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The w-Function of the KdV Hierarchy

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ABSTRACT. In this paper we construct a family of commuting multidimensional differential operators of order 3, which is closely related to the KdV hierarchy. We find a common eigenfunction of this family and an algebraic relation between these operators. Using these operators we associate a hyperelliptic curve to any solution of the stationary KdV equation. A basic generating function of the solutions of stationary KdV equation is introduced as a special polarization of the equation of the hyperelliptic curve. We also define and discuss the notion of a w-function of a solution of the stationary g-KdV equation.

Introduction

At the present time various forms of solutions of the stationary g-KdV equations are known, including the representations with the τ-function ([**13**]), θ-function ([**14, 15**]), and σ-function ([**2, 3**]); rational solutions can be expressed in terms of Adler-Moser polynomials ([**1**]). All these functions satisfy the equation

$$2\partial_x^2 \log f = -u, \tag{0.1}$$

where $u = u(x, t_2, \ldots, t_g)$ is a solution of the stationary g-KdV equation.

In this paper we construct a family of commuting multidimensional differential operators of third order starting with an arbitrary solution of the stationary g-KdV equation. Using these operators we solve the following well-known

PROBLEM 1. Supplement (0.1) with natural conditions so that the problem have a unique solution.

We call this solution a w-function of the KdV hierarchy.

In [**20**] Novikov observed that each solution of the stationary g-KdV equation is a g-gap potential of the Schrödinger operator. It was shown in [**2**], [**3**] that the Kleinian σ-function $\sigma(x, t_2, \ldots, t_g)$ provides a solution of the g-KdV equation. This fact follows from a general result describing all algebraic relations between the higher logarithmic derivatives of the σ-function.

We are going to discuss also the following natural

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PROBLEM 2. Describe all the relations between the higher logarithmic derivatives

$$\frac{\partial^{i_1+\cdots+i_g}}{\partial x^{i_1}\partial t_2^{i_2}\cdots\partial t_g^{i_g}}\log w(x,t_2,\dots,t_g), \quad \text{where } i_1+\cdots+i_g\geq 2,$$

following from the construction of the w-function of the KdV hierarchy.

A solution of this problem is given in Section 8.

In [**16**] Krichever introduced the concept of the Baker–Akhiezer function as a common eigenfunction of the operators $\mathcal{L}$ and A (see Section 1 for definitions). This function is characterized by its analytic properties, including the behavior at singular points. In Subsection 10.2 we express this function in terms of the common eigenfunction of our family of commuting differential operators.

The results of this paper were partially announced in [**5**], [**6**].

1. Preliminaries

This section is a brief review of basic facts about the KdV hierarchy. See [**19**] for more details.

The classical KdV (Korteweg–de Vries) equation is

$$\frac{\partial}{\partial t}u=\frac{1}{4}(u'''-6uu'), \tag{1.1}$$

where u is a function of real variables x and t; the prime means differentiation with respect to x.

Denote $\mathcal{L}=\partial_x^2-u$ the Schrödinger operator with the potential u. The second term here means *the operator of multiplication* by the function u; we will use similar notation throughout the paper. Let also

$$A_1=\partial_x^3-\frac{3}{4}(u\partial_x+\partial_x u)=\partial_x^3-\frac{3}{2}u\partial_x-\frac{3}{4}u'. \tag{1.2}$$

Then, as it was first noticed in [**18**], the KdV equation is equivalent to the condition

$$[A_1,\mathcal{L}]=-\frac{1}{4}(u'''-6uu').$$

Denote $\mathfrak{D}$ a ring of differential operators with coefficients in the ring of smooth functions in variables x and t. Consider the action of the operator $\partial/\partial t$ on the ring $\mathfrak{D}$ defined by the formula

$$\frac{\partial}{\partial t}\left(\sum_{k\geq 0}f_k(t,x)\partial_x^k\right)=\sum_{k\geq 0}\frac{\partial f_k(t,x)}{\partial t}\,\partial_x^k. \tag{1.3}$$

Then for the operator $\mathcal{L}$ we obtain the equality

$$\frac{\partial}{\partial t}\mathcal{L}=-\frac{\partial}{\partial t}u.$$

So, equation (1.1) is equivalent to

$$\frac{\partial}{\partial t}\mathcal{L}=[A_1,\mathcal{L}].$$

For every differential operator $B\in\mathfrak{D}$ define its formal conjugate B^* as follows: take, by definition,

$$\partial_x^*=-\partial_x,\quad f^*=f, \tag{1.4}$$

where f is an operator of multiplication by the function f, and assume * to be a ring anti-homomorphism:

$$(B_1 B_2)^* = B_2^* B_1^*, \quad (B_1 + B_2)^* = B_1^* + B_2^*, \tag{1.5}$$

for all $B_1, B_2 \in \mathfrak{D}$.

We call an operator B *symmetric* if $B^* = B$, and *anti-symmetric* if $B^* = -B$. Thus, the operator $\mathcal{L}$ is symmetric whereas the operator A_1 is anti-symmetric.

Consider the subring $\mathfrak{D}_1 \subset \mathfrak{D}$ generated by ∂_x and the multiplication operator u. Supply the ring $\mathfrak{D}_1$ with the grading such that

$$\deg u = 2, \quad \deg \partial_x = 1. \tag{1.6}$$

Thus, $\deg u^{(k)} = \deg \partial_x^k u = k + 2$. The operators $\mathcal{L}$ and A_1 are then homogeneous of degree 2 and 3, respectively.

DEFINITION 1.1. Denote by $\mathfrak{A}$ the linear space of anti-symmetric differential operators A such that the commutator $[A, \mathcal{L}]$ is an operator of multiplication by a function.

THEOREM 1.1 ([**18**]). *The space* $\mathfrak{A}$ *has a basis* $A_0, A_1, \dots,$ *where* $A_k = \partial_x^{2k+1} + \sum P_{k,i} \partial_x^i$ *is a homogeneous degree* $2k+1$ *differential operator of order* $2k+1$*, and* $P_{k,i}$ *is a differential polynomial in* u *of degree* $2k + 1 - i$.

The recurrence relation for the operators A_k can be found in [**8**]. The operator A_1 is given by (1.2). The operators A_0 and A_2 are

$$\begin{aligned} A_0 &= \partial_x, \\ A_2 &= \partial_x^5 - \frac{5}{4}(u\partial_x^3 + \partial_x^3 u) + \frac{15}{8} u \partial_x u + \frac{5}{16}(u'' \partial_x + \partial_x u''). \end{aligned}$$

Denote $r_k[u] = [A_k, \mathcal{L}]$, so that $r_1[u] = \frac{1}{4}(u''' - 6uu')$, $r_2[u] = \frac{1}{16}(u^{(5)} - 10uu''' - 20u'u'' + 30u^2 u')$, etc. Suppose now that u depends on x and an *infinite* set of variables $t_1, t_2, \dots$. The equation

$$\partial_{t_g} u = r_g[u] \tag{1.7}$$

is called the gth higher KdV equation.

The family of equations (1.7) is called the KdV hierarchy.

The action of differential operators ∂_{t_k} on the ring $\mathfrak{D}_u$ is defined similarly to (1.3).

LEMMA 1.1. *The operators* A_k *satisfy the following "zero curvature" condition:* $\partial_{t_k} A_m - \partial_{t_m} A_k = [A_k, A_m]$*, or, equivalently,* $[\partial_{t_k} - A_k, \partial_{t_m} - A_m] = 0$.

The expression

$$\mathcal{R} = \frac{1}{4}\partial_x^2 - \frac{1}{2} u' \partial_x^{-1} - u, \tag{1.8}$$

is called the Lenard operator; here ∂_x^{-1} is an operator of integration with respect to x. Note that the Lenard operator $\mathcal{R}$ is multivalued, and to fix its value we need to choose the integration constant.

THEOREM 1.2. *Functions* $r_k[u]$ *are related by the Lenard operator:*

$$r_{k+1}[u] = \mathcal{R}(r_k[u]).$$

For example, $r_0 = u'$, and $r_1 = \frac{1}{4}u''' - \frac{3}{2}uu' = \frac{1}{4}\partial_x^2 u' - \frac{1}{2}u'u - uu' = \mathcal{R}(r_0)$.

DEFINITION 1.2. The equations

$$r_g[u] + \sum_{k=0}^{g-1} a_k r_k[u] = 0, \tag{1.9}$$

where a_k are constants, are called higher stationary g-KdV (or Novikov) equations.

THEOREM 1.3. *A function u is a solution of (1.9) if and only if it satisfies the relation $\mathcal{R}^g(u') = 0$ for some choice of the integration constants; this choice depends on the constants a_k.*

See [9] for a proof.

If one replaces the function u with $u+c$ where c is a constant, then the operator A_k becomes $A_k + c\sum_{i=0}^{k-1} c_{k;i}A_i$ for some constants $c_{k;i}$ where $c_{k;k-1} \neq 0$. We can choose the constant c so that to achieve the equality $a_{g-1} = 0$ in the decomposition of the operator $A = A_g + \sum_{k=0}^{g-1} a_k A_k$.

2. A family of commuting multidimensional differential operators of order 3

In this section we describe a family $\{\mathcal{U}_k\}$ of differential operators commuting with each other and with the Schrödinger operator. In this aspect they resemble the operators $\partial_{k+1} - A_k$, but unlike $\{A_k\}$ they are multidimensional operators of the third order.

Let $\{u_1, u_2, \dots, u_g\}$ be a sequence of functions of variables $t_1 = x, t_2, \dots, t_g$. Denote $\partial_i = \partial/\partial t_i$. Suppose that the first derivatives of the function u_1 are linearly independent, i.e., $\sum_{i=1}^{g} c_i\partial_i(u_1) \equiv 0$ only if $c_1 = \dots = c_g = 0$. This condition means that the function $u_1 = u_1(t_1, \dots, t_g)$ essentially depends on all its arguments, i.e., there is no linear projection $\pi : \mathbb{C}^g \to \mathbb{C}^{g-1}$ such that $u = \pi^*\widetilde{u}$, where $\widetilde{u}$ is a function on $\mathbb{C}^{g-1}$.

Denote

$$\begin{aligned}
\mathcal{L} &= \partial_x^2 - u_1, \\
\mathcal{A}_k &= \partial_x^2\partial_k - \frac{1}{2}(u_1\partial_k + \partial_k u_1) - \frac{1}{4}(u_k\partial_x + \partial_x u_k),
\end{aligned}$$

where $k = 1, \dots, g$. Note that $\mathcal{A}_1$ coincides with the operator A_1 given by (1.2).

Define the formal conjugation * on the space of multidimensional differential operators by formulas (1.5) together with the rule $\partial_i^* = -\partial_i$ for $i > 1$. The operators $\mathcal{A}_k$ are anti-symmetric: $\mathcal{A}_k^* = -\mathcal{A}_k$.

In the ring of differential operators in variables $t_1, \dots, t_g$ consider the subring $\mathfrak{D}_g$ generated by the operators $\partial_1, \dots, \partial_g$ and $u_1, \dots, u_g$. Define the grading on $\mathfrak{D}_g$ using formulas (1.6) and assuming also that $\deg u_k = 2k$ and $\deg \partial_k = 2k - 1$. It is clear that the operators $\mathcal{L}$ and $\mathcal{A}_k$ are homogeneous, $\deg \mathcal{L} = 2$, $\deg \mathcal{A}_k = 2k + 1$.

LEMMA 2.1. *The commutator $[\mathcal{L}, \mathcal{A}_k]$ is a multiplication operator if and only if $u_k' = \partial_k u_1$ for all k. If this condition is satisfied, then*

$$[\mathcal{L}, \mathcal{A}_k] = \frac{1}{4}(u_k''' - 2u_1'u_k - 4u_1u_k'). \tag{2.1}$$

Proof. The assertion follows from the formula

$$[\mathcal{L}, \mathcal{A}_k] = (-2u_k'' + 2\partial_k u_1')\partial_x + (-u_k' + \partial_k u_1)\partial_x^2 + \partial_k u_1'' - u_1\partial_k u_1 - \frac{1}{2}u_k u_1' - \frac{3}{4}u_k'''. \qquad \square$$

LEMMA 2.2. *Let K be an anti-symmetric differential multidimensional operator of order* 3. *Suppose the commutator $[K,\mathcal{L}]$ is a multiplication operator, and the coefficients in derivatives of order* 3 *are constants. Then $K = \sum_{1\le i\le g} c_i\mathcal{A}_i + \psi_i\partial_i + \phi$, where c_i are constants, and the functions ψ_i and ϕ do not depend on x.*

Proof. Let

$$K = \sum_{1\le i,j,k\le g} s_{ijk}\,\partial_i\partial_j\partial_k + \sum_{1\le i,j\le g} f_{ij}\,\partial_i\partial_j + \sum_{1\le i\le g} g_i\,\partial_i + h,$$

where s_{ijk} are constants such that $s_{ijk} = s_{ikj} = s_{jik}$, and all the f_{ij}, g_i, h are functions of $t_1,\dots,t_g$. The anti-symmetry implies that $f_{ij} = 0$ for all i,j, and $\sum_{1\le i\le g}\frac{\partial g_i}{\partial t_i} = 2h$. We have

$$[\mathcal{L},K] = \sum_{1\le i\le g} g_i''\partial_i + 2\sum_{1\le i\le g} g_i'\partial_x\partial_i + h'' + 2h'\partial_x + \sum_{1\le i\le g} g_i\frac{\partial}{\partial t_i}u_1$$
$$+\sum_{1\le i,j,k\le g} s_{ijk}\left(\frac{\partial^3}{\partial t_i\partial t_j\partial t_m}u_1 + 3\frac{\partial^2}{\partial t_i\partial t_j}u_1\partial_k + 3\frac{\partial}{\partial t_i}u_1\partial_j\partial_k\right).$$

Since the commutator $[\mathcal{L},K]$ is a multiplication operator, the coefficients of $\partial_i\partial_j$ and ∂_i in the last formula are zeros.

It follows from the linear independence of the first derivatives of the function u_1 that $s_{ijk} = 0$ when $i,j \ne 1$. If $i \ne 1$, then one has $2g_i' = -3s_{11i}\frac{\partial}{\partial x}u_1$.

From the condition that the coefficient of ∂_x vanishes, we obtain

$$g_1'' + 2h' + 3\sum_{1\le i\le g} s_{11i}\frac{\partial^2 u}{\partial_i\partial x} = 0.$$

Put $c_i = 3s_{11i} = s_{11i} + s_{1i1} + s_{i11}$; $\psi_1 = g_1 + 1/2\sum_{i=1}^g c_iu_i$, $\psi_k = g_k + \sum_{i=1}^g c_iu_1$ for $k \ne 1$, and $\phi = h + \sum_{i=1}^g c_iu_i'$. Then the functions ψ_i, $i = 1,\dots,g$, and ϕ do not depend on x and $K = \sum_{1\le i\le g} c_i\mathcal{A}_i + \psi_i\partial_i + \phi$. □

Denote

$$\mathcal{U}_i = \mathcal{A}_i - \partial_{i+1}, \quad \text{for } i < g;$$
$$\mathcal{U}_g = \mathcal{A}_g.$$

The operators $\mathcal{U}_i$ are anti-symmetric and homogeneous.

LEMMA 2.3. *The following conditions are equivalent:*

(1) $[\mathcal{L},\mathcal{U}_k] = 0$.
(2) $-\partial_{k+1}\mathcal{L} = [\mathcal{L},\mathcal{A}_k]$.
(3) $\partial_{k+1}u_1 = u_{k+1}' = \frac{1}{4}(u_k''' - 2u_1'u_k - 4u_1u_k')$ *for $k < g$, and* $(u_g''' - 2u_1'u_g - 4u_1u_g') = 0$.

Proof. The asertion follows from the equality

$$[\mathcal{L},\mathcal{U}_k] = [\mathcal{L},\mathcal{A}_k] + \partial_{k+1}\mathcal{L} = [\mathcal{L},\mathcal{A}_k] - \partial_{k+1}u_1. \quad \square$$

The last condition in Lemma 2.3 allows us to express the functions u_k by recursion in terms of u_1 and its x-derivative, up to the choice of a function that does not depend on x.

COROLLARY 2.1. *Under the hypotheses of Lemma 2.3, functions u_i are related by the Lenard operator $\mathcal{R}$ (see (1.8)):*

$$\partial_{i+1} u_1 = \mathcal{R}(u_i') = \mathcal{R}^{i+1}(u_1'). \tag{2.2}$$

For $i = g$ one has $0 = \partial_{g+1} u_1 = \mathcal{R}^g(u_1')$.

COROLLARY 2.2. *The operators $\{\mathcal{U}_k\}$ commute with $\mathcal{L}$ if and only if the function $u_1(x)$ is a solution of the stationary g-KdV equation.*

LEMMA 2.4. *The operators $\mathcal{U}_i$, $\mathcal{U}_j$ commute for all $1 \le i, j \le g$ if and only if the functions $\{u_i\}$ satisfy condition* (3) *of Lemma 2.3 and the following equalities:*

$$\partial_i u_j = \partial_j u_i, \tag{2.3}$$

$$u_k' u_i - u_i' u_k + 2\partial_{i+1} u_k - 2\partial_{k+1} u_i = 0, \quad 1 \le i, k \le g. \tag{2.4}$$

The lemma is proved by direct calculation.

Note that (2.3) implies the existence of a function $z(t_1, \dots, t_g)$ that satisfies $\partial_i z = u_i$.

The fact that the operators $\mathcal{U}_i$ commute is equivalent to the zero curvature conditions for the operators $\mathcal{A}_i$:

$$\partial_{j+1} \mathcal{A}_i - \partial_{i+1} \mathcal{A}_j + [\mathcal{A}_i, \mathcal{A}_j] = 0. \tag{2.5}$$

3. A generalized translation associated with the KdV hierarchy

In this section we develop the technique of a generalized translation from [**3**].

For $\eta \in \mathbb{R}$ define an operator D^η acting on the space of functions of one variable as $(D^\eta f)(\xi) = \frac{\xi\eta}{2(\xi-\eta)} f(\eta)$. Define the operator $\mathcal{B}$ by the rule

$$\mathcal{B}(f, h)(\xi, \eta) = \frac{\xi\eta}{2(\xi - \eta)}(f(\xi)h(\eta) - f(\eta)h(\xi)) = f(\xi)(D^\eta h)(\xi) - g(\xi)(D^\eta f)(\xi).$$

It possesses the following properties:

- $\mathcal{B}(f, h)(\xi, \eta) = -\mathcal{B}(h, f)(\xi, \eta)$.
- $\mathcal{B}(f, h)(\xi, \eta) = \mathcal{B}(f, h)(\eta, \xi)$.
- $\mathcal{B}$ is a bilinear operator.
- If $f(\xi)$ and $h(\xi)$ are polynomials, then $\mathcal{B}(f, h)(\xi, \eta)$ is also a polynomial.
- $\mathcal{B}(f, \xi^{-1})(\xi, \eta) = \dfrac{f(\xi)\xi - f(\eta)\eta}{2(\xi - \eta)}$.
- $\mathcal{B}(1, 2\xi^{-1}) = 1$.

Define also an operator $\mathcal{B}_k$ acting on the set of k-tuples of functions of one variable as follows:

$$\begin{aligned}
&\mathcal{B}_k(f_1, \dots, f_k)(\xi_1, \dots, \xi_k) \\
&\qquad = \frac{\prod_{i=1}^k \xi_i^{k-1}}{2^{k-1}\prod_{1\le i<j\le k}(\xi_i - \xi_j)} \left(\sum_{\sigma \in S_k} (-1)^\sigma f_1(\xi_{\sigma(1)}) \cdots f_k(\xi_{\sigma(k)}) \right) \\
&\qquad = \frac{\prod_{i=1}^k \xi_i^{k-1}}{2^{k-1} W(\xi_1, \xi_2, \dots, \xi_k)} \begin{vmatrix} f_1(\xi_1) & f_2(\xi_1) & \dots & f_k(\xi_1) \\ f_1(\xi_2) & f_2(\xi_2) & \dots & f_k(\xi_2) \\ \vdots & \vdots & & \vdots \\ f_1(\xi_k) & f_2(\xi_k) & \dots & f_k(\xi_k) \end{vmatrix},
\end{aligned}$$

where $W(\xi_1, \dots, \xi_k)$ is the Vandermonde determinant. Note that $\mathcal{B}_1(f) = f$, $\mathcal{B}_2(f, g) = \mathcal{B}(f, g)$.

Let f_i be a function of variables ξ; $t_1, \dots, t_g$, so that one has

$$\partial_j \mathcal{B}_k(f_1, \dots, f_k)(\xi_1, \dots, \xi_k) = \sum_{i=1}^{k} \mathcal{B}_k(f_1, \dots, \partial_j f_i, \dots, f_k)(\xi_1, \dots, \xi_k).$$

For a fixed function h define $(T_\xi^\eta f)(\xi, \eta) = (T(h)_\xi^\eta f)(\xi, \eta) = \mathcal{B}(f, h)(\xi, \eta)$.

LEMMA 3.1. *The operators T_ξ^η satisfy the associativity condition $T_\xi^\eta T_\xi^\tau = T_\eta^\tau T_\xi^\eta$ and the commutativity condition $T_\xi^\eta T_\xi^\tau = T_\xi^\tau T_\xi^\eta$.*

Proof. Calculate $T_\xi^\tau T_\xi^\eta f$:

$$\begin{aligned} T_\xi^\tau T_\xi^\eta f &= \frac{\xi\tau}{2(\xi-\tau)} \left(\xi\eta \frac{f(\xi)h(\eta) - f(\eta)h(\xi)}{2(\xi-\eta)} h(\tau) - \tau\eta \frac{f(\tau)h(\eta) - f(\eta)h(\tau)}{2(\tau-\eta)} h(\xi) \right) \\ &= \xi^2\eta^2\tau^2 \\ &\times \frac{f(\xi)h(\eta)h(\tau)(\tau^{-1} - \eta^{-1}) + f(\eta)h(\tau)h(\xi)(\xi^{-1} - \tau^{-1}) + f(\tau)h(\xi)h(\eta)(\eta^{-1} - \xi^{-1})}{4(\xi-\tau)(\xi-\eta)(\eta-\tau)} \\ &= \mathcal{B}_3(f(\xi), h(\xi), h(\xi)\xi^{-1})(\xi, \eta, \tau). \end{aligned}$$

This expression is invariant under all the permutations of the variables ξ, η, τ. The lemma is proved. □

COROLLARY 3.1. *The operator T_ξ^η is an operator of commutative generalized translation and*

$$T_\xi^\eta 1 = \frac{\xi\eta}{2(\xi-\eta)}(h(\eta) - h(\xi)).$$

In particular, $T_\xi^\eta 1 = 1$ if and only if $h(\xi) = 2/\xi$.

REMARK 3.1. The generalized translation operator $\mathcal{D}_\xi^\eta(f) = \frac{\xi f(\xi) - \eta f(\eta)}{\xi - \eta}$ from [**3**] is equal to T_ξ^η when $h = 2/\xi$.

REMARK 3.2. Let $h(\xi) = h_{-1}/\xi + \widetilde{h}(\xi)$, where $\widetilde{h}(\xi)$ is a function regular in a neighbourhood of the origin. Then for a function $f(\xi)$ regular in a neighbourhood of the origin the function $f(\xi, \eta) = T_\xi^\eta f$ is regular in a neighbourhood of the point $(\xi, \eta) = (0, 0)$.

DEFINITION 3.1. A polarization of a smooth function $f(\xi)$ is a symmetric function of two variables $f(\xi, \eta)$ such that $f(\xi, \xi) = 2f(\xi)$.

LEMMA 3.2. *Let $f(\xi, \eta)$ be a polarization of a function $f(\xi)$. Then*

$$\left. \frac{\partial f(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta} = \frac{\partial f(\xi)}{\partial \xi}. \tag{3.1}$$

Proof. For a symmetric function $f(\xi, \eta)$ there exists a function $h(s_1, s_2)$ such that $f(\xi, \eta) = h(\xi + \eta, \xi\eta)$. Since $2f(\xi) = f(\xi, \xi) = h(2\xi, \xi^2)$, one has

$$\left. \frac{\partial f(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta} = \left. \frac{\partial h}{\partial s_1}(\xi + \eta, \xi\eta) + \frac{\partial h}{\partial s_2}(\xi + \eta, \xi\eta)\eta \right|_{\xi=\eta} = \frac{\partial h}{\partial s_1}(2\xi, \xi^2) + \frac{\partial h}{\partial s_2}(2\xi, \xi^2)\xi.$$

On the other hand,

$$\frac{\partial f}{\partial \xi}(\xi) = \frac{1}{2} \frac{\partial h(2\xi, \xi^2)}{\partial \xi} = \frac{\partial h}{\partial s_1}(2\xi, \xi^2) + \frac{\partial h}{\partial s_2}(2\xi, \xi^2)\xi = \left. \frac{\partial f(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta}. \qquad \square$$

EXAMPLE 1. *Let* $f(\xi) = \sum_i g_i(\xi)h_i(\xi)$. *Then the function*

$$f(\xi, \eta) = \sum_i (g_i(\xi)h_i(\eta) + g_i(\eta)h_i(\xi))$$

is a polarization of $f(\xi)$.

Let F_n be a set of smooth functions of n variables.

DEFINITION 3.2. Let $G : F_1^k \to F_1$ and $\widehat{G} : F_1^k \to F_2$. The operator $\widehat{G}$ is called a polarization of the operator G if the function $\widehat{G}(f_1, \dots, f_k)$ is a polarization of the function $G(f_1, \dots, f_k)$ for any $f_1, \dots, f_k$.

Recall ([**13**]) that the one-variable Hirota operator H_ξ is given by

$$\mathrm{H}_\xi[f(\xi), g(\xi)] = f'(\xi)g(\xi) - f(\xi)g'(\xi).$$

LEMMA 3.3. *The operator* $\frac{2}{\xi\eta}\mathcal{B}(f,g)(\xi,\eta)$ *gives a polarization of the Hirota operator*

Proof. We need to prove that

$$\lim_{\eta\to\xi} \mathcal{B}(f,g)(\xi,\eta) = \frac{\xi^2}{2}\mathrm{H}_\xi[f(\xi), g(\xi)].$$

Let $\eta = \xi + \varepsilon$. Then $f(\eta) = f(\xi) + \varepsilon f'(\xi) + O(\varepsilon^2)$ and $g(\eta) = g(\xi) + \varepsilon g'(\xi) + O(\varepsilon^2)$. Hence

$$\begin{aligned}\mathcal{B}(f,g)(\xi, \xi+\varepsilon) &= \frac{\xi(\xi+\varepsilon)}{-2\varepsilon}\left(f(\xi)g'(\xi)\varepsilon - g(\xi)f'(\xi)\varepsilon + O(\varepsilon^2)\right) \\ &= \frac{\xi^2}{2}\mathrm{H}_\xi[f(\xi), g(\xi)] + O(\varepsilon). \qquad \square\end{aligned}$$

Define the operators D_i by the expansion

$$(D^\eta f)(\xi) = \sum_{i\in\mathbb{Z}} (D_i f)(\xi)\eta^i.$$

LEMMA 3.4. *Let* $f(\xi) = \dots + f_0 + f_1\xi + f_2\xi^2 + \cdots$. *Then*

$$(D_k f)(\xi) = \frac{1}{2}(\dots + f_0\xi^{-k+1} + \dots + f_{k-1}).$$

If $f(\xi) = f_0 + f_1\xi + \dots + f_n\xi^n$ *is a polynomial, then* $(D_1 f)(\xi) = \frac{1}{2}f_0$ *and* $(D_{n+1}f)(\xi) = \frac{1}{2}\xi^{-n}f(\xi)$.

It is clear that

$$(D_{k+1}f)(\xi) = \xi^{-1}(D_k f)(\xi) + \frac{1}{2}f_k. \tag{3.2}$$

Note one more property of the operators D_i.

LEMMA 3.5. *Let* $f(\xi)$ *be a polynomial. Then*

$$\begin{aligned}\sum_{k\ge0, m\ge0} D_{k+m+1}(f(\xi))\eta^k\zeta^m &= \frac{\xi}{2(\eta-\zeta)}\left(\frac{\eta}{\xi-\eta}f(\eta) - \frac{\zeta}{\xi-\zeta}f(\zeta)\right) \\ &= \frac{2}{\eta\zeta}\mathcal{B}\left(1, D_\xi^\eta f(\xi)\right)(\eta,\zeta).\end{aligned}$$

Define the operators d_i by the formula $T_\xi^\eta f(\xi) = \sum (d_i f(\xi))\eta^i$. Then

$$d_i f(\xi) = f(\xi) D_i h(\xi) - h(\xi) D_i f(\xi).$$

From Lemma 3.1 using the standard methods we obtain the following result.

LEMMA 3.6. *The linear space spanned by the operators d_i, $i = 1, \dots,$ is an associative and commutative algebra with the following multiplication:*

$$d_i d_j = \sum_{k=0}^{i+j} c_{ij}^k d_k,$$

where the structure constants c_{ij}^k are found from the expansion

$$T_\xi^\eta \xi^k = \frac{\xi\eta}{2(\xi-\eta)}\left(\xi^k h(\eta) - \eta^k h(\xi)\right) = \sum_{i+j\geq k} c_{ij}^k \xi^i \eta^j.$$

For the sequence of function $\{u_1, \dots, u_g\}$ of variables $t_1 = x, t_2, \dots, t_g$ we introduce the generating functions

$$\mathbf{u}(\xi) = \sum_{i=1}^g u_i \xi^i, \quad \mathbf{u}'(\xi) = \sum_{i=1}^g u_i' \xi^i, \quad \dots, \quad \mathbf{u}^{(k)}(\xi) = \sum_{i=1}^g u_i^{(k)} \xi^i$$

(the prime here, as usual, means differentiation with respect to x). The following statement gives an expression of the third derivatives $u_1''', \dots, u_g'''$ in terms of the functions $u_1, \dots, u_g$ and their first derivatives. Moreover, it allows us to express these derivatives by recursion as a differential polynomial in u_1. Here is one of the key results of the paper.

THEOREM 3.1. *The sequence $\{u_1, u_2, \dots, u_g\}$ satisfies condition* (3) *of Lemma 2.3 if and only if the generating function $\mathbf{u}(\xi)$ is a solution of the following equation:*

$$\mathbf{u}'''(\xi) + 2u_1'(2 - \mathbf{u}(\xi)) - 4(\xi^{-1} + u_1)\,\mathbf{u}'(\xi) = 0. \tag{3.3}$$

Proof. We have

$$\mathbf{u}'''(\xi) + 2u_1'(2 - \mathbf{u}(\xi)) - 4(\xi^{-1} + u_1)\,\mathbf{u}'(\xi) = \sum_{i=1}^g (u_i''' - 2u_1' u_i - 4u_1 u_i' - 4u_{i+1}')\xi^i.$$

The coefficients of ξ^i on the right-hand side of this formula are all zero if and only if condition (3) of Lemma 2.3 holds. □

Take, by definition,

$$\partial_k \mathbf{u}(\xi) = \sum_{i=1}^g \partial_k u_i \xi^i.$$

LEMMA 3.7. *Equations* (2.3) *and* (2.4) *together are equivalent to the following equation:*

$$\partial_{k+1}\mathbf{u}(\xi) = \xi^{-1}\partial_k \mathbf{u}(\xi) - \frac{1}{2}u_k \mathbf{u}'(\xi) + \frac{1}{2}u_k'(\mathbf{u}(\xi)) - u_k'.$$

This equation allows us to determine by recursion the partial derivatives $\partial_k \mathbf{u}(\xi)$:

$$\partial_k \mathbf{u}(\xi) = D_k(2 - \mathbf{u}(\xi))\mathbf{u}'(\xi) - D_k(\mathbf{u}'(\xi))(2 - \mathbf{u}(\xi)). \tag{3.4}$$

In what follows we assume that (3.3) and (3.4) hold for the function $\mathbf{u}(\xi)$.

COROLLARY 3.2.

$$\partial_k \mathbf{u}'(\xi) = D_k(2-\mathbf{u}(\xi))\mathbf{u}''(\xi) - D_k(\mathbf{u}''(\xi))(2-\mathbf{u}(\xi)), \tag{3.5}$$

$$\begin{aligned}\partial_k \mathbf{u}''(\xi) &= 4(\xi^{-1}+u)\partial_k \mathbf{u}(\xi) - 2u_k'(2-\mathbf{u}(\xi))\\ &\quad + D_k(\mathbf{u}''(\xi))\mathbf{u}'(\xi) - D_k(\mathbf{u}'(\xi))\mathbf{u}''(\xi).\end{aligned} \tag{3.6}$$

Let $\partial(\eta) = \sum_{i=1}^{g} \eta^i \partial_i$. Note that for a fixed η the operator $\partial(\eta)$ is an operator of differentiation in the direction of the vector $(\eta, \eta^2, \dots, \eta^g)$, i.e.,

$$\partial(\eta) f(t_1, \dots, t_g) = \frac{\partial}{\partial \tau} f(t_1 + \tau\eta, \dots, t_g + \tau\eta^g)\Big|_{\tau=0}.$$

COROLLARY 3.3.

$$(\partial_x^2 \partial(\xi) + 2(2-\mathbf{u}(\xi))\partial_x - 4(\xi^{-1}+u_1)\partial(\xi))u = 0.$$

Proof. Recall that $\partial_i u_1 = u_i'$, and therefore $\mathbf{u}'(\xi) = \partial(\xi)u_1$ and $\mathbf{u}'''(\xi) = \partial_x^2\partial(\xi)u$. Now the statement follows from (3.3). □

Denote $\mathcal{T}_\xi^\eta = T(2-\mathbf{u}(\xi))_\xi^\eta$. In the sequel, the operator $\mathcal{T}_\xi^\eta$ plays a special role, as is shown by the following theorem.

THEOREM 3.2.

$$\partial(\eta)\mathbf{u}(\xi) = \mathcal{T}_\xi^\eta \partial_x \mathbf{u}(\xi). \tag{3.7}$$

$$\partial(\eta)\mathbf{u}'(\xi) = \mathcal{T}_\xi^\eta \partial_x^2 \mathbf{u}(\xi). \tag{3.8}$$

$$\partial(\eta)\mathbf{u}''(\xi) = \mathcal{T}_\xi^\eta \partial_x^3 \mathbf{u}(\xi) - \mathcal{B}_\xi^\eta(\mathbf{u}'(\xi), \mathbf{u}''(\xi)). \tag{3.9}$$

Proof. These formulas follow from the definition of the operator $\mathcal{T}_\xi^\eta$ and equations (3.4), (3.5), (3.6). □

Note that (3.7), (3.8) imply that

$$[\partial_x, \mathcal{T}_\xi^\eta]\partial_x \mathbf{u}(\xi) = 0.$$

The associativity condition for the operator $\mathcal{T}_\xi^\eta$ is equivalent to the following relation, which will be used later:

$$\partial(\zeta)\frac{\xi\eta}{2(\xi-\eta)}\frac{2-\mathbf{u}(\eta)}{2-\mathbf{u}(\xi)} = \frac{1}{(2-\mathbf{u}(\xi))^2}\mathcal{T}_\xi^\eta \mathcal{T}_\xi^\zeta \mathbf{u}'(\xi). \tag{3.10}$$

Now we describe the family of differential operators $\{\mathcal{U}_i\}$ using the method of generating function.

LEMMA 3.8. *The generating function of the sequence of operators $\mathcal{U}_i$ is*

$$\sum_{i=1}^{g} \mathcal{U}_i \xi^i = \frac{1}{2}((\mathcal{L}-\xi^{-1})\partial(\xi) + \partial(\xi)(\mathcal{L}-\xi^{-1})) + \frac{1}{4}((2-\mathbf{u}(\xi))\partial_x + \partial_x(2-\mathbf{u}(\xi))).$$

4. The hyperelliptic curve associated with a solution of KdV

THEOREM 4.1. *Suppose the generating function $\mathbf{u}(\xi)$ satisfies* (3.3) *and* (3.4). *Let*

$$4\mu(\xi) = \mathbf{u}'(\xi)^2 + 2\mathbf{u}''(\xi)\,(2-\mathbf{u}(\xi)) + 4(\xi^{-1}+u_1)\,(2-\mathbf{u}(\xi))^2. \tag{4.1}$$

Then $\mu(\xi) = 4\xi^{-1} + \sum_{i=1}^{2g} \mu_i \xi^i$, where μ_i are constants, $i = 1, \dots, 2g$.

Proof. It follows from (3.7), (3.8), (3.9) that

$$\partial(\eta)\mu(\xi) = 2\mathbf{u}'(\xi)\partial(\eta)\mathbf{u}'(\xi) + 2(2-\mathbf{u}(\xi))\partial(\eta)\mathbf{u}''(\xi) - 2\mathbf{u}''(\xi)\partial(\eta)\mathbf{u}(\xi) \\ + 4\mathbf{u}'(\eta)(2-\mathbf{u}(\xi))^2 - 8(\xi^{-1}+u_1)(2-\mathbf{u}(\xi))\partial(\eta)\mathbf{u}(\xi) = 0.$$

Therefore $\partial_i\mu_j = 0$, where $1 \le i \le g$, $1 \le j \le 2g$, and all the μ_i are constants. □

Suppose $u_k = 0$ for $k > g$. Equation (4.1) implies that

$$u_{k+1} = \frac{1}{4}\mu_k + J_k(u, u', u'', \dots, u_k, u_k', u_k''), \quad k = 1, \dots, 2g, \tag{4.2}$$

where J_k are polynomials. We see that the functions u_k, $k = 2, \dots, g$, can be expressed by recursion in terms of the function u_1, its derivatives, and the constants μ_i, namely

$$u_k = \Theta_k(u, u', \dots, u^{(2k-2)}, \mu_1, \dots, \mu_{k-1}), \tag{4.3}$$

where Θ_k are polynomials.

Note that the condition $J_g' = 0$ is equivalent to the stationary g-KdV equation. For $k > g$ one has $J_k = -(1/4)\mu_k$, which provides integrals of the higher KdV equation (see [**19**]).

Since $\partial_k u_1 = u_k'$, the partial derivative of u_1 with respect to t_k can also be expressed in terms of derivatives with respect to x. Therefore the behavior of the function u_1 along the coordinate axes $t_2, \dots, t_g$ can be reconstructed if its derivatives with respect to x are known.

LEMMA 4.1. *Let μ_i be constants. Then equation* (4.1) *implies equation* (3.3). *If equations* (2.3) *and* (4.1) *hold, then equation* (3.4) *also holds.*

Proof. The first statement of the lemma is clear. From the equality $u_{k+1}' = (1/4)(u_k''' - 2u'u_k - 4uu_k')$ one obtains

$$\partial_{k+1}u_m' = \partial_m u_{k+1}' = (1/4)(\partial_m u_k''' - 2u_m''u_k - 2u'\partial_m u_k - 4u_m'u_k' - 4u\partial_m u_k') \\ = \partial_k u_{m+1}' + (1/2)u_k''u_m - (1/2)u_m''u_k.$$

This equation proves that (3.5) holds. So (3.6) also holds. Integration of (3.5) with respect to x gives the formula $\partial_k\mathbf{u}(\xi) = D_k(2-\mathbf{u}(\xi))\mathbf{u}'(\xi) - D_k(\mathbf{u}'(\xi))(2-\mathbf{u}(\xi)) + \varphi(t_2, \dots, t_g)$, where the function φ does not depend on x. Combining the last equation with (4.1), we obtain

$$0 = 4\,\partial_k\mu(\xi) = \partial_k(\mathbf{u}'(\xi)^2 + 2\mathbf{u}''(\xi)\,(2-\mathbf{u}(\xi)) + 4(\xi^{-1}+u_1)\,(2-\mathbf{u}(\xi))^2) \\ = \big(8(\xi^{-1}+u_1)(\mathbf{u}(\xi)) + 2\mathbf{u}''(\xi)\big)\,\varphi(t_2, \dots, t_g).$$

The function in parentheses cannot vanish identically as a function of x, and thus $\varphi(t_2, \dots, t_g) \equiv 0$. □

Now we summarize the results obtained:

THEOREM 4.2. *The following statements are equivalent:*

(1) *The function u_1 is a solution of a stationary g-KdV equation.*
(2) *There exists a sequence of functions $\{u_1, \dots, u_g\}$ such that the operators $\mathcal{L} = \partial_x^2 - u_1$ and $\mathcal{U}_i = \partial_x^2\partial_i - \frac{1}{2}(u_1\partial_i + \partial_i u_1) - \frac{1}{4}(u_i\partial_x + \partial_x u_i) - \partial_{i+1}$, $1 \le i \le g$, commute.*
(3) *There exists a sequence of functions $\{u_1, \dots, u_g\}$ and a set of constants $\mu_1, \dots, \mu_{2g}$ such that the generating function $\mathbf{u}(\xi) = \sum_{i=1}^{g} u_i\xi^i$ satisfies* (4.1).

In order to find the relation between the constants μ_k and the coefficients a_i we need the following result.

LEMMA 4.2. *The operator* $\mathbf{U} = \mathcal{U}_1\mathcal{L}^{g-1} + \mathcal{U}_2\mathcal{L}^{g-2} + \cdots + \mathcal{U}_g$

(1) *commutes with the operator* $\mathcal{L}$;

(2) *is an operator of order* $2g+2$ *with the leading coefficient* 1;

(3) *contains the differentiation with respect to* x *only.*

Proof. The first statement of the lemma is obvious. The leading term of $\mathbf{U}$ is a composition of the leading terms of the operators $\mathcal{U}_1$ and $\mathcal{L}^{g-1}$, so it is equal to ∂_x^{2g+2}. This proves the second statement. Since $\mathcal{U}_i = \partial_i\mathcal{L} - \partial_{i+1} - (1/2)u_i\partial_x + (1/4)u_i'$, the sum $\mathcal{U}_i\mathcal{L} - \mathcal{U}_{i+1}$ does not contain the differentiation with respect to t_{i+1}. By recurrence, we get the third statement of the lemma. □

THEOREM 4.3. *Under the hypotheses of Theorem 4.2 the operator* A *can be decomposed as* $A = \mathcal{U}_1\mathcal{L}^{g-1} + \mathcal{U}_2\mathcal{L}^{g-2} + \cdots + \mathcal{U}_g$, *where* $A = A_g + \sum_{i=0}^{g-2} a_iA_i$. *The coefficients* μ_k *and* a_i *satisfy the following relation:*

$$\mu_k = 8a_{g-k-1} + 4\sum_{i=1}^{k-2} a_{g-i-1}a_{g-k+i} \tag{4.4}$$

for $k = 1, \dots, g-1$.

Proof. The first statement of the theorem follows from Lemma 4.2 and uniqueness of the operators A_k (see Theorem 1.1).

The function u_k is a differential polynomial

$$u_k = \Theta_k(u_1,\ u_1', \dots, u_1^{(2k)}, \mu_1, \dots, \mu_{k-1}).$$

Let ε_k be a constant term of Θ_k. Then

$$\begin{aligned}\mathcal{U}_1\mathcal{L}^{g-1} + \mathcal{U}_2\mathcal{L}^{g-2} + \cdots + \mathcal{U}_g &= \partial_x^{2g+1} - (1/2)\sum u_k\partial_x^{2g-2k-1} + \sum \vartheta_i\partial_x^i \\ &= \partial_x^{2g+1} - (1/2)\sum \varepsilon_k\partial_x^{2g-2k-1} + \sum \widetilde{\vartheta}_i\partial_x^i,\end{aligned}$$

where ϑ_i and $\widetilde{\vartheta}_i$ are differential polynomials in u_1 without constant terms. On the other hand, $A = \partial_x^{2g+1} - (1/2)\sum a_k\partial_x^{2k+1} + \sum \widetilde{\vartheta}_i\partial_x^i$. Thus, $a_k = -(1/2)\varepsilon_{g-k-1}$, and so it remains to find ε_k. The result now follows from (4.1). □

The following corollary is one of the main results of the paper.

COROLLARY 4.1. *There is a canonical way to associate a solution* u_1 *of the stationary* g*-KdV equation with a hyperelliptic curve*

$$\Gamma = \{(\xi, y) \in \mathbb{C}^2 \mid y^2 = 4\mu(\xi)\}. \tag{4.5}$$

The coefficients $\mu_1, \dots, \mu_{g-1}$ *are expressed in terms of the constants* a_i *as in equation* (4.4), *and* $\mu_g, \dots, \mu_{2g}$ *are found from* (4.3) *in terms of the values of* $u_1^{(k)}(t_0)$, $k = 0, 1, \dots,$ *at some point* $t_0 \in \mathbb{C}^g$.

REMARK 4.1. In the case where the solution u_1 is periodic as a function of x, the hyperelliptic curve constructed above coincides with the spectral curve introduced in [**12**]. Our construction uses only the local properties of the function u_1, while in [**12**] only periodic or rapidly decreasing functions are discussed.

REMARK 4.2. The number of singular points on Γ is an important characteristic of the solution u_1. This number can be expressed in terms of $u_1^{(k)}(t_0)$ using the resultant.

5. Fiber bundles associated with the stationary g-KdV equations

The equations described by (1.9) are ordinary differential equations of order $2g+1$, and so their solution u_1 is *uniquely* determined in a neighbourhood of a given point x_0 by the values $c_k = u_1^{(k)}(x_0)$, $k = 0, \dots, 2g$. Since the coefficients of the KdV equations are constants, we can take $x_0 = 0$. The stationary g-KdV equations depend on the numbers $a_0, \dots, a_{g-2}$, and so the space of all such equations is isomorphic to $\mathbb{C}^{g-1}$.

The space $\mathcal{M}_g$ of all hyperelliptic curves $\Gamma = \{(\xi, y) \in \mathbb{C}^2 \mid y^2 = 4\mu(\xi)\}$ can be parametrized by the numbers $\mu_1, \dots, \mu_{2g}$, so it is isomorphic to $\mathbb{C}^{2g}$.

Denote by $\mathbf{R}_g$ the space of solutions u of all stationary g-KdV equations such that u is regular at the point x_0. As was explained above, we can identify the space $\mathbf{R}_g$ with $\mathbb{C}^{3g}$ using coordinates $(c_0, c_1, \dots, c_{2g}, a_0, \dots, a_{g-2})$.

There exists a canonical map $\pi_{\mathcal{M}} : \mathbf{R}_g \to \mathcal{M}_g$, which sends a solution u to the hyperelliptic curve Γ described by (4.5).

Denote by $\mathfrak{U}_g$ the space of gth symmetric powers of hyperelliptic genus g curves. We consider the universal bundle $(\mathfrak{U}_g, \mathcal{M}_g, \pi_{\mathfrak{U}})$, where the natural projection $\pi_{\mathfrak{U}} : \mathfrak{U}_g \to \mathcal{M}_g$ is given by $\pi_{\mathfrak{U}_g}(x \in Sym^g\Gamma) = \Gamma$.

THEOREM 5.1. *There is a canonical fiber-preserving birational equivalence* $\mathbf{R}_g \to \mathcal{U}_g$.

Proof. Let Γ be a hyperelliptic curve associated with the solution u_1 of the stationary g-KdV equation (see Corollary 4.1). Let $\xi_1, \dots, \xi_g$ be the roots of the equation $2 - \mathbf{u}(0, \xi) = 0$. Denote $y_i = \mathbf{u}'(0, \xi_i)$. Equation (4.1) implies that $y_i^2 = \mathbf{u}'(0, \xi_i)^2 = 4\mu(\xi)$, so the point (ξ_i, y_i) belongs to Γ. Thus we have a map $\upsilon : \mathbf{R}_g \to \mathfrak{U}_g$ given by the formula $\upsilon(u_1) = (\Gamma, [(\xi_1, y_1), \dots, (\xi_g, y_g)])$, where $(\xi_i, y_i) \in \Gamma$. Apparently, υ is fiber preserving.

On the other hand, if a curve Γ and a point $[(\xi_1, y_1), \dots, (\xi_g, y_g)] \in Sym^g\Gamma$ are given, then in the case of distinct points $(\xi_1, \dots, \xi_g)$, it is possible to construct the point $(c_0, \dots, c_{2g}, a_0, \dots, a_{g-2})$ as follows. The constants a_k are a solution of (4.4) where the parameters μ_i are known. The values $u_i(0)$ are the symmetric functions of $\xi_1, \dots, \xi_g$, namely $u_i(0) = 2\,(-1)^{g-i}\,\sigma_{g-i+1}(\xi_1, \dots, \xi_g)/\sigma_g(\xi_1, \dots, \xi_g)$. Then the values $u_i'(0)$ can be found as the coefficients of the generating function $\mathbf{u}'(0, \xi)$, from the equations $\mathbf{u}'(0, \xi_i) = y_i$. All higher derivatives $c_k = u_1^{(k)}(0)$ can be found by recursion using equation (4.3). Thus the inverse rational map υ^{-1} is constructed. $\square$

In the case of the universal bundle of Jacobians over the moduli space of genus g hyperelliptic curves this theorem gives the famous results of Dubrovin and Novikov; see [**12**].

6. Algebraic relations between the operators $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$

The Burchnall–Chaundy lemma ([**7**]) says that two commuting differential operators of one variable are always connected by an algebraic relation. In [**16**] the case of commuting differential operators of n variables was considered. In the same paper the author (I. M. Krichever) introduced a class of n-algebraic families of operators, i.e., families of commuting operators characterized by finite-dimensional algebraic manifolds. The family $\{\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g\}$ gives an example of n-algebraic operators from [**16**].

LEMMA 6.1. *The operators $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$ satisfy the following algebraic relation:* $4(\mathcal{U}_1\mathcal{L}^{g-1}+\mathcal{U}_2\mathcal{L}^{g-2}+\dots+\mathcal{U}_{g-1}\mathcal{L}+\mathcal{U}_g)^2 = (4\mathcal{L}^{2g+1}+\mu_1\mathcal{L}^{2g-1}+\mu_2\mathcal{L}^{2g-2}+\dots+\mu_{2g})$.

Using the notation $\mathcal{U}(z) = \mathcal{U}_1 z^{g-1} + \mathcal{U}_2 z^{g-2} + \dots + \mathcal{U}_g$ and $\widetilde{\mu}(z) = 4z^{2g+1} + \mu_1 z^{2g-1} + \dots + \mu_{2g}$, one can write down this relation as $4\mathcal{U}(\mathcal{L})^2 = \widetilde{\mu}(\mathcal{L})$.

Proof. Denote
$$S_i = \frac{1}{2}\,u_i\partial_x - \frac{1}{4}\,u_i'.$$
Then
$$\mathcal{U}_i = \partial_i\mathcal{L} - S_i - \partial_{i+1}. \tag{6.1}$$
We have
$$[\mathcal{L}, S_i] = u_i'\mathcal{L} - u_{i+1}',$$
which implies the equation
$$\begin{aligned}\sum_{i+j=k} \mathcal{U}_i\mathcal{U}_j &= \sum_{i+j=k} (\partial_i\mathcal{L} - S_i - \partial_{i+1})(\partial_j\mathcal{L} - S_j - \partial_{j+1}) \\ &= \sum_{i+j=k} \partial_i\partial_j\mathcal{L}^2 - 2\partial_i\partial_{j+1}\mathcal{L} + \partial_{i+1}\partial_{j+1} \\ &\qquad - (\partial_i S_j + S_i\partial_j)\mathcal{L} + (\partial_{i+1}S_j + S_i\partial_{j+1}) + S_iS_j.\end{aligned}$$
A direct calculation gives that
$$S_iS_j = (1/4)u_iu_j\,\partial_x^2 + (1/8)(u_iu_j' - u_ju_i')\partial_x - (1/8)u_iu_j'' + (1/16)u_i'u_j',$$
$$\partial_x S_i + S_i\partial_x = u_i\partial_x^2 - (1/4)u_i''.$$
Therefore,
$$\begin{aligned}\sum_{1\le i,j\le g} \mathcal{U}_i\mathcal{U}_j\mathcal{L}^{2g-i-j+1} &= \partial_x^2\mathcal{L}^{2g} - \sum_{1\le i\le g} (u_i\partial_x^2 - (1/4)u_i'')\mathcal{L}^{2g-i} \\ &\quad + \sum_{1\le i,j\le g} ((1/4)u_iu_j\partial_1^2 - (1/2)u_iu_j'' + (1/16)u_i'u_j')\mathcal{L}^{2g-i-j} \\ &= \mathcal{L}^{2g+1} + u_1\mathcal{L}^{2g} - \sum_{1\le i\le g} u_i\mathcal{L}^{2g-i+1} \\ &\quad - \sum_{1\le i\le g} (u_iu_1 - (1/4)u_i'')\mathcal{L}^{2g-i-1} \\ &\quad + \sum_{1\le i,j\le g} ((1/4)u_iu_j\mathcal{L}^{2g-i-j+1}) \\ &\quad + \sum_{1\le i,j\le g} ((1/4)u_iu_ju_1 - (1/8)u_iu_j'' + (1/16)u_i'u_j')\mathcal{L}^{2g-i-j}.\end{aligned}$$
We see that the coefficient of $\mathcal{L}^{2g-i}$ in this formula is exactly the coefficient of ξ^i in the expression $(1/16)(\mathbf{u}'(\xi)^2 + 2\mathbf{u}''(\xi)(2-\mathbf{u}(\xi)) + 4(u_1+\xi^{-1})(2-\mathbf{u}(\xi))^2 = (1/4)\mu(\xi)$. □

COROLLARY 6.1. *Let $\Psi(t_1, t_2, \dots, t_g)$ be a common eigenfunction of the operators $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$, with the eigenvalues $E, \alpha_1, \dots, \alpha_g$. Let $\xi = E^{-1}$, $\alpha(\xi) = \sum_{i=1}^g \alpha_i\xi^i$. Then*
$$4\alpha(\xi)^2 = \mu(\xi). \tag{6.2}$$

7. A common eigenfunction of the family $\{\mathcal{U}_i\}$

In this section we construct a common eigenfunction of the family of commuting differential operators $\{\mathcal{U}_i\}$.

LEMMA 7.1.

$$\frac{\partial}{\partial t_i}\left(\frac{D_j(2-\mathbf{u}(\xi))}{2-\mathbf{u}(\xi)}\right)=\frac{\partial}{\partial t_j}\left(\frac{D_i(2-\mathbf{u}(\xi))}{2-\mathbf{u}(\xi)}\right).$$

Proof. It follows from the definition of the operators $\partial(\eta)$ and D_i that the expression $\frac{\partial}{\partial t_i}\left(\frac{D_j(2-\mathbf{u}(\xi))}{2-\mathbf{u}(\xi)}\right)$ equals the coefficient of $\zeta^i\eta^j$ in the expansion of the function $\partial(\zeta)\frac{\xi\eta}{2(\xi-\eta)}\frac{2-\mathbf{u}(\eta)}{2-\mathbf{u}(\xi)}$ with respect to η and ζ. This function is equal to $\frac{1}{(2-\mathbf{u}(\xi))^2}\mathcal{T}_\xi^\eta\mathcal{T}_\xi^\zeta\mathbf{u}'(\xi)$ (see (3.10)). Since the generalized translation $\mathcal{T}_\xi^\eta$ is commutative, this function is symmetric with respect to the variables ζ and η. Consequently the coefficients of $\zeta^i\eta^j$ and $\zeta^j\eta^i$ are equal. □

COROLLARY 7.1. *There exists a function $F(\xi)=F(t_1,\dots,t_g,\xi)$ such that $\partial_iF=\frac{D_i(2-\mathbf{u}(\xi))}{2-\mathbf{u}(\xi)}$, $1\le i\le g$. The function $F(\xi)$ is uniquely determined up to an additive constant in a neighborhood of any point $(\bar t_0,\xi_0)=(t_1^0,\dots,t_g^0,\xi_0)$ such that $2-\mathbf{u}(\bar t_0;\xi_0)\neq 0$.*

Consider also the function $\Phi=\Phi(t_1,\dots,t_g;E=\xi^{-1},\alpha_1,\dots,\alpha_g)$ given by

$$\Phi=\sqrt{2-\mathbf{u}(\xi)}\exp\left(2\alpha(\xi)F(\xi)\right)\exp(-2\sum_{i=1}^{g}D_i(\alpha(\xi))t_i), \tag{7.1}$$

where $\alpha(\xi)=\sum_{i=1}^g\alpha_i\xi^i$. The function Φ is uniquely determined up to a multiplicative constant in a neighborhood of any point $(\bar t_0,\xi_0,\bar\alpha)$ such that $2-\mathbf{u}(\bar t_0;\xi_0)\neq 0$.

Let us find the derivatives of the function Φ with respect to $x=t_1$ and t_k, $k\ge 2$:

$$\Phi'=\frac{4\alpha(\xi)-\mathbf{u}'(\xi)}{2(2-\mathbf{u}(\xi))}\Phi, \tag{7.2}$$

$$\begin{aligned}\Phi''&=\left(\frac{-\mathbf{u}''(\xi)(\mathbf{u}(\xi)+\mathbf{u}'(\xi))(4\alpha(\xi)-\mathbf{u}'(\xi))}{2(2-\mathbf{u}(\xi))}+\frac{(4\alpha(\xi)-\mathbf{u}'(\xi))^2}{4(2-\mathbf{u}(\xi))^2}\right)\Phi\\&=\frac{16\alpha(\xi)^2-2\mathbf{u}''(\xi)(2-\mathbf{u}(\xi))-\mathbf{u}'(\xi)^2}{4(2-\mathbf{u}(\xi))^2}\Phi,\end{aligned} \tag{7.3}$$

$$\partial_k\Phi=\left(\frac{4\alpha(\xi)D_k(2-\mathbf{u}(\xi))-\partial_k\mathbf{u}(\xi)}{2(2-\mathbf{u}(\xi))}-2D_k(\alpha(\xi))\right)\Phi. \tag{7.4}$$

LEMMA 7.2. *The function Φ is an eigenfunction of the operator $\mathcal{L}$ with the eigenvalue $E\neq 0$ if and only if $\xi=E^{-1}$ and $\{\xi,4\alpha(\xi)\}\in\Gamma$, where Γ is a curve defined by equation* (4.5).

Proof. Equation (4.1) implies that

$$(\mathcal{L}-E)\Phi=\left(\frac{4\alpha(\xi)^2-\mu(\xi)}{2-\mathbf{u}(\xi)}-(\xi^{-1}-E)\right)\Phi.$$

The function in parentheses vanishes identically if and only if $4\alpha(\xi)^2-\mu(\xi)=(\xi^{-1}-E)(2-\mathbf{u}(\xi))$. Differentiating the last formula with respect to x, we obtain that $(\xi^{-1}-E)\mathbf{u}'(\xi)=0$. Hence $\xi^{-1}=E$ and $4\alpha(\xi)^2=\mu(\xi)$. □

THEOREM 7.1. *Suppose that* $4\alpha(\xi)^2 = \mu(\xi)$. *Then*

(1) *The function* Φ *is a common eigenfunction of the family* $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$, *with eigenvalues* $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$.
(2) *The space of common eigenfunctions of the operators* $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$ *with eigenvalues* $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$ *is one-dimensional.*

Proof. Express the operators $\mathcal{U}_k$ as

$$\mathcal{U}_k = \partial_k(\partial_x^2 - (u_1 + \xi^{-1})) + \xi^{-1}\partial_k - (1/2)u_k\partial_x + (1/4)u_k' - \partial_{k+1}. \tag{7.5}$$

Let Ψ be a common eigenfunction of $\mathcal{L}$, $\mathcal{U}_k$ with the eigenvalues indicated. Then

$$\left(\xi^{-1}\partial_k - (1/2)u_k\partial_x + (1/4)u_k' - \partial_{k+1}\right)\Psi = \alpha_k\Psi. \tag{7.6}$$

This allows us to express all partial derivatives $\partial_k\Psi$ in terms of Ψ and Ψ', namely

$$\partial_k\Psi = D_k(2 - \mathbf{u}(\xi))\Psi' + \frac{1}{2}D_k(\mathbf{u}'(\xi))\Psi - 2D_k(\alpha(\xi))\Psi, \quad 1 \le k \le g-1.$$

For $k = g-1$ one gets from (7.6) that

$$\xi^{-1}\partial_{g-1}\Psi - (1/2)u_{g-1}\Psi' + (1/4)u_{g-1}'\Psi = \alpha_g\Psi.$$

Therefore,

$$\xi^{-1}\left(D_g(2 - \mathbf{u}(\xi))\Psi' + \frac{1}{2}D_g(\mathbf{u}'(\xi))\Psi - 2D_g(\alpha(\xi))\right) - (1/2)u_g\Psi' + (1/4)u_g'\Psi - \alpha_g\Psi = 0.$$

Using Lemma 3.4 and (3.2) we obtain

$$(2 - \mathbf{u}(\xi))\Psi' = ((1/2)\mathbf{u}'(\xi) + \alpha(\xi))\Psi.$$

Thus,

$$\frac{\Psi'}{\Psi} = \frac{-(1/2)\mathbf{u}'(\xi) + \alpha(\xi)}{2 - \mathbf{u}(\xi)},$$

and

$$\begin{aligned}\frac{\partial_k\Psi}{\Psi} &= D_k(2 - \mathbf{u}(\xi))\frac{-(1/2)\mathbf{u}'(\xi) + \alpha(\xi)}{2 - \mathbf{u}(\xi)} + (1/2)D_k(\mathbf{u}'(\xi)) - 2D_k(\alpha(\xi)) \\ &= \frac{\alpha(\xi)D_k(2 - \mathbf{u}(\xi)) - (1/2)\partial_k\mathbf{u}(\xi)}{2 - \mathbf{u}(\xi)} - 2D_k(\alpha(\xi)), \quad k = 2, \dots, g.\end{aligned}$$

We see that $\frac{\partial_k\Psi}{\Psi} = \frac{\partial_k\Phi}{\Phi}$, $k = 1, \dots, g$. Therefore, $\Psi = \lambda\Phi$, where λ is a constant. $\square$

Consider now the special case $E = 0$.

THEOREM 7.2. *The space of common eigenfunctions of the operators* $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$ *with eigenvalues* $0, \alpha_1, \dots, \alpha_g$, *where* $4\alpha_g^2 = \mu_{2g}$, *is one-dimensional.*

Proof. Let Φ^0 be a common eigenfunction of operators $\mathcal{L}$, $\mathcal{U}_1, \dots, \mathcal{U}_g$ with eigenvalues $0, \alpha_1, \dots, \alpha_g$. Since $\mathcal{L}\Phi^0 = 0$, equation (6.1) implies that

$$\mathcal{U}_k\Phi^0 = (-(1/2)u_k\partial_x + (1/4)u_k' - \partial_{k+1})\Phi^0 = \alpha_k\Phi^0.$$

Therefore,

$$\partial_k\Phi^0 = -((1/2)u_{k-1}\partial_x - (1/4)u_{k-1}' + \alpha_{k-1})\Phi^0, \quad k = 2, \dots, g, \tag{7.7}$$

and

$$\partial_x\Phi^0 = \frac{u_g' - 4\alpha_g}{2u_g}\Phi^0. \tag{7.8}$$

Note that (7.8) and (4.1) imply

$$\partial_x^2\Phi^0 = \frac{2u_g''u_g - (u_g')^2 + 16\alpha_g^2}{4u_g^2}\Phi_0 = \frac{2u_g''u_g - (u_g')^2 + 4\mu_{2g}}{4u_g^2}\Phi^0 = u\Phi^0.$$

It follows from (7.7) and (7.8) that

$$\partial_k\Phi^0 = \frac{u_{k-1}(4\alpha_g - u_g') - u_g(4\alpha_{k-1} - u_{k-1}')}{4u_g}\Phi^0.$$

Since the logarithmic derivatives are uniquely defined, the space of eigenfunctions with eigenvalues $E = 0, \alpha_1, \dots, \alpha_g = (1/4)\sqrt{\mu_{2g}}$ is one-dimensional. □

Note that the function Φ can be expressed as

$$\Phi = \exp \widetilde{F}, \tag{7.9}$$

where

$$\widetilde{F} = \left(2\alpha(\xi)F - 2\sum D_i(\alpha(\xi))t_i + \frac{1}{2}\log(2 - \mathbf{u}(\xi))\right). \tag{7.10}$$

We have

$$\partial_i\widetilde{F} = \frac{D_i(2 - \mathbf{u}(\xi))(4\alpha(\xi) - \mathbf{u}'(\xi)) - D_i(4\alpha(\xi) - \mathbf{u}'(\xi))(2 - \mathbf{u}(\xi))}{2(2 - \mathbf{u}(\xi))}. \tag{7.11}$$

Using the notation $\partial(\eta) = \sum_{i=1}^g \eta^i\partial_i$, we can rewrite these formulas as

$$\begin{aligned}\partial(\eta)\widetilde{F} &= \frac{\xi\eta}{\xi - \eta}\frac{(2 - \mathbf{u}(\eta))(4\alpha(\xi) - \mathbf{u}'(\xi)) - (4\alpha(\eta) - \mathbf{u}'(\eta))(2 - \mathbf{u}(\xi))}{4(2 - \mathbf{u}(\xi))} \\ &= \frac{\mathcal{T}_\xi^\eta(4\alpha(\xi) - \mathbf{u}'(\xi))}{2(2 - \mathbf{u}(\xi))}.\end{aligned}$$

LEMMA 7.3. *The function* $\chi(\xi) = \partial_1\widetilde{F} = \frac{4\alpha(\xi) - \mathbf{u}'(\xi)}{2(2 - \mathbf{u}(\xi))}$ *satisfies the Riccati equation* $\chi'(\xi) + \chi(\xi)^2 = u_1 + \xi^{-1}$. *Moreover,*

$$\frac{1}{2 - \mathbf{u}(\eta)}\partial(\eta)\widetilde{F} = \frac{\xi\eta}{2(\xi - \eta)}(\chi(\xi) - \chi(\eta)).$$

Denote by V the hypersurface in $\mathbb{C}^{g+1} = \{(\xi, \alpha_1, \dots, \alpha_g)\}$ defined by equation (6.2). Recall that $\Gamma = \{\xi, y \in \mathbb{C}^2 \mid y^2 = 4\mu(\xi)\}$. In coordinates E, α_i the hypersurface V is given by the equation

$$4\left(\alpha_1E^{g-1} + \alpha_2E^{g-2} + \cdots + \alpha_g\right)^2 = 4E^{2g+1} + \mu_1E^{2g-1} + \cdots + \mu_{2g},$$

and the curve Γ is given by the equation $\eta^2 = 4(4E^{2g+1} + \mu_1E^{2g-1} + \cdots + \mu_{2g})$, where $\eta = y\xi^{-g}$.

Define the projection $\pi : V \to \Gamma$ by the formula $\pi(\xi, \alpha_1, \dots, \alpha_g) = (\xi, 2\alpha(\xi))$. In what follows we will regard the curve Γ as a subvariety of V using the canonical embedding $i : \Gamma \hookrightarrow V$ defined as $i(\xi, \eta) = (\xi, 0, 0, \dots, \eta)$. Let $V^* = \pi^{-1}(\Gamma^*)$, where $\Gamma^* = \{(\xi, y) \in \Gamma;\ \xi \neq 0\}$.

Recall that $\mathbb{C}^* = \{\xi \in \mathbb{C} \mid \xi \neq 0\}$. The function Φ in equation (7.1) is defined on the space $\mathbb{C}^g \times \mathbb{C}^* \times \mathbb{C}^g$ parametrized by coordinates $t_1, \dots, t_g, \xi, \alpha_1, \dots, \alpha_g$. Consider this space as a graded space using the following grading: $\deg t_k = 1 - 2k$, $\deg \xi = -2$, $\deg \alpha_k = 2k + 1$. Take also $\deg \mu_i = 2i + 2$. Then the equation $4\alpha(\xi) = \mu(\xi)$ defining the variety V^* is homogeneous.

LEMMA 7.4. *Let Φ be a common eigenfunction of the operators $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$ with eigenvalues $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$. Let $\gamma_2, \dots, \gamma_g \in \mathbb{C}$ be arbitrary constants. Then the function $\widetilde{\Phi} = \Phi \exp(\gamma_2 t_2 + \dots + \gamma_g t_g)$ is also a common eigenfunction of these operators with eigenvalues given by $E = \xi^{-1}$, $\widetilde{\alpha}_1 = \alpha_1 - \gamma_2$, $\widetilde{\alpha}_2 = \alpha_2 - \gamma_3 + \xi\gamma_2, \dots, \widetilde{\alpha}_i = \alpha_i - \gamma_{i+1} + \xi\gamma_i, \dots, \widetilde{\alpha}_g = \alpha_g + \xi\gamma_g$.*

Proof. Take $\gamma_1 = 0$. It is obvious that $\frac{\partial_k \widetilde{\Phi}}{\widetilde{\Phi}} = \frac{\partial_k \Phi}{\Phi} + \gamma_k$, $k = 1, \dots, g$, and $\mathcal{L}\widetilde{\Phi} = E\widetilde{\Phi}$. From (7.5) one obtains that

$$\begin{aligned}\mathcal{U}_k \widetilde{\Phi} &= (\xi^{-1}\partial_k - \frac{1}{2}u_k\partial_x + \frac{1}{4}u_k' - \partial_{k+1})\widetilde{\Phi} \\ &= \exp(\gamma_2 t_2 + \dots + \gamma_g t_g)\left(\mathcal{U}_k\Phi + (\xi^{-1}\gamma_k - \gamma_{k+1})\Phi\right).\end{aligned}$$

Therefore,

$$\frac{\mathcal{U}_k \widetilde{\Phi}}{\widetilde{\Phi}} = \widetilde{\alpha}_k,$$

where

$$\widetilde{\alpha}_k = \alpha_k + (\xi^{-1}\gamma_k - \gamma_{k+1}). \qquad \square \tag{7.12}$$

Note that $\sum_{i=1}^g \widetilde{\alpha}_i \xi^i = \sum_{i=1}^g \alpha_i \xi^i$.
Assume that $\deg \gamma_k = 2k - 1$.

COROLLARY 7.2. *Equation (7.12) defines a free action of the graded additive group $\mathbb{C}^{g-1}$ with coordinates $\gamma_2, \dots, \gamma_g$ on the variety V^*. The quotient space $V^*/\mathbb{C}^{g-1}$ is Γ^*. The vector bundle $V^* \to \Gamma^*$ is trivial.*

Proof. Define the map $s : \Gamma^* \times \mathbb{C}^{g-1} \to V^*$ by the formula $s(\xi, y, \gamma_2, \dots, \gamma_{g-2}) = (\xi, t, -\gamma_2, \xi^{-1}\gamma_2 - \gamma_3, \dots, \xi^{-1}\gamma_g)$. This is the required trivialization. $\square$

Consider the case $u_1 \equiv 0$. In this case the operators $\mathcal{U}_k$ and $\mathcal{L}$ are

$$\mathcal{L} = \partial_x^2, \ \ \mathcal{U}_k = \partial_x^2\partial_k - \partial_{k+1}. \tag{7.13}$$

LEMMA 7.5. *Let $\alpha_1, \dots, \alpha_g$ and ξ satisfy the equation $\left(\sum_{i=1}^g \alpha_i \xi^i\right)^2 = \xi^{-1}$. Then the function*

$$\Phi_0 = \exp\left(\sum_{1 \le k \le i \le g} \alpha_i \, t_k \, \xi^{i-k+1}\right) \tag{7.14}$$

is a common eigenfunction of operators (7.13) with eigenvalues $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$.

Proof. The logarithmic derivatives of the function Φ_0 are given by

$$\frac{\partial_k \Phi_0}{\Phi_0} = \sum_{i=k}^g \alpha_i \xi^{i-k+1}.$$

It is clear that $\partial_x^2 \Phi_0 = \xi^{-1}\Phi_0$ and $\mathcal{U}_k \Phi_0 = (\xi^{-1}\partial_k - \partial_{k+1})\Phi_0 = \alpha_k \Phi_0$. $\square$

The function Φ_0 can be obtained from the formula (7.1) by rescaling. This fact will be proved in Subsection 10.2.

8. Basic generating function for the solution of the stationary g-KdV equation

Denote $\mu(\xi,\eta) = 4\xi^{-1} + 4\eta^{-1} + 2\sum_{i=1}^{g} \mu_{2i}\xi^i\eta^i + \sum_{i=0}^{g-1} \mu_{2i+1}(\xi+\eta)\xi^i\eta^i$. We have $\mu(\xi,\xi) = 2\mu(\xi)$ and $\mu(\xi,\eta) = \mu(\eta,\xi)$, so $\mu(\xi,\eta)$ is a polarization of $\mu(\xi)$ (see Definition 3.1).

Consider the function

$$Q(\xi,\eta) = \mathbf{u}'(\xi)\mathbf{u}'(\eta) + (2-\mathbf{u}(\xi))\mathbf{u}''(\eta) + \mathbf{u}''(\xi)(2-\mathbf{u}(\eta)) \\ + 2(2-\mathbf{u}(\xi))(2-\mathbf{u}(\eta))\left(\xi^{-1}+\eta^{-1}+2u_1\right).$$

The function $Q(\xi,\eta)$ is a polarization of the function on the right-hand side of (4.1). Therefore $Q(\xi,\xi) = \mu(\xi,\xi)$. Equations (3.1) and (4.1) imply that

$$\left.\frac{\partial\mu(\xi,\eta)}{\partial\xi}\right|_{\xi=\eta} = \left.\frac{\partial Q(\xi,\eta)}{\partial\xi}\right|_{\xi=\eta}.$$

Denote also

$$P(\xi) = \frac{\xi^4}{8}\left(\left.\frac{\partial^2\mu(\xi,\eta)}{\partial\xi\,\partial\eta}\right|_{\xi=\eta} - \left.\frac{\partial^2 Q(\xi,\eta)}{\partial\xi\,\partial\eta}\right|_{\xi=\eta}\right).$$

LEMMA 8.1. *The function*

$$P(\xi,\eta) = \frac{\xi^2\,\eta^2}{4(\xi-\eta)^2}\left(2\mu(\xi,\eta) - Q(\xi,\eta)\right) \tag{8.1}$$

is a polarization of $P(\xi)$.

Proof. It is obvious that $P(\xi,\eta)$ is symmetric. Direct calculations show that

$$P(\xi) = \frac{\xi^4}{8}\left(2\left.\frac{\partial^2\mu(\xi,\eta)}{\partial\xi^2}\right|_{\xi=\eta} - 2\frac{\partial^2\mu(\xi)}{\partial\xi^2} + \left(\frac{\partial\mathbf{u}'(\xi)}{\partial\xi}\right)^2 - 2\frac{\partial\mathbf{u}''(\xi)}{\partial\xi}\frac{\partial\mathbf{u}(\xi)}{\partial\xi}\right. \\ \left. +4\xi^{-2}\frac{\partial\mathbf{u}(\xi)}{\partial\xi}(2-\mathbf{u}(\xi)) + 4(\xi^{-1}+u_1)\left(\frac{\partial\mathbf{u}(\xi)}{\partial\xi}\right)^2\right) = \frac{1}{2}\lim_{\xi\to\eta} P(\xi,\eta). \quad \square$$

COROLLARY 8.1. *$P(\xi,\eta)$ is a polynomial of degree g in the variables ξ and η.*

Define functions p_{ij} as coefficients in the expansion

$$P(\xi,\eta) = \sum_{i=1}^{g}\sum_{j=1}^{g} p_{ij}\xi^i\eta^j. \tag{8.2}$$

LEMMA 8.2. $p_{1i} = p_{i1} = u_i$.

Proof. The assertion follows from the formula $\sum_{i=1}^{g} p_{1i}\xi^i = \frac{P(\xi,\eta)}{\eta}\big|_{\eta\to 0} = \mathbf{u}(\xi)$. $\square$

This result motivates the following definition.

DEFINITION 8.1. The function $P(\xi,\eta)$ is called the basic generating function for the solution u_1 of the stationary KdV equation.

The coefficient of η^2 in (8.1) is equal to

$$2\sum_{i=1}^{g} p_{2i}\xi^i = -3(2-\mathbf{u}(\xi))(u_1+2\xi^{-1}) - \mathbf{u}''(\xi) + \mu_1\xi + 12\xi^{-1}. \tag{8.3}$$

Therefore,

$$u_i'' = 3u_iu_1 + 6u_{i+1} - 2p_{2,i} + \mu_1\delta_{1i}, \tag{8.4}$$

where δ_{ij} is the Kronecker symbol.

It will be shown later (see Subsection 10.1) that if $u_1 = 2\wp_{gg}$ is a solution of the stationary KdV equation from [**3**], then $u_i = 2\wp_{g,g-i+1}$, $u_i'' = 2\wp_{ggg,g-i+1}$, $p_{2,i} = 2\wp_{g-1,g-i+1}$. In this case equation (8.4) becomes the basic relation for $\wp$-functions (see (4.1) in [**3**]). All the results of [**3**] for the $\wp$-functions, derived from the basic relation, are thus true for an arbitrary solution of the stationary KdV.

LEMMA 8.3. $\partial(\zeta)P(\xi,\eta) = \partial(\xi)P(\zeta,\eta)$.

Proof. We have

$$\begin{aligned}
\partial(\zeta)P(\xi,\eta) = {} & \frac{\xi^2\eta^2\zeta^2}{8(\xi-\eta)(\xi-\zeta)(\eta-\zeta)}\Big(\mathbf{u}''(\xi)\big(\mathbf{u}'(\eta)(2-\mathbf{u}(\zeta)) - \mathbf{u}'(\zeta)(2-\mathbf{u}(\eta))\big) \\
& - \mathbf{u}''(\eta)\big(\mathbf{u}'(\xi)(2-\mathbf{u}(\zeta)) - \mathbf{u}'(\zeta)(2-\mathbf{u}(\xi))\big) \\
& + \mathbf{u}''(\zeta)\big(\mathbf{u}'(\xi)(2-\mathbf{u}(\eta)) - \mathbf{u}'(\eta)(2-\mathbf{u}(\xi))\big)\Big) \\
& + \frac{\xi\eta\zeta^2}{4(\xi-\zeta)(\eta-\zeta)}\mathbf{u}'(\zeta)(2-\mathbf{u}(\xi))(2-\mathbf{u}(\eta)) \\
& - \frac{\xi\eta^2\zeta}{4(\xi-\eta)(\eta-\zeta)}\mathbf{u}'(\eta)(2-\mathbf{u}(\xi))(2-\mathbf{u}(\zeta)) \\
& + \frac{\xi^2\eta\zeta}{4(\xi-\eta)(\xi-\zeta)}\mathbf{u}'(\xi)(2-\mathbf{u}(\eta))(2-\mathbf{u}(\zeta)) \\
= {} & \frac{1}{2}\mathcal{B}_3(\mathbf{u}''(\xi),\mathbf{u}'(\xi),(2-\mathbf{u}(\xi))) + \mathcal{B}_3(\mathbf{u}'(\xi),2-\mathbf{u}(\xi),(2-\mathbf{u}(\xi))\xi^{-1}).
\end{aligned}$$

Thus, $\partial(\zeta)P(\xi,\eta)$ is symmetric as a function of variables ξ,η,ζ. □

COROLLARY 8.2. *There exists a function $\phi = \phi(t_1,\dots,t_g)$ such that $P(\xi,\eta) = \partial(\xi)\partial(\eta)\phi$.*

COROLLARY 8.3. $P'(\xi,\eta) = \partial(\eta)\mathbf{u}(\xi)$.

Proof. Indeed,

$$\begin{aligned}
P'(\xi,\eta) &= \frac{\partial(\zeta)P(\xi,\eta)}{\zeta}\Bigg|_{\zeta\to 0} \\
&= \frac{\xi\eta}{2(\xi-\eta)}(\mathbf{u}'(\xi)(2-\mathbf{u}(\eta)) - (2-\mathbf{u}(\xi))\mathbf{u}'(\eta)) = \partial(\eta)\mathbf{u}(\xi).
\end{aligned}$$ □

Note that it follows from Theorem 3.2 that $\partial_x P(\xi,\eta) = \mathcal{T}_\xi^\eta \partial_x \mathbf{u}(\xi)$.

9. A construction of the w-function

Consider the equation

$$2\partial_x^2 \log w = -u_1, \tag{9.1}$$

with the initial conditions

$$w(0) = 1, \qquad \partial_k w(0) = 0, \quad k = 1,\dots,g; \tag{9.2}$$

here $u_1 = u_1(t_1,\dots,t_g)$ is a solution of the stationary g-KdV equation with respect to $x = t_1$.

THEOREM 9.1. *There exists a differentiable solution w of* (9.1), (9.2) *such that the functions*

$$u_k = -2\partial_x\partial_k \log w, \quad k = 1, \dots, g, \tag{9.3}$$

satisfy the hypotheses of Theorem 4.2.

DEFINITION 9.1. The solutions of (9.1) described in Theorem 9.1 are called special.

THEOREM 9.2. *Let $p_{ij}(t)$ be as in* (8.1) *and* (8.2). *Then there is a unique special solution of* (9.1) *such that*

$$2\partial_i\partial_j \log w = -p_{ij} \tag{9.4}$$

for all i, j.

Proof. The existence of a required solution of (9.4) follows from Corollary 8.2. The function w is defined by (9.4) up to a factor $\exp(\lambda_0 + \lambda_1 t_1 + \dots + \lambda_g t_g)$. All the constants λ_i are uniquely determinated by the initial conditions (9.2). □

This result completed the solution of Problem 1.

DEFINITION 9.2. The special solution (9.1) described in Theorem 9.2 is called the w-function of the solution u of the stationary g-KdV equation.

The relations between the higher logarithmic derivatives of the w-function are obtained using the the technique of generating function. For example

$$\sum_{ijk} \xi^i \eta^j \zeta^k \partial_i\partial_j\partial_k \log w = \partial(\zeta) P(\xi, \eta).$$

This function was calculated in Lemma 8.3.

The solution u is a point of the space $\mathbf{R}_g$ (see Section 5). Consequently, we can treat the w-function as a function $w : \mathbb{C}^g \times \mathbf{R}_g \to \mathbb{C}$.

The rest of this section is devoted to an explicit construction of the w-function, starting with the given solution u of the KdV equation.

Denote $t = (x, t_2, \dots, t_g)$ and put

$$\varphi(t) = \frac{1}{2}\int_0^x \int_0^x u(t)\, \mathrm{d}x.$$

Then equation (9.1) implies that $w(t) = \exp(a(t) - \varphi(t))$ where $a''(t) = 0$. Therefore, $a(t) = a_1(\widetilde{t})x + a_0(\widetilde{t})$, where $\widetilde{t} = (t_2, \dots, t_g)$. The initial condition (9.2) now gives $a_0(0) = 0$, $a_1(0) = 0$, and $\partial_k a_0(0) = 0$, $k = 2, \dots, g$. It follows from (9.3) that

$$2\partial_k a_1(\widetilde{t}) = -u_k + 2\partial_k \int_0^x u(t)\mathrm{d}t, \quad k = 2, \dots, g. \tag{9.5}$$

The set of equations (9.5) with the initial condition $a_1(0) = 0$ has a unique solution $a_1(\widetilde{t})$. It follows from (9.4) that $2\partial_i\partial_j a_0(\widetilde{t}) = 2\partial_i\partial_j\varphi(t) - 2\partial_i\partial_j\, a_1(\widetilde{t})x - p_{ij}(t)$. These equations with the initial condition $a_0(0) = 0$, $\partial_k a_0(0) = 0$, $k = 1, \dots, g$, have a unique solution $a_0(\widetilde{t})$.

10. Applications

10.1. Kleinian σ-function. Consider hyperelliptic Kleinian functions $\sigma(t)$, $\zeta_i(t) = \partial_i \log \sigma(t)$, and $\wp_{ij}(t) = -2\partial_i\partial_j \log \sigma(t)$. The function $2\wp_{gg}(t)$ is a solution of the stationary KdV equation (see [**3**]).

COROLLARY 10.1. *Let $z \in \mathbb{C}^g$ be a point where $\sigma(z) \neq 0$. Then the function*

$$w(t) = \frac{\sigma(t+z)}{\sigma(z)} \exp\langle -\zeta(z), t\rangle$$

is the w-function of the solution $2\wp_{gg}(t+z)$.

Proof. The functions $u_i = 2\wp_{g,g-i+1}$ and $p_{ij} = 2\wp_{g-i+1,g-j+1}$ satisfy equations (8.1), (9.4) (see [**2**]). The corollary now follows from the uniqueness of the w-function. $\square$

Let θ_g be the polynomials from [**1**]. The second logarithmic derivatives of θ_g give solutions of the higher KdV equations. As was proved in [**4**], the polynomial θ_g is, up to a linear change of variables, a rational limit $\widehat{\sigma}_g$ of the σ-function of genus g. Denote $\widehat{\zeta}_i(t) = \partial_i \log \widehat{\sigma}_g(t)$.

COROLLARY 10.2. *Let $z \in \mathbb{C}^g$ be a point where $\widehat{\sigma}_g(z) \neq 0$. Then the function*

$$w(t) = \frac{\widehat{\sigma}_g(t+z)}{\widehat{\sigma}_g(z)} \exp\left\langle -\widehat{\zeta}(z), t\right\rangle$$

is the w-function of the solution $u = -2(\log\theta_g)''$.

10.2. The homogeneity condition. The results obtained in this subsection follow from the uniqueness theorems for the w-functions.

LEMMA 10.1. *Suppose that $u(x, t_2, \dots, t_g)$ is a solution of the stationary g-KdV equation with respect to x. Take $\kappa \in \mathbb{C}^*$. Then the function $\widehat{u}(x, t_2, \dots, t_g) = \kappa^2 u(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g)$ is also a solution of the stationary g-KdV equation. Under the transformation $u \to \widehat{u}$ the constants μ_i and a_i change to $\widehat{\mu}_i = \mu_i \kappa^{2i+2}$, $\widehat{a}_i = \kappa^{2g-2i} a_i$.*

Proof. Let $\{u_1 = u, u_2, \dots, u_g\}$ be a sequence of functions from Theorem 4.2. Then the functions

$$\widehat{u}_i(x, t_2, \dots, t_g) = \kappa^{2i} u_i(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g) \tag{10.1}$$

satisfy the hypotheses of Lemmas 2.3 and 2.4. Therefore by Theorem 4.2 the function $\widehat{u}$ is a solution of the stationary KdV equation. The values $\widehat{\mu}_i$ are determined by (4.1); the values $\widehat{a}_i$ are found from (4.4). $\square$

Thus we have an action of the group $\mathbb{C}^*$ on the space $\mathbf{R}_g$. It is obvious that under this action the initial values $c_j = u^{(j)}(0)$ are transformed as $\widehat{c}_j = \kappa^{2j+1} c_j$.

Denote by $\widehat{w}$ the w-function of the solution $\widehat{u}$.

LEMMA 10.2. *The w-functions w and $\widehat{w}$ of the solutions u and $\widehat{u}$ are related as follows:*

$$\widehat{w}(t_1, \dots, t_g) = w(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g).$$

Proof. The functions u_i, $\widehat{u}_i$ are related by equation (10.1). It follows from (8.1), (9.4) that

$$-2\partial_i\partial_j \widehat{w}(t_1, \dots, t_g) = \widehat{p}_{ij} = \kappa^{2i+2j-2} p_{ij} = -2\partial_i\partial_j w(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g).$$

Since the w-function is unique, this completes the proof. $\square$

Consider now the w-function as a function on the space $\mathbb{C}^g \times \mathbf{R}_g$.

THEOREM 10.1. *The w-function satisfies the following homogeneity condition:*

$$\begin{aligned} &w(t_1, \dots, t_g, a_0, \dots, a_{g-2}, c_0, \dots, c_{2g}) \\ &\qquad = w(\kappa t_1, \dots, \kappa^{2g-1} t_g, \kappa^{-2g} a_0, \dots, \kappa^{-4} a_{g-2}, \kappa^{-1} c_0, \dots, \kappa^{-2g-1} c_{2g}). \end{aligned}$$

Proof. The theorem follows directly from Lemmas 10.1 and 10.2. $\square$

Consider the function $\widehat{\mathbf{u}}(\xi) = \sum_{i=1}^{g} \widehat{u}_i \xi_i$. It follows from (10.1) that

$$\widehat{\mathbf{u}}(x, t_2, \dots, t_g; \xi) = \mathbf{u}(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g; \kappa^{-2} \xi). \tag{10.2}$$

THEOREM 10.2. *Let $\Phi(t_1, \dots, t_g; \xi^{-1}, \alpha_1, \dots, \alpha_g)$ be a common eigenfunction of the operators $\mathcal{L}$, $\mathcal{U}_1, \dots, \mathcal{U}_g$ with the eigenvalues $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$ (see Section 7). Then the function*

$$\begin{aligned} &\widetilde{\Phi}(t_1, \dots, t_g; \xi^{-1}, \alpha_1, \dots, \alpha_g; \kappa) \\ &\qquad = \Phi(t_1 \kappa, \dots, t_g \kappa^{2g-1}; \xi^{-1} \kappa^{-2}, \alpha_1 \kappa^{-3}, \dots, \alpha_g \kappa^{-2g-1}) \end{aligned} \tag{10.3}$$

is regular as a function of κ in the vicinity of the origin, and

$$\widetilde{\Phi} = \exp\left(\sum_{1 \le i \le j \le g} \alpha_j \, \xi^{j-i+1} \, t_i \right) + O(\kappa).$$

Proof. Denote $\widehat{t} = (t_1 \kappa, \dots, t_g \kappa^{2g-1})$. Using (7.9), (7.11), one gets

$$\begin{aligned} \frac{\partial_i \widetilde{\Phi}}{\widetilde{\Phi}} &= \kappa^{2i-1} (\partial_i \widetilde{F})(\widehat{t}; \xi^{-1} \kappa^{-2}, \alpha_1 \kappa^{-3}, \dots, \alpha_g \kappa^{-2g-1}) \\ &= \frac{\xi^{1-i}}{4(2 - u_1(\widehat{t}) \xi \kappa^2 - \cdots)} \Bigg(2 \sum_{j \ge i} (4 \alpha_j \xi^j - u_j'(\widehat{t}) \xi^j \kappa^{2j+1}) \\ &\qquad - \sum_{j<i} u_j(\widehat{t}) \xi^j \kappa^{2j} \sum_{j \ge i} (4 \alpha_j \xi^j - u_j'(\widehat{t}) \xi^j \kappa^{2j+1}) \\ &\qquad + \sum_{j \ge i} u_j(\widehat{t}) \xi^j \kappa^{2j} \sum_{j<i} (4 \alpha_j \xi^j - u_j'(\widehat{t}) \xi^j \kappa^{2j+1}) \Bigg). \end{aligned}$$

Note that $\widehat{t} \to (0, \dots, 0)$ and $u_i(\widehat{t}) \to u_i(0)$ as $\kappa \to 0$. Therefore we obtain

$$\frac{\partial_i \widetilde{\Phi}}{\widetilde{\Phi}} = \sum_{j \ge i} \alpha_j \xi^{j-i+1} + O(\kappa). \qquad \square$$

This allows us to obtain a deformation of the function Φ. We see that $\widetilde{\Phi}$ tends to the function Φ_0 of (7.14) as $\kappa \to 0$.

Consider the space $L = \mathbb{C}^g \times \mathbb{C}^* \times \mathbb{C}^g$ with coordinates $(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g)$. Consider also an action of the group $\mathbb{C}^*$ on the space L given by the formula

$$\kappa(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) = (t_1 \kappa,\ t_2 \kappa^3, \dots, t_g \kappa^{2g-1}; \xi \kappa^2, \alpha_1 \kappa^{-3}, \dots, \alpha_g \kappa^{-2g-1}).$$

This defines a projection $p : L \to M$ where $M = L/\mathbb{C}^*$. Take a small $\varepsilon > 0$ and denote

$$L_\varepsilon = \{(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) \in L : |\xi| \geq \varepsilon\},$$
$$\partial L_\varepsilon = \{(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) \in L \ : \ |\xi| = \varepsilon\}.$$

Glue the boundary ∂L_ε to the space M using the projection p to obtain the space $Z_\varepsilon = L_\varepsilon \cup_p M$.

Let $\varepsilon_2 < \varepsilon$. Then there is a map $L_\varepsilon \to L_{\varepsilon_2}$ defined by the formula

$$(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) \to \kappa(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g),$$

where $\kappa = \varepsilon^{-1/2}\varepsilon_2^{1/2}$. This map sends the boundary ∂L_ε to the boundary $\partial L_{\varepsilon_2}$, so it can be lifted to a map $Z_\varepsilon = L_\varepsilon \cup M \to Z_{\varepsilon_2} = L_{\varepsilon_2} \cup M$. Denote $Z = \lim_{\varepsilon \to 0} Z_\varepsilon$ and recall that $\mathbb{C}^* \times V^* \subset L$.

Consider the embedding $\mathbb{C}^g \times V^* \to Z$. This embedding covers the embedding $\Gamma^* \to \Gamma$. Approaching the limit point in Γ corresponds to $\xi \to 0$ in the space $M \subset Z$. So we get the following result:

THEOREM 10.3. *On the space Z there is a function $\widehat{\Phi}$ such that $\widehat{\Phi}|_L = \Phi$ and $\widehat{\Phi}|_M = \Phi_0$.*

If $\xi \to 0$, then for the restriction $\Phi|_\Gamma = \Phi(t_1, \dots, t_g, \xi^{-1}, 0, 0, \dots, \alpha_g)$ one has $\Phi \sim \exp\left(\sum_{1 \leq j \leq g} \alpha_g \, \xi^{g-j+1} \, t_j\right)$. Take a local parameter $k = \alpha_g \xi^g$. It now follows from the equation $(a_g \xi^g)^2 = \mu(\xi) = 4\xi^{-1} + O(\xi)$ that $\Phi \sim \exp(\sum_{j=1}^g k^{2j-1} t_j)$.

So, the restriction $\Phi|_\Gamma$ has the same analytic properties as the Baker–Akhiezer function ([**16**]) of the solution u. By the uniqueness of the Baker–Akhiezer function we conclude that $\Phi|_\Gamma$ coincides with the Baker–Akhiezer function.

11. Examples

In this section we demonstrate the key constructions of the paper in the cases $g = 1$ and $g = 2$.

11.1. $g = 1$. We start with a solution u of the classical stationary KdV equation $u''' - 6uu' = 0$. Suppose that $x = 0$ is a regular point of the function u. Then the solution u with given values $c_0 = u(0)$, $c_1 = u'(0)$, $c_2 = u''(0)$ is unique in a neighbourhood of the point $x = 0$.

The key equation (4.1) becomes

$$4(4\xi^{-1} + \mu_1 \xi + \mu_2 \xi^2) = (u')^2 \xi^2 + 2u'' \xi (2 - u\xi) + 4(\xi^{-1} + u)(2 - u\xi)^2;$$

hence $\mu_1 = u'' - 3u^2$, $\mu_2 = (1/4)((u')^2 - 2u''u + 4u^3)$. It is easy to see that $\mu_1' = 0$ and $\mu_2' = 0$. Therefore μ_1, μ_2 are constants and so $\mu_1 = c_2 - 3c_0^2$, $\mu_2 = (1/4)(c_1^2 - 2c_2 c_0 + 4c_0^3)$.

The equation of the hyperelliptic curve is

$$4(4\xi^{-1} + \mu_1 \xi + \mu_2 \xi^2) = y^2.$$

The w-function is $w = \exp(-\phi(x))$, where $\phi(x) = \frac{1}{2} \int_0^x \int_0^x u(x) \mathrm{d}x$.

The birational equivalence $v : \mathbf{R}_1 \to \mathcal{U}_1$ is given by the formula $v(u_1) = (\Gamma, (\xi, y))$, where $\xi = 2/c_0$, $y = 2c_1/c_0$.

Let Φ be a common eigenfunction of the operators $\mathcal{L}$ and $\mathcal{U}_1 = A_1$ with the eigenvalues $E = \xi^{-1}$ and α, respectively. Then the logarithmic derivative of the function Φ is

$$\frac{\Phi'}{\Phi} = \frac{4\,\alpha\,\xi - u'(x)\xi}{2(2 - u(x)\xi)},$$

where $4\alpha\xi = 2\sqrt{4\xi^{-1} + \mu_1\xi + \mu_2\xi^2}$. Therefore $\partial_x \log\Phi \sim \xi^{-1/2}$ as $\xi \to 0$. Let $z \in \mathbb{C}$ be such that $u(z) = 2\xi^{-1}$ and $u'(z) = 4\alpha$. Then

$$\widetilde{F}'(x;\xi^{-1},\alpha) = \widetilde{F}'(x;z) = \frac{1}{2}\frac{u'(z) - u'(x)}{u(z) - u(x)}.$$

11.2. $g = 2$. It follows from the equation $[\mathcal{L}, A_2 + a_0A_0] = 0$ that $u^{(5)} - 10uu^{(3)} - 20u''u' + 30\,u^2u' + 16a_0u' = 0$. Equation (4.4) implies that $\mu_1 = 8a_0$. We have $\mathbf{u}(\xi) = u\xi + u_2\xi^2$. The equation (4.1) gives

$$\begin{aligned}
4(4\xi^{-1} &+ \mu_1\xi + \mu_2\xi^2 + \mu_3\xi^3 + \mu_4\xi^4)\\
&= \xi(-12u^2 - 16u_2 + 4u'') + \xi^2(4u^3 - 8uu_2 + (u')^2 - 2uu'' + 4u_2'')\\
&+ \xi^3(8u^2u_2 + 4u_2^2 + 2u'u_2' - 2u_2u'' - 2uu_2'') + \xi^4(4uu_2^2 + (u_2')^2 - 2u_2u_2'').
\end{aligned}$$

Therefore, $u_2 = \frac{1}{4}(u'' - 3u^2 - 8a_0)$. Now we can describe μ_2, μ_3 and μ_4 as constants in the following ordinary differential equation for u:

$$\begin{aligned}
\mu_2 &= \frac{1}{4}\left(4u^{(4)} - 10uu'' - 5(u')^2 + 10u^3 + 16a_0u\right),\\
\mu_3 &= \frac{1}{16}(2u'u''' - 2uu^{(4)} - 2(u'')^2 - 15u^4 + 8u^2u'' - 16u^2a_0 + 12u^2u' + 64a_0),\\
\mu_4 &= \frac{1}{64}((u''')^2 + 16(u'')^2u - 2u''u^{(4)} + 12(u')^2u'' + 6u^2u^{(4)} + 32u^5 - 30u''u^3\\
&\quad - 12u'''u'u - 160a_0uu'' + 132a_0u^3 + 16a_0u^{(4)} - 96a_0(u')^2 + 256a_0^2u).
\end{aligned}$$

Finally, the birational equivalence $v : \mathbf{R}_2 \to \mathcal{U}_2$ is given by the formula $v(u_1) = v(c_0,\ldots,c_4,a_0) = (\Gamma, [(\xi_1,y_1),(\xi_2,y_2)])$. Here for the construction of Γ the coefficients μ_1,μ_2,μ_3,μ_4 are used, obtained from the formula above by substitution c_k for $u^{(k)}$. The pairs (ξ_i,y_i) are the following ones: ξ_1,ξ_2 are the roots of the equation $2 - c_0\xi - \frac{1}{4}(c_2 - 3c_0^2 - 8a_0)\xi^2 = 0$ and y_1,y_2 are defined by the formula $y_i = c_1\xi_i + \frac{1}{4}(c_3 - 6c_1c_0)$.

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