{H} \in \mathcal{H} : \mathrm{rank}\,\mathrm{H} \leqslant 3\}$. For each complex symmetric $(g\times g)$-matrix $\mathrm{A} = \{a_{k,s}\}$ of rank not greater 1, there exists a g-dimensional column vector $\mathbf{z} = \mathbf{z}(\mathrm{A})$, defined up to sign such that $\mathrm{A} = -4\mathbf{z}\cdot\mathbf{z}^T$.

Let us introduce the vectors $\mathbf{h}_k = \{\mathrm{h}_{k,s};\ s = 1,\dots,g\} \in \mathbb{C}^g$.

LEMMA 4.1. *The map*

$$\gamma\colon K\mathcal{H} \to (\mathbb{C}^g/\pm)\times\mathbb{C}^g\times\mathbb{C}^g\times\mathbb{C}^1,$$
$$\gamma(\mathrm{H}) = -(\mathbf{z}(\mathrm{A(H)}), \mathbf{h}_{g+1}, \mathbf{h}_{g+2}, \mathrm{h}_{g+1,g+1})$$

is a homeomorphism.

PROOF. The claim follows from the relation:

$$4\widehat{\mathrm{H}} = 4\mathbf{z}\cdot\mathbf{z}^T + 2(\mathbf{h}_{g+2}\mathbf{h}_{g+1}^T + \mathbf{h}_{g+1}\mathbf{h}_{g+2}^T) - \mathrm{h}_{g+1,g+1}\,\mathbf{h}_{g+2}\mathbf{h}_{g+2}^T,$$

where $\widehat{\mathrm{H}}$ is the matrix composed of the column vectors $\mathbf{h}_k$, $k = 1,\dots,g$, and $\mathbf{z} = \mathbf{z}(\mathrm{A(H)})$. □

Let us introduce the 2-sheeted ramified covering $\pi\colon J\mathcal{H} \to K\mathcal{H}$, which is induced by the map γ from the covering $\mathbb{C}^g \to (\mathbb{C}^g/\pm)$.

COROLLARY 4.1.1. $\widehat{\gamma}\colon J\mathcal{H} \cong \mathbb{C}^{3g+1}$.

Now let us consider the universal space W_g of gth symmetric powers of hyperelliptic curves

$$V = \left\{(y,x)\in\mathbb{C}^2 : y^2 = 4x^{2g+1} + \sum_{k=0}^{2g}\lambda_{2g-k}x^{2g-k}\right\}$$

as an algebraic subvariety in $(\mathbb{C}^2)^g \times \mathbb{C}^{2g+1}$ with coordinates

$$\{((y_1,x_1),\dots,(y_g,x_g)), \lambda_{2g},\dots,\lambda_0\},$$

where $(\mathbb{C}^2)^g$ is gth symmetric power of the space $\mathbb{C}^2$. Define the map

$$\lambda\colon J\mathcal{H} \cong \mathbb{C}^{3g+1} \to (\mathbb{C}^2)^g \times \mathbb{C}^{2g+1}$$

in the following way:

- for $G = (\mathbf{z}, \mathbf{h}_{g+1}, \mathbf{h}_{g+2}, \mathrm{h}_{g+1,g+1}) \in \mathbb{C}^{3g+1}$ construct the matrix $\pi(G) = \mathrm{H} = \{\mathrm{h}_{k,s}\} \in K\mathcal{H}$ using Lemma 4.1;

- put
$$\lambda(G) = \{(y_k, x_k), \lambda_r;\ k = 1, \dots, g,\ r = 0, \dots, 2g\},$$
where $\{x_1, \dots, x_g\}$ is the set of roots of the equation $2x^g + \mathbf{h}_{g+2}^T \mathbf{X} = 0$, and $y_k = \mathbf{z}^T \mathbf{X}_k$, and $\lambda_r = \sum_{i+j=r+2} \mathrm{h}_{i,j}$.

Here $\mathbf{X}_k = (1, x_k, \dots, x_k^{g-1})^T$.

THEOREM 4.2. *The map* λ *induces a map* $J\mathcal{H} \cong \mathbb{C}^{3g+1} \to W_g$.

PROOF. A direct verification shows that

$$\mathbf{X}_k^T \,\mathrm{A}\, \mathbf{X}_s + 4 \sum_{i,j=1}^{g+2} \mathrm{h}_{i,j}\, x_k^{i-1} x_s^{j-1} = 0,$$

where $\mathrm{A} = \mathrm{A}(\mathrm{H})$ and $\mathrm{H} = \pi(G)$. Putting $k = s$ and using $\mathrm{A} = 4\mathbf{z} \cdot \mathbf{z}^T$, we obtain

$$y_k^2 = 4x_k^{2g+1} + \sum_{s=0}^{2g} \lambda_{2g-s} x_k^{2g-s}. \quad \square$$

Now everything is ready to give the description of our realization of the varieties $T^g = \mathrm{Jac}(V)$ and $K^g = \mathrm{Kum}(V)$ of hyperelliptic curves.

For each nonsingular curve

$$V = \left\{ (y, x), y^2 = 4x^{2g+1} + \sum_{s=0}^{2g} \lambda_{2g-s} x^{2g-s} \right\}$$

define the map

$$\gamma \colon T^g \setminus (\sigma) \to \mathcal{H} \colon \gamma(u) = \mathrm{H} = \{\mathrm{h}_{k,s}\},$$

where

$$\begin{aligned} \mathrm{h}_{k,s} &= 4\wp_{k-1,s-1} - 2(\wp_{s,k-2} + \wp_{s-2,k}) \\ &\quad + \tfrac{1}{2}[\delta_{ks}(\lambda_{2s-2} + \lambda_{2k-2}) + \delta_{k+1,s}\lambda_{2k-1} + \delta_{k,s+1}\lambda_{2s-1}]. \end{aligned}$$

THEOREM 4.3. *The map* γ *induces a map* $T^g \setminus (\sigma) \to K\mathcal{H}$ *such that* $\wp_{ggk}\wp_{ggs} = \frac{1}{4} a_{ks}(\gamma(u))$, *i.e.,* γ *is lifted to*

$$\tilde{\gamma} \colon T^g \setminus (\sigma) \to J\mathcal{H} \cong \mathbb{C}^{3g+1} \quad \textit{with } \mathbf{z} = (\wp_{gg1}, \dots, \wp_{ggg})^T.$$

The composition of maps $\lambda\tilde{\gamma} \colon T^g \setminus (\sigma) \to W_g$ *defines the inversion of the Abel map* $\mathfrak{A} \colon (V)^g \to T^g$ *and, therefore, the map* $\tilde{\gamma}$ *is an embedding.*

So we have obtained an explicit realization of the Kummer variety $T^g \setminus (\sigma)/\pm$ of the hyperelliptic curve V of genus g as a subvariety in the variety of matrices $K\mathcal{H}$. In particular, as a consequence of Theorem 4.3 we get a new proof of the theorem by Dubrovin and Novikov about the rationality of the universal space of the Jacobians of hyperelliptic curves V of genus g with fixed branching point $e_{2g+2} = \infty$ (**[27]**).

4.2. **Matrix operators satisfying the zero curvature condition and the generalized shift.** The theory of Kleinian functions developed above yields the explicit description of a family of operators satisfying the zero curvature condition (see Theorem 4.4 below). Further in this subsection, we present a new construction that assigns to this family a parametric family with the same property. This construction is based on the generalized shift operator approach.

We introduce the family of matrices

$$\mathcal{A}_k = \begin{pmatrix} B_k & A_k \\ C_k & -B_k \end{pmatrix}$$

where

$$A_k = \delta_{k,g} - \wp_{g,k+1}(\boldsymbol{u}),$$

$$B_k = -\frac{1}{2}\frac{\partial}{\partial u_g} A_k = \frac{1}{2}\wp_{gg,k+1}(\boldsymbol{u}), \qquad C_k = -\frac{1}{2}\frac{\partial^2}{\partial u_g^2} A_k + \left(2\wp_{gg}(\boldsymbol{u}) + \frac{\lambda_{2g}}{4}\right) A_k .$$

THEOREM 4.4. *Let $\partial_k = \partial/\partial u_k$. Then the family of matrices $\{\mathcal{A}_j\}$ satisfies the zero curvature condition*:

$$[\mathcal{A}_k, \mathcal{A}_i] = \partial_k \mathcal{A}_i - \partial_i \mathcal{A}_k .$$

PROOF. The required relations are verified directly using identities (3.7) and (3.8). For example, according to (3.7),

$$\begin{aligned} &\partial_k A_i - \partial_i A_k - 2(A_i B_k - A_k B_i) \\ &\quad = \wp_{gk,i+1} - \wp_{gi,k+1} - \wp_{gg,k+1}(\delta_{g,i} - \wp_{g,i+1}) + \wp_{gg,i+1}(\delta_{g,k} - \wp_{g,k+1}) \equiv 0. \end{aligned}$$

COROLLARY 4.4.1. *Let $L(\xi) = \sum_{k\geqslant 0} \xi^k \mathcal{A}_k$ and $\partial^\xi = \sum_{k=1}^{g} \xi^k \partial_k$. Then*

$$[L(\xi_1), L(\xi_2)] = \partial^{\xi_1} L(\xi_2) - \partial^{\xi_2} L(\xi_1) .$$

Note that Theorem 4.4 has a direct geometric interpretation in terms of the embeddings (constructed in the previous section) of the Jacobians in matrix spaces.

Let us introduce the shift operator $\mathcal{D}_x^\xi$ by setting

$$\mathcal{D}_x^\xi G(x) = \frac{xG(x) - \xi G(\xi)}{x - \xi},$$

where the subscript indicates the shifted argument and the superscript indicates the size of the shift.

LEMMA 4.5. *The operator $\mathcal{D}_x^\xi$ defines a commutative generalized shift, i.e., it satisfies the associativity equation*

$$\mathcal{D}_{\xi_1}^{\xi_2} \mathcal{D}_x^{\xi_1} = \mathcal{D}_x^{\xi_1} \mathcal{D}_x^{\xi_2} .$$

The proof follows immediately from the definition.

PROPOSITION 4.6. *On the space of functions regular at $x=0$, the action of $\mathcal{D}_x^\xi G(x)$ can be expressed as*

$$\mathcal{D}_x^\xi G(x) = \sum_{k\geqslant 0} \xi^k D_k G(x),$$

where the operators D_k are invariant with respect to the shift $\mathcal{D}_x^\xi$ and coincide with those introduced in Definition 2.

Note that $D_k = D_1^k$. The action of $\mathcal{D}_x^\xi$ can be extended to matrices whose entries are functions of x and is given by the same formula.

Let us introduce the matrix $\mathcal{L}(\xi,x)$ of the form

$$\mathcal{L}(\xi,x) = \mathcal{D}_x^\xi(L(x)+G(\xi,x)), \quad \text{where } G(\xi,x) = \left[\sum_{i>0}(x^i-\xi^i)A_i\right]\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}.$$

The coefficients of the decomposition of $\mathcal{L}(\xi,x)$ determine the operators $L_k(x)$:

$$\mathcal{L}(\xi,x) = \sum_{k\geqslant 0}\xi^k L_k(x); \tag{4.1}$$

note that $L_k(x)$ vanishes for $k>g$.

THEOREM 4.7. *For such a matrix $\mathcal{L}(\xi,x)$ and for the vector field*

$$\partial^\xi = \sum_{k=1}^{g} \xi^k \frac{\partial}{\partial u_k}$$

we have

$$\partial^{\xi_1}\mathcal{L}(\xi_2,x) - \partial^{\xi_2}\mathcal{L}(\xi_1,x) = [\mathcal{L}(\xi_1,x),\mathcal{L}(\xi_2,x)].$$

PROOF. The definition of the generalized shift implies

$$\begin{aligned}
[\mathcal{L}(\xi_1,x_1),\mathcal{L}(\xi_2,x_2)] &= \mathcal{D}_{x_1}^{\xi_1}\mathcal{D}_{x_2}^{\xi_2}[L(x_1)+G(\xi_1,x_1),L(x_2)+G(\xi_2,x_2)],\\
\partial^{\xi_1}\mathcal{L}(\xi_2,x_2) &= \mathcal{D}_{x_1}^{\xi_1}\mathcal{D}_{x_2}^{\xi_2}\partial^{\xi_1}(L(x_2)+G(\xi_2,x_2)),\\
\partial^{\xi_2}\mathcal{L}(\xi_1,x_1) &= \mathcal{D}_{x_1}^{\xi_1}\mathcal{D}_{x_2}^{\xi_2}\partial^{\xi_2}(L(x_1)+G(\xi_1,x_1)).
\end{aligned}$$

Put

$$\begin{aligned}
F(\xi_1,x_1,\xi_2,x_2) = {}&[L(x_1)+G(\xi_1,x_1),L(x_2)+G(\xi_2,x_2)]\\
&- \partial^{\xi_1}(L(x_2)+G(\xi_2,x_2)) + \partial^{\xi_2}(L(x_1)+G(\xi_1,x_1)).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathcal{D}_{x_1}^{\xi_1}\mathcal{D}_{x_2}^{\xi_2}F(\xi_1,x_1,\xi_2,x_2)\\
&\quad= \frac{1}{(x_1-\xi_2)(x_2-\xi_1)}\,[x_1x_2F(\xi_1,x_1,\xi_2,x_2) - \xi_1x_2F(\xi_1,\xi_1,\xi_2,x_2)\\
&\qquad - x_1\xi_2F(\xi_1,x_1,\xi_2,\xi_2) + \xi_1\xi_2F(\xi_1,\xi_1,\xi_2,\xi_2)].
\end{aligned}$$

Obviously, we have

LEMMA 4.8.
$$\mathcal{D}_{x_1}^{\xi_1}\mathcal{D}_{x_2}^{\xi_2}F(\xi_1,x_1,\xi_2,x_2)|_{x_1=x_2=x}=0$$
if and only if
$$F(\xi_1,\xi_1,\xi_2,\xi_2)=0$$
and at the same time
$$xF(\xi_1,x,\xi_2,x)-\xi_1F(\xi_1,\xi_1,\xi_2,x)-\xi_2F(\xi_1,x,\xi_2,\xi_2)=0\,.$$

The proof of the theorem is a direct verification of the assumptions of Lemma 4.8. First, the condition $F(\xi_1,\xi_1,\xi_2,\xi_2)=0$ holds by Corollary 4.4.1. The second assumption of the lemma is equivalent to
$$\sum x\{(\partial^{\xi_2}-\partial^{\xi_1})(L(x)+G(\xi_2,\xi_1))+[L(x),G(\xi_2,\xi_1)]\}=0,$$
where the sum is over all the cyclic permutations of the triples $\{x,\xi_1,\xi_2\}$. The matrix elements (i,j) of the functions under the summation sign can be rewritten in the form
$$\begin{aligned}
(1,1):\quad & x^{g+1}(\xi_1^{g+1}-\xi_2^{g+1})+(x^{g+1}(\partial^{\xi_2}-\partial^{\xi_1})+(\xi_1^{g+1}-\xi_2^{g+1})\partial^x)\zeta_g\\
&+((\partial^{\xi_2}-\partial^{\xi_1})\zeta_g)\partial^x\zeta_g-(1/2)(\partial^{\xi_2}-\partial^{\xi_1})\partial^x\wp_{gg};\\
(1,2):\quad & (\partial^{\xi_2}-\partial^{\xi_1})\partial^x\zeta_g;\\
(2,1):\quad & (\partial^{\xi_2}-\partial^{\xi_1})[x(2\wp_{gg}+\lambda_{2g}/4)+(1/2)\partial_g\partial^x(\wp_{gg})]\\
&+(x(\partial^{\xi_2}-\partial^{\xi_1})-(\partial^x\wp_{gg}))[\xi_1^{g+1}-\xi_2^{g+1}+(\partial^{\xi_2}-\partial^{\xi_1})\zeta_g];\\
(2,2)&=-(1,1),
\end{aligned}$$
where we have used the relation $A_i=-\partial_{i+1}\zeta_g(\boldsymbol{u})$; after summation, we see that the assumptions of the lemma hold. The theorem is proved. □

COROLLARY 4.8.1. *The parametric family of matrices* $\{L_j(x)\}$ *also satisfies the zero curvature condition*:
$$[L_k(x),L_i(x)]=\partial_kL_i(x)-\partial_iL_k(x)\,.$$

Thus, applying the generalized shift $\mathcal{D}_x^{\xi}$ to the generating function $L(x)$ of the matrix family $\mathcal{A}_1,\dots,\mathcal{A}_g$ shifted by the gauge summand $G(\xi,x)$, we obtain the generating function of the matrix family $L_0(x),\dots,L_g(x)$ that depends on a parameter, and the family thus obtained also satisfies the zero curvature condition.

The above result also solves the following problem: for a given family of operators satisfying the zero curvature condition, construct a generalized shift operator, which (after a gauge correction) takes this family to a new family satisfying the same condition for all values of the parameter.

4.3. **The hyperelliptic Φ-function.** In this section we construct linear differential operators for which the hyperelliptic curve $V(y,x)$ is the spectral variety.

DEFINITION 3. The Φ-function of the curve $V(y,x)$ with fixed point a is
$$\Phi\colon\mathbb{C}\times\mathrm{Jac}(V)\times V\to\mathbb{C},$$
$$\Phi(u_0,\boldsymbol{u};(y,x))=\frac{\sigma(\boldsymbol{\alpha}-\boldsymbol{u})}{\sigma(\boldsymbol{\alpha})\sigma(\boldsymbol{u})}\exp\Big(-\frac{1}{2}yu_0+\boldsymbol{\zeta}^T(\boldsymbol{\alpha})\boldsymbol{u}\Big),$$
where $\boldsymbol{\zeta}^T(\boldsymbol{\alpha})=(\zeta_1(\boldsymbol{\alpha}),\dots,\zeta_g(\boldsymbol{\alpha}))$ and $(y,x)\in V$, $\boldsymbol{u}$ and $\boldsymbol{\alpha}=\int_a^x\mathbf{du}\in\mathrm{Jac}(V)$.

In particular, $\Phi(0, \boldsymbol{u}; (y,x))$ is the Baker function (see [**11**, p. 421] and [**28**]). The following theorem recovers the fundamental result of finite-gap integration theory ([**29**]) in the framework of the Kleinian σ-functions.

THEOREM 4.9. *The function* $\Phi = \Phi(u_0, \boldsymbol{u}; (y,x))$ *solves the Schrödinger equation with the potential* $2\wp_{gg}$,

$$(\partial_g^2 - 2\wp_{gg})\,\Phi = (x + \lambda_{2g}/4)\,\Phi, \tag{4.2}$$

with respect to u_g *for all* $(y,x) \in V$.

PROOF. From (3.5),

$$\partial_g \Phi = \frac{y + \partial_g \mathcal{P}(x;\boldsymbol{u})}{2\mathcal{P}(x;\boldsymbol{u})}\,\Phi,$$

where $\mathcal{P}(x;\boldsymbol{u})$ is given by (2.12). Hence,

$$\frac{\partial_g^2\Phi}{\Phi} = \frac{y^2 - (\partial_g\mathcal{P}(x;\boldsymbol{u}))^2 + 2\mathcal{P}(x;\boldsymbol{u})\,\partial_g^2\mathcal{P}(x;\boldsymbol{u})}{4\mathcal{P}^2(x;\boldsymbol{u})},$$

and by (3.9) and (3.6) we obtain the theorem. □

Let us introduce the vector $\boldsymbol{\Psi}^T = (\Phi, \Phi_g)$, where Φ_g stands for $\partial_g\Phi$.

THEOREM 4.10. *For every genus* $g \geqslant 1$, *the vectors* $\boldsymbol{\Psi} = \boldsymbol{\Psi}(u_0, \boldsymbol{u}; (y,x))$ *satisfy the equations*

$$\partial_k \boldsymbol{\Psi} = L_k(x)\,\boldsymbol{\Psi}, \qquad k = 1, \dots, g, \tag{4.3}$$

where $L_k(x)$ *is defined by* (4.1).

The system (4.3) has a solution by Corollary 4.8.1.

PROOF. According to the results of the previous subsection, the entries of the matrix $L_k(x)$ have the form

$$L_0(x) = \begin{pmatrix} V_0 & U_0 \\ W_0 & -V_0 \end{pmatrix}, \qquad L_k(x) = D_k L_0(x) - \begin{pmatrix} 0 & 0 \\ \wp_{gk} & 0 \end{pmatrix},$$

where

$$U_0 = x^g - \sum_{i=1}^{g} x^{i-1}\wp_{gi}, \qquad V_0 = -\frac{1}{2}\,\partial_g U_0, \tag{4.4}$$

and

$$W_0 = -\frac{1}{2}\,\partial_g^2 U_0 + \left(x + 2\wp_{gg} + \frac{1}{4}\,\lambda_{2g}\right) U_0 .$$

Note that $U_0 = \mathcal{P}(x;\boldsymbol{u})$. On the other hand, (3.5) implies

$$\frac{\Phi_j}{\Phi} = \frac{(yD_j + \partial_j)\,\mathcal{P}(x;\boldsymbol{u})}{2\mathcal{P}(x;\boldsymbol{u})} = \frac{(yD_j + \partial_j)\,U_0}{2U_0},$$

so that using the identity

$$(\partial_j U_0) + (\partial_g U_0)(D_j U_0) - U_0(\partial_g(D_j U_0)) = 0,$$

which is valid by (3.7), we obtain

$$\Phi_j = U_j \Phi_g + \Phi V_j .$$

Differentiating the last equation with respect to u_g and expressing Φ_{gg} using (4.2), we conclude that

$$\Phi_{gj} = -V_j \Phi_g + \Phi W_j ,$$

which proves the theorem.

COROLLARY 4.10.1. *Let the vector* $\boldsymbol{u}^T = (u_1, \dots, u_g)$ *be identified with the vector* $(t_g, \dots, t_3, t_2 = t, t_1 = z)$. *Then the function* $\Phi = \Phi(u_0, \boldsymbol{u}; (y, x))$ *solves the problem*

$$\left[\begin{pmatrix} 0 & \partial/\partial z \\ \partial/\partial t & 0 \end{pmatrix} - \begin{pmatrix} \mathcal{U}(z,t) & 0 \\ -\mathcal{U}_z(z,t)/4 & -\mathcal{U}(z,t)/2 + \lambda_{2g}/8 \end{pmatrix} \right] \begin{pmatrix} \Phi \\ \Phi_z \end{pmatrix} = x \begin{pmatrix} \Phi \\ \Phi_z \end{pmatrix},$$

where $\mathcal{U}(z,t) = 2\wp_{gg} + \lambda_{2g}/4$ *and subscript means differentiation.*

THEOREM 4.11. *The function* $\Phi = \Phi(u_0, \boldsymbol{u}; (y, x))$ *solves the system of equations*

$$(\partial_k \partial_l - \gamma_{kl}(x, \boldsymbol{u})\, \partial_g + \beta_{kl}(x, \boldsymbol{u}))\, \Phi = \frac{1}{4}\, D_{k+l}(f(x))\, \Phi$$

with polynomials in x

$$\gamma_{kl}(x, \boldsymbol{u}) = \frac{1}{4}\, [\partial_k D_l + \partial_l D_k] \sum_{i=1}^{g+2} x^{i-1} h_{g+2,i},$$

$$\beta_{kl}(x, \boldsymbol{u}) = \frac{1}{8}\, [(\partial_g \partial_k + h_{g+2,k})\, D_l + (\partial_g \partial_l + h_{g+2,l})\, D_k] \sum_{i=1}^{g+2} x^{i-1} h_{g+2,i}$$
$$- \frac{1}{4} \sum_{j=k+l+2}^{2g+2} x^{j-(k+l+2)} \left[\left(\sum_{\nu=1}^{k+1} h_{\nu, j-\nu} \right) + \left(\sum_{\mu=1}^{l+1} h_{j-\mu, \mu} \right) \right]$$

for all $k, l \in 0, \dots, g$ *and arbitrary* $(y, x) \in V$.

The most remarkable aspect of the equations of Theorem 4.11 is the balance of degrees of the polynomials γ_{kl}, β_{kl} and of the "spectral part", the umbral derivative $D_{k+l}(f(x))$:

$$\deg_x \gamma_{kl}(x, \boldsymbol{u}) \leqslant g - 1 - \min(k, l),$$
$$\deg_x \beta_{kl}(x, \boldsymbol{u}) \leqslant 2g - (k + l),$$
$$\deg_x D_{k+l}(f(x)) = 2g + 1 - (k + l) .$$

Here $f(x)$ is as given in (1.1) with $\lambda_{2g+2} = 0$ and $\lambda_{2g+1} = 4$.

Note that the definition of the functions $\{h_{i,k}\}$ and the help of relations (3.14)–(3.15) readily yields the following formula, which re-expresses the functions $\{h_{i,k}\}_{i,k \leqslant g}$ in terms of the basis functions $\{\wp_{gj}, \wp_{ggj}, \wp_{gggj}\}$ and of the constant λ_{2g}:

$$h_{i,k} = (8\wp_{gg} + \lambda_{2g})\, \wp_{gi} \wp_{gk} + 2\wp_{gi} \wp_{g,k-1} + 2\wp_{g,i-1} \wp_{gk}$$
$$+ \wp_{ggi} \wp_{ggk} - \wp_{gggi} \wp_{gk} - \wp_{gi} \wp_{gggk} .$$

Thus all the coefficients in the differential equations from Theorem 4.11 are polynomials in x and in the basis functions $\{\wp_{gj}, \wp_{ggj}, \wp_{gggj}\}$.

PROOF. The construction of the operators L_k yields

$$\Phi_{lk} = \tfrac{1}{2}(\partial_l U_k + \partial_k U_l)\,\Phi_g + (V_l V_k + \tfrac{1}{2}(\partial_l V_k + \partial_k V_l + U_k W_l + W_k U_l))\,\Phi\,.$$

To prove the theorem, we use (4.3) and notice that (cf. Lemma 4.1):

$$\begin{aligned} &D_k(V_0)\,D_l(V_0) + \tfrac{1}{2}D_k(U_0)\,D_l(W_0) + \tfrac{1}{2}D_l(U_0)\,D_k(W_0) \\ &\quad = -\tfrac{1}{16}\big(\det H\big[\begin{smallmatrix} g+1,\, g+2 \\ g+1,\, g+2 \end{smallmatrix}\big]\big)(1, x, \ldots, x^{g+1-k})H\big[\begin{smallmatrix} l,\, \ldots,\, g+2 \\ k,\, \ldots,\, g+2 \end{smallmatrix}\big](1, x, \ldots, x^{g+1-l})^T; \end{aligned}$$

having in mind the relations $h_{g+2,g+2} = 0$ and $h_{g+2,g+1} = 2$, we obtain the theorem from properties of the matrix H. □

As an example of the application of Theorem 4.11, consider the genus 2 case:

$$\begin{aligned} &(\partial_2^2 - 2\wp_{22})\,\Phi = \tfrac{1}{4}(4x + \lambda_4)\,\Phi, \\ (4.5)\qquad &(\partial_2\partial_1 + \tfrac{1}{2}\wp_{222}\partial_2 - \wp_{22}(x + \wp_{22} + \tfrac{1}{4}\lambda_4) + 2\wp_{12})\,\Phi = \tfrac{1}{4}(4x^2 + \lambda_4 x + \lambda_3)\,\Phi, \\ &(\partial_1^2 + \wp_{122}\partial_2 - 2\wp_{12}(x + \wp_{22} + \tfrac{1}{4}\lambda_4))\,\Phi = \tfrac{1}{4}(4x^3 + \lambda_4 x^2 + \lambda_3 x + \lambda_2)\,\Phi\,. \end{aligned}$$

Now the function $\Phi = \Phi(u_0, u_1, u_2; (y, x))$ of the curve

$$y^2 = 4x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

solves these equations for all x.

Excluding x from the left-hand sides of equations (4.5) one after the other, we obtain the equivalued system

(4.6)

$$\begin{aligned} \Lambda_{22}\Phi &= (\partial_2^2 - 2\wp_{22})\,\Phi = \tfrac{1}{4}(4x + \lambda_4)\,\Phi, \\ \Lambda_{12}\Phi &\doteq (\partial_2\partial_1 - \wp_{22}\partial_2^2 + \tfrac{1}{2}\wp_{222}\partial_2 + \wp_{22}^2 - 2\wp_{12})\,\Phi = \tfrac{1}{4}(4x^2 + \lambda_4 x + \lambda_3)\,\Phi, \\ \Lambda_{11}\Phi &= (\partial_1^2 - 2\wp_{12}\partial_2^2 + \wp_{122}\partial_2 + 2\wp_{22}\wp_{12})\Phi = \tfrac{1}{4}(4x^3 + \lambda_4 x^2 + \lambda_3 x + \lambda_2)\,\Phi\,. \end{aligned}$$

This example illustrates the general fact that for the systems of equations described in Theorem 4.11, one can iteratively exclude the dependence on x in their left-hand sides.

COROLLARY 4.11.1. *Calculating the commutators of the triple of operators* Λ_{22}, Λ_{12}, Λ_{11}, *we find*

$$[\Lambda_{12}, \Lambda_{22}] = -2\wp_{222}A, \quad [\Lambda_{11}, \Lambda_{22}] = -4\wp_{122}A, \quad [\Lambda_{11}, \Lambda_{12}] = -6\wp_{112}A,$$

where

$$A = \partial_1 - \partial_2^3 + (3\wp_{22} + \tfrac{1}{4}\lambda_4)\partial_2 + \tfrac{3}{2}\wp_{222}\,.$$

The operator A *is the difference between* ∂_1 *and the positive part of the formal fractional power* $[(L)^{3/2}]_+$ *of the operator* $L = \partial_2^2 - (2\wp_{22} + \frac{1}{6}\lambda_4)$, *i.e., it is the* A-*operator of the* (L, A)-*pair of the classical KdV equation with respect to the function* $2\wp_{22} + \frac{1}{6}\lambda_4$.

This corollary illustrates the following general property of the operators $\{\Lambda_{k,l}\}$ obtained by excluding x from the equations of Theorem 4.11: commutation yields operators from Lax pairs of integrable systems.

4.4. **Solution of KdV system by Kleinian functions.** The KdV system is the infinite hierarchy of differential equations

$$\mathcal{U}_{t_k} = \mathcal{X}_k[\mathcal{U}],$$

for the function $\mathcal{U}(t_1, t_2, t_3, \dots)$. One usually puts $t_1 = z$ and $t_2 = t$. The first two equations of the hierarchy have the form

$$\mathcal{U}_{t_1} = \mathcal{U}_z \quad \text{and} \quad \mathcal{U}_{t_2} = \tfrac{1}{2}(\mathcal{U}_{zzz} - 6\mathcal{U}\mathcal{U}_z),$$

the second being the classical Korteweg–de Vries equation. The higher KdV equations are given by the relations

$$\mathcal{X}_{k+1}[\mathcal{U}] = \mathcal{R}\mathcal{X}_k[\mathcal{U}],$$

where $\mathcal{R} = \frac{1}{4}\partial_z^2 - (\mathcal{U} + c) - \frac{1}{2}\mathcal{U}_z\partial_z^{-1}$ is the Lenard recursion operator, and c is a constant.

The KdV hierarchy is a well-known object in the theory of integrable systems, studied in many papers, beginning with [**34**].

The theory of Kleinian functions constructed above yields explicit solutions to the KdV equations so that these solutions directly depend on the canonical coordinates of the Jacobian of a hyperelliptic curve. Identifying time variables $(t_1, t_2, \dots, t_g) \to (u_g, u_{g-1}, \dots, u_1)$ and the constant $c = \lambda_{2g}/12$, we have

THEOREM 4.12. *The function $\mathcal{U} = 2\wp_{gg}(\boldsymbol{u}) + \lambda_{2g}/6$ is a g-gap solution of the KdV system.*

PROOF. Indeed, we have $\mathcal{U}_z = \partial_g 2\wp_{gg}$ and by (3.6)

$$\mathcal{U}_{t_2} = \partial_{g-1} 2\wp_{gg} = \tfrac{1}{2}(\wp_{ggggg} - (12\wp_{gg} + \lambda_{2g})\,\wp_{ggg}).$$

The action of $\mathcal{R}$

$$\partial_{g-i-1} 2\wp_{gg} = [\tfrac{1}{4}\partial_g^2 - (2\wp_{gg} + \tfrac{1}{4}\lambda_{2g})]\,2\wp_{gg,g-i} - 2\wp_{ggg}\wp_{g,g-i}$$

is verified by (3.6) and (3.7).

On the gth step of recursion, the “times” u_i are exhausted and the stationary equation $\mathcal{X}_{g+1}[\mathcal{U}] = 0$ appears. A periodic solution of $g+1$ higher stationary equation is a g-gap potential (see [**29**]).

4.5. **Solution of the Sine-Gordon equation in Kleinian functions.** Consider the following system

$$(4.7) \qquad \begin{cases} \dfrac{\partial^2}{\partial z_1 \partial t}\,\varphi(z_1, z_2, t) = 2\beta \sin\varphi(z_1, z_2, t) - \alpha\psi(z_1, z_2, t)\, e^{\{-2i\varphi(z_1, z_2, t)\}}, \\ \dfrac{\partial}{\partial z_1}\,\psi(z_1, z_2, t) = \dfrac{1}{2}\,\beta e^{\{i\varphi(z_1, z_2, t)\}}\,\dfrac{\partial}{\partial z_2}\,\varphi(z_1, z_2, t) \end{cases}$$

for the two functions $\varphi(z_1, z_2, t)$ and $\psi(z_1, z_2, t)$. For $\alpha = 0$ this system splits into two independent equations, the first of which is the well-known Sine-Gordon (SG) equation in the coordinates of the light cone. In the general case ($\alpha \neq 0$) this system is a weak two-dimensional analog of the SG equation.

The theory of Kleinian functions yields explicit solutions of this system.

Consider the correspondence of coordinates $(z_1, z_2, t) \to (u_1, u_2, u_g)$.

THEOREM 4.13. *Let* $\alpha = \lambda_0/\lambda_1$ *and* $\beta = \sqrt{\lambda_1}$. *Then the pair of functions* $\varphi = -i\log(2\wp_{1,g}(\boldsymbol{u})/\sqrt{\lambda_1})$ *and* $\psi = i\wp_{2,g}(\boldsymbol{u})$ *is* $2g$*-periodic finite-gap solution to system* (4.5) *for every genus* $g \geqslant 2$.

PROOF. By (3.6)–(3.9) we have

$$\partial_1\partial_g \log\wp_{1,g} = 2\wp_{1,g} - \frac{\lambda_1}{2\wp_{1,g}} + \frac{\lambda_0}{\wp_{1,g}^2}\wp_{2,g},$$

which ensures that the first equation in (4.5) holds; the second one is equivalent to the relation $\partial_1\wp_{2,g} = \partial_2\wp_{1,g}$. □

The results of the last subsections demonstrate the possibilities of Kleinian functions in the contemporary theory of integrable systems. Note that the natural identification of the independent variables (see Theorems 4.12 and 4.13) with the canonical coordinates on the Jacobians yields explicit solutions in Kleinian functions of well-known systems in a form convenient for the study of these solutions and their applications. This is one of the important incentives for the further development of the theory of Kleinian functions.

§5. Concluding remarks

Several remarkable properties of the Kleinian functions are beyond the scope of our paper. We give some instructive examples for the case of genus 2 when $\boldsymbol{u}^T = (u_1, u_2)$:

- The addition theorem

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma^2(\boldsymbol{u})\sigma^2(\boldsymbol{v})} = \wp_{22}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) - \wp_{12}(\boldsymbol{u})\wp_{22}(\boldsymbol{v}) + \wp_{11}(\boldsymbol{v}) - \wp_{11}(\boldsymbol{u});$$

- The following expression, which can be interpreted as the Hirota bilinear relation:

$$\left\{\frac{1}{3}\Delta\Delta^T + \Delta^T\begin{pmatrix}0 & 0 & 1\\ 0 & -1/2 & 0\\ 1 & 0 & 0\end{pmatrix}\epsilon_{\eta,\eta}\epsilon_{\eta,\eta}\cdot\epsilon_{\eta,\eta}^T - (\xi-\eta)^4\epsilon_{\eta,\xi}\epsilon_{\eta,\xi}^T\right\}\sigma(\boldsymbol{u})\sigma(\boldsymbol{u}')\bigg|_{\boldsymbol{u}'=\boldsymbol{u}},$$

is identically 0, where $\Delta^T = (\Delta_1^2, 2\Delta_1\Delta_2, \Delta_2^2)$ with $\Delta_i = \partial/\partial u_i - \partial/\partial u_i'$ and also $\epsilon_{\xi,\eta}^T = (1, \eta+\xi, \eta\xi)$. After evaluation, the powers of parameters η and ξ are replaced according to the rules $\eta^k, \xi^k \to \lambda_k k!(6-k)!/6!$ by the constants defining the curve;

- For Kleinian σ-functions, the following operation is defined

$$\sigma(u_1, u_2) = \exp\left\{\frac{u_2}{u_1}\sum_{k=1}^{6} k\lambda_k\frac{\partial}{\partial\lambda_{k-1}}\right\}\sigma(u_1, 0),$$

which resembles the function executed by vertex operators.

A reference for these formulas is [**12**].

Another interesting problem is the reduction of hyperelliptic $\wp$-functions to lower genera. In the case of genus 2, this happens according to the Weierstrass theorem when the period matrix τ can be transformed to the form (see e.g. [**11**, **24**])

$$\tau = \begin{pmatrix} \tau_{11} & 1/N \\ 1/N & \tau_{22} \end{pmatrix},$$

where the so-called *Picard number* $N > 1$ is a positive integer. The associated Kummer surface in this case turns out to be the *Plücker surface.* Similar reductions were studied in [**30**] in order to single out elliptic potentials among the finite gap ones. Problems of this kind were treated in [**31–33**] by means of the spectral theory. We remark that the formalism of Kleinian functions makes the related calculations much easier and the solution more descriptive.

These and other problems of hyperelliptic Abelian functions will be discussed in our forthcoming publications.

In the forthcoming papers we intend to develop the following topics:

- For any genus g, to express

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma^2(\boldsymbol{u})\sigma^2(\boldsymbol{v})}$$

as a polynomial in the basis meromorphic functions on the hyperelliptic Jacobian;

- Using the relations between the $\wp$-functions and their derivatives obtained above, to obtain the differential addition theorem for $\wp = (\wp_{1,g}, \ldots, \wp_{gg})^T$;
- To continue an in-depth study of the relationship between the theory of generalized shift operators and algebro-geometric methods in the theory of integrable systems.

In conclusion we emphasize that the Kleinian construction of the hyperelliptic Abelian functions complements the theta functional realization; according to the authors' experience, the combination of the two approaches makes the whole picture more complete and descriptive.

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