GEOMETRIC METHODS IN PHYSICS

Proceedings of the XXVIII Workshop on Geometric Methods in Physics

Białowieża, Poland *28 June – 4 July 2009*

EDITORS

Piotr Kielanowski
University of Białystok, Poland
and
Cinvestav, México City, México

S. Twareque Ali
Concordia University, Montréal, Quebec, Canada

Anatol Odzijewicz
University of Białystok, Poland

Martin Schlichenmaier
University of Luxembourg, Luxembourg

Theodore Voronov
University of Manchester, United Kingdom

All papers have been peer reviewed

SPONSORING ORGANIZATION
University of Białystok, Poland

AMERICAN INSTITUTE OF PHYSICS

Melville, New York, 2009
AIP CONFERENCE PROCEEDINGS ▌ 1191

Heat Equations and Sigma Functions

V. M. Buchstaber

Steklov Mathematical Institute, Russian Academy of Sciences
buchstab@mi.ras.ru
School of Mathematics, University of Manchester, Manchester
Victor.Buchstaber@manchester.ac.uk

Abstract. A review of solutions of some important differential equations in terms of sigma function is given. In particular we discuss the heat equation and the Chazy type equations.

Keywords: heat equation, Chazy equation, sigma functions
PACS: 02.30.Jr, 02.30Gp

INTRODUCTION

The theory of Abelian functions was a central topic of the 19th century mathematics. In mid-seventies of the last century a new wave of investigation arose in this field in response to the discovery that Abelian functions provide solutions to a number of challenging problems of modern Theoretical and Mathematical Physics.

In the cycle of our joint papers with V. Enolski and D. Leykin we have developed a theory of multivariate sigma function, an analogue of the classical Weierstrass sigma function.

Considerable progress in the development of the theory of sigma functions was made with the construction of operators annihilating sigma functions and the calculation of Lie algebras of such operators (see [1]). In the hyperelliptic case, this calculation is very effective (see [2, 3]).

A number of applications of the theory of multivariate sigma functions have been found (see [4–17]).

This survey is based on the joint papers with D. Leykin and E. Bunkova. We will describe in terms of sigma functions solutions of some important differential equations related to the classical heat equations.

HEAT EQUATION AND CHAZY EQUATION

The elliptic theta function $\theta(z,\tau)$ is an entire function of z defined by the series

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z},$$

where $\tau \in \mathbb{C}$ is a parameter with $\operatorname{Im}\tau > 0$.

CP1191, *Geometric Methods in Physics—Proceedings of the XXVIII Workshop on Geometric Methods in Physics*
edited by P. Kielanowski, S. Twareque Ali, A. Odzijewicz, M. Schlichenmaier, and T. Voronov

We need the following properties of the theta function.

1. This function is quasi periodic in z with respect to the lattice generated by $(1,\tau)$:

$$\theta(z+1,\tau)=\theta(z,\tau), \quad \theta(z+\tau,\tau)=e^{-2\pi i(z+\frac{1}{2}\tau)}\theta(z,\tau).$$

2. The theta function satisfies the heat equation

$$\frac{\partial}{\partial\tau}\theta(z,\tau)=\frac{1}{4\pi i}\frac{\partial^2}{\partial z^2}\theta(z,\tau). \tag{1}$$

We also need the function

$$\theta_1(z,\tau)=e^{\frac{\pi i}{4}\tau+\pi i(z+\frac{1}{2})}\theta\left(z+\frac{1}{2}(1+\tau),\tau\right)=\sum_{n\in\mathbb{Z}}e^{\pi i(n+\frac{1}{2})^2\tau+2\pi i(n+\frac{1}{2})(z+\frac{1}{2})},$$

which satisfies the heat equation (1) as well as the quasi periodicity condition $\theta_1(z+1,\tau)=-\theta_1(z,\tau)$.

Each elliptic curve whose affine part has the Weierstrass form

$$V_{(g_2,g_3)}=\{(x,y)\in\mathbb{C}^2 \mid y^2=4x^3-g_2x-g_3\}$$

determines a sigma function $\sigma(u,g_2,g_3)$. This function is an entire function of $u\in\mathbb{C}$ with parameters $(g_2,g_3)\in\mathbb{C}^2$; it has a series expansion in powers of u over the polynomial ring $\mathbb{Q}[g_2,g_3]$ in the vicinity of $u=0$ (see, e.g., [3]). An initial segment of the series has the form

$$\sigma(u)=u-\frac{g_2u^5}{2\cdot 5!}-\frac{6g_3u^7}{7!}-\frac{g_2^2u^9}{4\cdot 8!}-\frac{18g_2g_3u^{11}}{11!}+(u^{13}). \tag{2}$$

We need the following properties of the sigma function.

Set

$$\begin{pmatrix} l_0\\ l_2\end{pmatrix}=T\begin{pmatrix}\frac{\partial}{\partial g_2}\\ \frac{\partial}{\partial g_3}\end{pmatrix}, \quad T=\begin{pmatrix}4g_2 & 6g_3\\ 6g_3 & \frac{1}{3}g_2^2\end{pmatrix}, \tag{3}$$

$$H_0=u\frac{\partial}{\partial u}-1, \quad H_2=\frac{1}{2}\frac{\partial^2}{\partial u^2}+\frac{1}{24}g_2u^2. \tag{4}$$

1. The elliptic sigma-function satisfies the system of equations

$$Q_0\sigma=0, \quad Q_2\sigma=0, \tag{5}$$

   where $Q_0=l_0-H_0$, $Q_2=l_2-H_2$.
2. The equation $Q_0\sigma=0$ implies that the function σ is homogeneous of degree 1 in the variables (u,g_2,g_3) with respect to the grading $\deg u=1$, $\deg g_2=-4$, $\deg g_3=-6$.
3. The discriminant of the elliptic curve $V_{(g_2,g_3)}$ equals $\Delta=g_2^3-27g_3^2$. Let $\mathcal{B}=\{(g_2,g_3)\in\mathbb{C}^2 \mid \Delta\neq 0\}$; then the curve $V_{(g_2,g_3)}$, where $(g_2,g_3)\in\mathcal{B}$, is non degenerate.

The function $\sigma(u)$ is quasi periodic in u with respect to the lattice generated by $(2\omega, 2\omega')$:

$$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u), \quad \sigma(u+2\omega') = -e^{2\eta'(u+\omega')}\sigma(u).$$

The parameters ω, ω', η and η' are determined by the relations

$$2\omega = \oint_a \frac{dx}{y}, \quad 2\omega' = \oint_b \frac{dx}{y}, \quad 2\eta = -\oint_a \frac{xdx}{y}, \quad 2\eta' = -\oint_b \frac{xdx}{y},$$

where $\frac{dx}{y}$ and $\frac{xdx}{y}$ form a basis of holomorphic differentials on $V_{(g_2,g_3)}$ and a and b are basis cycles on the curve such that the integrals satisfy the Legendre identity $\eta\omega' - \omega\eta' = \frac{\pi i}{2}$.

In this notation, we have

$$\sigma(u, g_2, g_3) = \sqrt{\frac{\pi}{\omega}} \frac{1}{\sqrt[8]{\Delta}} \exp(2\omega\eta z^2)\theta_1(z, \tau), \tag{6}$$

where $u = 2\omega z$, $\omega' = \omega\tau$, and $\eta = 2\varkappa\omega$.

Theorem 1 (uniqueness conditions for the sigma function).
The entire function $\sigma(u, g_2, g_3)$ is uniquely determined by the conditions

$$Q_0\sigma = 0, \quad Q_2\sigma = 0, \quad \sigma(0, g_2, g_3) = 0, \quad \left(\frac{\partial}{\partial u}\sigma(u, g_2, g_3)\right)\Big|_{u=0} = 1.$$

Proof. Consider the series expansion of the sigma function $\sigma(u, g_2, g_3)$ in powers of u in the form

$$\sigma(u, g_2, g_3) = \sum_k \gamma_k(g_2, g_3)u^k.$$

The conditions of the theorem give us a recursion for the coefficients $\gamma_k(g_2, g_3)$ as polynomials of g_2 and g_3. The explicit form of this recursion see in [3]. □

The vector fields l_0, l_2 are tangent to the discriminant variety

$$\{(g_2, g_3) \in \mathbb{C}^2 \,|\, \Delta(g_2, g_3) = 0\}$$

(see, e.g., [18]) because

$$l_0\Delta = 12\Delta, \quad l_2\Delta = 0.$$

Let $\big(g_2(t), g_3(t)\big)$ be the integral curve of the field

$$\frac{\partial}{\partial t} = \dot{g}_2\frac{\partial}{\partial g_2} + \dot{g}_3\frac{\partial}{\partial g_3} = \gamma(t)l_0 + l_2$$

on $\mathcal{B}$. Then along the curve $\big(g_2(t), g_3(t)\big)$ we have

$$0 = (\gamma Q_0 + Q_2)\sigma = (\gamma l_0 + l_2)\sigma + (-\gamma u\partial_u + \gamma - \frac{1}{2}\partial_u^2 - \frac{g_2}{24}u^2)\sigma.$$

Therefore

$$\frac{\partial}{\partial t}\sigma = \frac{1}{2}\partial_u^2\sigma + \gamma(t)u\partial_u\sigma + \frac{g_2}{24}u^2\sigma - \gamma(t)\sigma.$$

and

$$\dot{g_2} = 4\gamma g_2 + 6g_3, \qquad \dot{g_3} = 6\gamma g_3 + \frac{1}{3}g_2^2. \tag{7}$$

Let $\gamma(t)$ be a solution of the Chazy equation (the theory of this equation see e.g. in [19]).

$$\dddot{\gamma}(t) = 12\gamma(t)\ddot{\gamma}(t) - 18\dot{\gamma}(t)^2. \tag{8}$$

Lemma 1. *The system* (7) *has the solution*

$$g_2(t) = 12\gamma(t)^2 - 12\dot{\gamma}(t), \qquad g_3(t) = 12\gamma(t)\dot{\gamma}(t) - 2\ddot{\gamma}(t) - 8\gamma(t)^3.$$

Set

$$\sigma(u, g_2(t), g_3(t)) = e^{s(u,t)}\phi(u,t), \tag{9}$$

where $s(u,t) = -\frac{\gamma(t)}{8\omega^2}u^2 + b(t)$ and $\dot{b}(t) = -\frac{3}{2}\gamma(t)$.

Lemma 2. *Let $\sigma(u,g_2,g_3)$ be the sigma function with parameters ω, ω', η, and η'. Then the function $\phi(u,t)$ defined by* (9)*, satisfies the periodicity condition $\phi(u+2\omega,t) = -\phi(u,t)$ if and only if $\gamma(t) = -4\omega\eta$.*

Proof. We have

$$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u) = -e^{2\eta(u+\omega)}e^{s(u,t)}\phi(u).$$

On the other hand

$$\sigma(u+2\omega) = e^{s(u+2\omega,t)}\phi(u+2\omega) = -e^{s(u+2\omega,t)}\phi(u).$$

Therefore, $-\frac{\gamma}{2\omega}(u+\omega) = 2\eta(u+\omega)$ and $\gamma = -4\omega\eta$. □

Theorem 2. *The function $\phi(u,t)$ defined by* (9)*, is a solution of the heat equation*

$$\frac{\partial}{\partial t}\phi = \frac{1}{2}\partial_u^2\phi \tag{10}$$

if an only if $\gamma(t)$ is a solution of the Chazy equation (8).

Example (a nonquasipereodic solution $\phi(u,t)$ to equation (10), defined by (9)**).**
Set $\tilde{g_2} = \rho^4 g_2$, $\tilde{g_3} = \rho^6 g_3$, $\rho = 2\omega$ and $u = \rho z$.

Then a constant solution $\gamma(t) \equiv \beta$, $\beta \neq 0$, of the Chazy equation (8) gives $\tilde{g_2}(t) \equiv 48\beta^2$, $\tilde{g_3}(t) \equiv -64\beta^3$, and $r(t) \equiv -3\beta t + r_0$. In this case, the curve V degenerates ($\Delta = 0$), and we have

$$\varphi(z,t) = e^{3\beta t - r_0}e^{\beta z^2}\sigma(z, 48\beta^2, -64\beta^3).$$

Set $\varphi_1(t) = e^{3\beta t - r_0}$, $\varphi_2(z) = e^{\beta z^2}\sigma(z, 48\beta^2, -64\beta^3)$. Equation (10) takes the form $\varphi_1'\varphi_2 = \frac{1}{2}\varphi_1\varphi_2''$. This equation is equivalent to the system

$$\frac{\varphi_1'}{\varphi_1} = \frac{1}{2}\kappa, \qquad \frac{\varphi_2''}{\varphi_2} = \kappa.$$

We obtain $\kappa = 6\beta$ and $\varphi_2(z) = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$, where $\lambda^2 = 6\beta$. It follows from the conditions $\sigma(0) = 0$ and $\sigma'(0) = 1$ that

$$\sigma\left(z, \frac{4}{3}\lambda^4, -\frac{8}{27}\lambda^6\right) = \frac{1}{2\lambda} e^{3/2}\left(e^{-(1/6)(\lambda z - 3)^2} - e^{-(1/6)(\lambda z + 3)^2}\right).$$

In his study of third order ordinary differential equations having the Painlevé property, Chazy (1911) was led to the remarkable family of equations

$$y_{xxx} = 2yy_{xx} - 3y_x^2 + \alpha(6y_x - y^2)^2.$$

He showed that when $\alpha = 0$ or $\alpha = \frac{4}{36-k^2}$ where $6 < k \in \mathbb{N}$, then the nontrivial solutions $y = f(x)$ to this equations have a movable circular natural boundary.

The Chazy equation also arises as a reduction of the stationary, incompressible Prandtl boundary layer equations

$$\psi_{\eta\eta\eta} = \psi_\eta \psi_{\xi\eta} - \psi_\xi \psi_{\xi\eta}.$$

The Chazy equation is a reduction of the self-dual Yang–Mills equations when the Yang–Mills potentials take values in the infinite-dimensional Lie algebra $\mathfrak{sdiff}(SU(2))$ of all "divergence-free" vector fields on $SU(2)$ (see details in [19]).

The problem of differentiation of elliptic functions over parameters ω, ω' was solved by Frobenius and Stickelberger (1882). An affine connection on the modular curve, named the FS-connection, was introduced in terms of basic operators giving this differentiation. The coefficient of this connection satisfies the Chazy equation. Solutions of the Chazy equation are used to construct Frobenius structures (see [20] for details).

Weierstrass (1894) found the operators l_0, l_2 considered above solving the problem of differentiation of elliptic functions over parameters g_2, g_3. See above how solutions of the Chazy equation appear in this case.

Our works are devoted to the construction of operators of differentiation of Abelian functions of higher genus over parameters (see [21]) and their applications to the construction of an analogues of the FS connection (see [22]).

TWO-DIMENSIONAL CHAZY TYPE EQUATIONS

The theta function $\theta(\mathbf{z}, \tau)$ of genus 2 is an entire function of $\mathbf{z} = (z_1, z_3)^\top$ determined by the series

$$\theta(\mathbf{z}, \tau) = \sum_{\mathbf{n} \in \mathbb{Z}^2} e^{\pi i \mathbf{n}^\top \tau \mathbf{n} + 2\pi i \mathbf{n}^\top \mathbf{z}},$$

where $\tau = \begin{pmatrix} \tau_{1,1} & \tau_{1,3} \\ \tau_{1,3} & \tau_{3,3} \end{pmatrix}$ is a symmetric matrix, $\tau_{i,j} \in \mathbb{C}$, and $\operatorname{Im}\tau$ is a positive definite matrix. We need the following properties of the two-dimensional theta function.

1. This function is quasi periodic in $\mathbf{z}$ with respect to the lattice generated by $(1_2, \tau)$:

$$\theta(\mathbf{z}+\mathbf{m},\tau) = \theta(\mathbf{z},\tau), \quad \theta(\mathbf{z}+\tau\mathbf{m},\tau) = e^{-\pi i \mathbf{m}^\top \tau \mathbf{m} - 2\pi i \mathbf{m}^\top \mathbf{z}} \theta(\mathbf{z},\tau),$$

where $\mathbf{m} \in \mathbb{Z}^2$ and 1_2 is the identity (2×2) matrix.

2. The function $\theta(\mathbf{z},\tau)$ satisfies the system of equations

$$\nabla_\tau \theta(\mathbf{z},\tau) = \frac{1}{4\pi i} \nabla_z \nabla_z^\top \theta(\mathbf{z},\tau), \tag{11}$$

where

$$\nabla_\tau = \begin{pmatrix} \frac{\partial}{\partial \tau_{1,1}} & \frac{1}{2}\frac{\partial}{\partial \tau_{1,3}} \\ \frac{1}{2}\frac{\partial}{\partial \tau_{1,3}} & \frac{\partial}{\partial \tau_{3,3}} \end{pmatrix}, \quad \nabla_z = \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_3} \end{pmatrix}.$$

The function $\theta[\hat{\varepsilon}](\mathbf{z},\tau)$ is defined by the series

$$\theta[\hat{\varepsilon}](\mathbf{z},\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^2} e^{\pi i (\mathbf{n}+\varepsilon')^\top \tau (\mathbf{n}+\varepsilon') + 2\pi i (\mathbf{n}+\varepsilon')^\top (\mathbf{z}+\varepsilon)}$$

and satisfies system (11). Here, $\hat{\varepsilon} = (\varepsilon, \varepsilon') \in \frac{1}{2}\mathbb{Z}^2 \times \frac{1}{2}\mathbb{Z}^2$ is the vector of half-integer characteristics.

Any hyperelliptic curve with affine part of the form

$$V_\lambda = \left\{ (x,y) \in \mathbb{C}^2 \,|\, y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10} \right\}$$

determines a sigma function $\sigma(\mathbf{u},\lambda)$ (see [23]). This function is an entire function of $\mathbf{u} = (u_1, u_3)^\top \in \mathbb{C}^2$ with parameters $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})^\top \in \mathbb{C}^4$; it has a series expansion in powers of $\mathbf{u}$ over the polynomial ring $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ in the vicinity of $\mathbf{u} = 0$. The initial segment of the series has the form

$$\sigma(\mathbf{u},\lambda) = u_3 - \frac{1}{3}u_1^3 + \frac{1}{6}\lambda_6 u_3^3 - \frac{1}{12}\lambda_4 u_1^4 u_3 - \frac{1}{6}\lambda_6 u_1^3 u_3^2 - \\ - \frac{1}{6}\lambda_8 u_1^2 u_3^3 - \frac{1}{3}\lambda_{10} u_1 u_3^4 + \left(\frac{1}{60}\lambda_4\lambda_8 + \frac{1}{120}\lambda_6^2\right) u_3^5 + (\mathbf{u}^7). \tag{12}$$

Here and in what follows, $(\mathbf{u}^k)$ denotes the ideal generated by the monomials $u_1^i u_3^j$ with $i + j = k$. The sigma function is odd with respect to $\mathbf{u}$.

We set $\nabla_\lambda = \left(\frac{\partial}{\partial \lambda_4} \frac{\partial}{\partial \lambda_6} \frac{\partial}{\partial \lambda_8} \frac{\partial}{\partial \lambda_{10}}\right)^\top$, $\nabla_u = \left(\frac{\partial}{\partial u_1} \frac{\partial}{\partial u_3}\right)^\top$, and $F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Let $E_{i,j}$ denote the (2×2) matrix in which the position (i,j) is occupied by 1 and all the other entries are zero. We need the following properties of the two-dimensional sigma function (see [23], [3] for details).

1. The following system of equations holds:

$$Q_i \sigma = 0, \quad \text{where} \quad Q_i = l_i - H_i, \quad i = 0,2,4,6, \tag{13}$$

$$(l_0\, l_2\, l_4\, l_6)^\top = T\nabla_\lambda, \tag{14}$$

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 & 4\lambda_6\lambda_{10} - \frac{8}{5}\lambda_8^2 \end{pmatrix},$$

$$H_0 = \mathbf{u}^\top F\nabla_u - 3, \tag{15}$$

$$H_2 = \frac{1}{2}\nabla_u^\top E_{1,1}\nabla_u + \mathbf{u}^\top\beta_2\nabla_u + \frac{1}{2}\mathbf{u}^\top\gamma_2\mathbf{u} - \delta_2,$$
$$H_4 = \nabla_u^\top E_{1,3}\nabla_u + \mathbf{u}^\top\beta_4\nabla_u + \frac{1}{2}\mathbf{u}^\top\gamma_4\mathbf{u} - \delta_4,$$
$$H_6 = \frac{1}{2}\nabla_u^\top E_{3,3}\nabla_u + \mathbf{u}^\top\beta_6\nabla_u + \frac{1}{2}\mathbf{u}^\top\gamma_6\mathbf{u} - \delta_6,$$

$$\beta_2 = \begin{pmatrix} 0 & 1 \\ -\frac{4}{5}\lambda_4 & 0 \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} 0 & 0 \\ -\frac{6}{5}\lambda_6 & \lambda_4 \end{pmatrix}, \quad \beta_6 = \begin{pmatrix} 0 & 0 \\ -\frac{3}{5}\lambda_8 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} -\frac{3}{5}\lambda_4 & 0 \\ 0 & -\frac{4}{5}\lambda_4^2 + 3\lambda_8 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} -\frac{2}{5}\lambda_6 & \lambda_8 \\ \lambda_8 & -\frac{6}{5}\lambda_4\lambda_6 + 6\lambda_{10} \end{pmatrix},$$

$$\gamma_6 = \begin{pmatrix} -\frac{1}{5}\lambda_8 & 2\lambda_{10} \\ 2\lambda_{10} & -\frac{3}{5}\lambda_4\lambda_8 \end{pmatrix}, \quad \delta_2 = 0, \quad \delta_4 = \lambda_4, \quad \delta_6 = \frac{1}{2}\lambda_6.$$

2. The equation $Q_0\sigma = 0$ implies that the function σ is homogeneous of degree 3 in variables $(\mathbf{u}, \lambda)$ with respect to the grading $\deg u_i = i$, $i = 1, 3$, $\deg\lambda_j = -j$, $j = 4, 6, 8, 10$.
3. The discriminant Δ of the curve V_λ equals $\frac{16}{5}\det T$. Let $\mathcal{B} = \{\lambda \in \mathbb{C}^4 | \Delta(\lambda) \neq 0\}$. Then the curve V_λ, where $\lambda \in \mathcal{B}$, is non degenerate.
   The matrices of the parameters ω, ω', η and η' are determined by the relations (see [23, pp. 8-9])

$$2\omega = \left(\oint_{a_i} du_j\right)_{i,j=1,3}, \qquad 2\omega' = \left(\oint_{b_i} du_j\right)_{i,j=1,3},$$

$$2\eta = -\left(\oint_{a_i} dr_j\right)_{i,j=1,3}, \qquad 2\eta' = -\left(\oint_{b_i} dr_j\right)_{i,j=1,3},$$

where $du_1 = \frac{xdx}{y}$, $du_3 = \frac{dx}{y}$, $dr_1 = \frac{x^2dx}{y}$, and $dr_3 = \frac{(3x^3+\lambda_4 x)dx}{y}$ form a basis of holomorphic differentials on V_λ and a_i and b_i are basis cycles on the curve chosen so as to satisfy the Legendre identity

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^T \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = \frac{\pi i}{2}.$$

The Legendre identity is equivalent to the existence of symmetric matrices τ and $\varkappa$ such that $\omega' = \omega\tau$, $\eta = 2\varkappa\omega$, and $\eta' = 2\varkappa\omega' - \frac{\pi i}{2}(\omega^\top)^{-1}$. In this notation, we have

$$\sigma(\mathbf{u},\lambda) = \frac{\pi}{\sqrt{\det(2\omega)}}\frac{1}{\sqrt[8]{\Delta}}\exp\left(\mathbf{z}^\top(2\omega)^\top\eta\mathbf{z}\right)\theta[\hat{\varepsilon}](\mathbf{z},\tau), \tag{16}$$

where $\mathbf{u} = (2\omega)\mathbf{z}$ and $\hat{\varepsilon}$ is the vector of half-integer characteristics corresponding to the vector of Riemann constants.

4. The function $\sigma(\mathbf{u},\lambda)$ is quasi periodic in $\mathbf{u}$ with respect to the lattice generated by the (2×4) matrix of periods $(2\omega, 2\omega')$.
   Set $\Omega_1(\mathbf{m},\mathbf{m}') = \omega\mathbf{m} + \omega'\mathbf{m}'$ and $\Omega_2(\mathbf{m},\mathbf{m}') = \eta\mathbf{m} + \eta'\mathbf{m}'$, where $\mathbf{m},\mathbf{m}' \in \mathbb{Z}^2$. Then

$$\begin{aligned}&\sigma\left(\mathbf{u}{+}2\Omega_1(\mathbf{m},\mathbf{m}'),\lambda\right)\\ &=\exp\left\{2\Omega_2^\top(\mathbf{m},\mathbf{m}')\left(\mathbf{u}{+}\Omega_1(\mathbf{m},\mathbf{m}')\right)+2\pi i\left(\mathbf{m}^\top\hat{\varepsilon}-\hat{\varepsilon}^\top\mathbf{m}'+\frac{1}{2}\mathbf{m}^\top\mathbf{m}'\right)\right\}\sigma(\mathbf{u},\lambda).\end{aligned}$$

   In particular, for $\mathbf{m}' = 0$, we obtain

$$\sigma(\mathbf{u}+2\omega\mathbf{m},\lambda) = \exp\left\{2\mathbf{m}^\top\eta^\top(\mathbf{u}+\omega\mathbf{m}) + 2\pi i\mathbf{m}^\top\hat{\varepsilon}'\right\}\sigma(\mathbf{u},\lambda).$$

Theorem 3 (uniqueness conditions for the two-dimensional sigma function). *The entire function $\sigma(\mathbf{u},\lambda)$ is uniquely determined by the system of equations* (13) *with the initial condition $\sigma(\mathbf{u},0) = u_3 - \frac{1}{3}u_1^3$.*

The linear recursion equations for the coefficients of the series expansion of $\sigma(\mathbf{u},\lambda)$ see in [3].

The vector fields l_0, l_2, l_4, l_6 are tangent to the discriminant variety

$$\{(\lambda) \in \mathbb{C}^4 \,|\, \Delta(\lambda) = 0\}$$

because

$$l_0\Delta = 40\Delta, \qquad l_2\Delta = 0, \qquad l_4\Delta = 12\lambda_4\Delta, \qquad l_6\Delta = 4\lambda_6\Delta.$$

We have

$$[l_0,l_2] = 2l_2, \qquad [l_0,l_4] = 4l_4, \qquad [l_0,l_6] = 6l_6, \qquad [l_2,l_4] = \frac{8}{5}\lambda_6 l_0 - \frac{8}{5}\lambda_4 l_2 + 2l_6,$$

$$[l_2,l_6] = \frac{4}{5}\lambda_8 l_0 - \frac{4}{5}\lambda_4 l_4, \qquad [l_4,l_6] = -2\lambda_{10}l_0 + \frac{6}{5}\lambda_8 l_2 - \frac{6}{5}\lambda_6 l_4 + 2\lambda_4 l_6.$$

Consider the integral curve $\lambda(t) = \left(\lambda_4(t),\lambda_6(t),\lambda_8(t),\lambda_{10}(t)\right)$ of the vector field $\gamma(t)l_0 + l_4$. Then along the curve $\lambda(t)$ we have

$$\frac{\partial}{\partial t} = \gamma(t)l_0 + l_4$$

and

$$\dot{\lambda}_4 = 4\gamma(t)\lambda_4 + 8\lambda_8$$
$$\dot{\lambda}_6 = 6\gamma(t)\lambda_6 + 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6$$
$$\dot{\lambda}_8 = 8\gamma(t)\lambda_8 + 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2$$
$$\dot{\lambda}_{10} = 10\gamma(t)\lambda_{10} + 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8$$

We obtain

$$\frac{\partial}{\partial t}\sigma = \partial_{u_1}\partial_{u_3}\sigma + l_u\sigma + V(u,t)\sigma - E(t)\sigma,$$

where $l_u = (3\gamma+\lambda_4)\,u_1\partial_{u_1} + \left(\gamma u_3 - \frac{6}{5}\lambda_6 u_1\right)\partial_{u_3}$, $V(u,t) = 3\left(\lambda_{10} - \frac{1}{5}\lambda_6\lambda_4\right)u_1^2 + \lambda_8 u_1 u_3 - \frac{1}{5}\lambda_6 u_3^2$ and $E(t) = 3\gamma + \lambda_4$.

Set

$$\sigma(\mathbf{u},\lambda(t)) = e^{s(\mathbf{u},t)}\phi(\mathbf{u},t),$$

where $s(\mathbf{u},t) = \frac{3}{5}\lambda_6 u_1^2 - \gamma(t)u_1u_3 + r(t)$ and $\lambda_4 = -2\gamma$.

Theorem 4. *The function $\phi(\mathbf{u},t)$ is a solution of the equation*

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1\partial u_3} + \sum_{i,j}\chi_{i,j}(t)u_iu_j\phi + \tau(t)\phi,$$

where

$$\chi_{3,3}(t) = -\frac{1}{5}\lambda_6;\quad \chi_{3,1}(t) = \lambda_8 + \dot{\gamma}(t) - \gamma(t)^2;\quad \chi_{1,1}(t) = \frac{3}{5}(-\dot{\lambda}_6 + 4\gamma(t)\lambda_6 + 5\lambda_{10});$$

and $\tau(t) = -2\gamma(t) - \dot{r}(t)$.

Here $r(t)$ is an additional function. If $r(t) \equiv 0$, then $\tau(t) = -2\gamma(t)$. Thus we obtain for λ_i:

$$\lambda_8 = -\frac{1}{4}\dot{\gamma}(t) + \gamma(t)^2;\qquad \lambda_6^2 = -\frac{5}{12}\dot{\lambda}_8;\qquad \lambda_{10} = \frac{1}{10}\dot{\lambda}_6 - \frac{23}{25}\gamma(t)\lambda_6.$$

Theorem 5. *The function $\gamma(t)$ is a solution of the equation $\dot{\gamma}(t) - 4\gamma(t)^2 = const$ if $\dot{\lambda}_8(t) \equiv 0$, and a solution of the equation*

$$\Big(\rho(t) + 130\gamma(t)\Big)\Big(\rho(t) + 42\gamma(t)\Big) + 10\dot{\rho}(t) = 0,$$

if $\dot{\lambda}_8(t) \neq 0$, where $\rho(t) = 5\nu(t) - 122\gamma(t)$, $\nu(t) = \frac{\dot{\varepsilon}(t)}{\varepsilon(t)}$, $\varepsilon(t) = \dot{\lambda}_8$.

If $\lambda_6 \equiv 0$ we have $\lambda_{10} = 0$. Then $\Delta = 2^{16}\cdot\frac{1}{5}\lambda_8^3(4\lambda_8 - \lambda_4^2)^2$. Thus if $\lambda_8 \neq 0$ and $4\lambda_8 \neq \lambda_4^2$ then our curve is non singular.

Let $\lambda_6 \equiv 0$ then $\lambda_{10} = 0$, $\lambda_4 = -2\gamma(t)$ and $-\frac{1}{4}\dot\gamma(t) + \gamma(t)^2 = \lambda_8 = const$. In this case

$$\chi_{3,3}(t) = 0; \quad \chi_{3,1}(t) = \frac{3}{4}\dot\gamma(t); \quad \chi_{1,1}(t) = 0; \quad \tau(t) = -2\gamma(t) - \dot r(t).$$

Corollary 1. *Let $\lambda_8(t) = a^2 = const$, $\lambda_{10}(t) = \lambda_6(t) = 0$ and $\lambda_4(t) = -2\gamma(t)$, where*

$$\gamma(t) = a\frac{1+ce^{8at}}{1-ce^{8at}}.$$

Set $\dot r(t) = -2\gamma(t)$. Then the equation

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1 \partial u_3} + 3\left(\gamma(t)^2 - a^2\right) u_1 u_3 \phi$$

has a solution

$$\phi(\mathbf{u}, t) = e^{\gamma(t)u_1u_3 - r(t)}\sigma(\mathbf{u}, \lambda(t)).$$

In the case $a = ib$, $b \in \mathbb{R}$, $c = -1$ we have $\lambda_8(t) = -b^2 = const$,

$$\gamma(t) = ib\frac{1-e^{8ibt}}{1+e^{8ibt}} = b\,\mathrm{tg}(4bt),$$

and $r(t) = \frac{1}{2}\ln\cos(4bt) + c_1$. Then the equation is

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1 \partial u_3} + 3\left(\frac{b}{\cos(4bt)}\right)^2 u_1 u_3 \phi.$$

In the case $a \in \mathbb{R}$, $c = -1$ we have $\lambda_8(t) = a^2 = const$,

$$\gamma(t) = a\frac{1-e^{8at}}{1+e^{8at}} = -a\,th(4at).$$

and $r(t) = \frac{1}{2}\ln\left(\mathrm{ch}(4at)\right) + c_2$. Then the equation is

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1 \partial u_3} - 3\left(\frac{a}{\mathrm{ch}(4at)}\right)^2 u_1 u_3 \phi.$$

Let $A \cdot B$ denote the tensor product of matrices A and B; i.e.,

$$\begin{pmatrix} A_{1,1} & A_{1,3} \\ A_{3,1} & A_{3,3} \end{pmatrix} \cdot \begin{pmatrix} B_{1,1} & B_{1,3} \\ B_{3,1} & B_{3,3} \end{pmatrix} = \begin{pmatrix} A_{1,1}B & A_{1,3}B \\ A_{3,1}B & A_{3,3}B \end{pmatrix}.$$

We set

$$K = \begin{pmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{pmatrix} \text{ and } N = \begin{pmatrix} E_{11} & E_{21}^\top \\ E_{12}^\top & E_{22} \end{pmatrix},$$

where $\boldsymbol{E}_{ij}$ as above, is the (2×2) matrix with 1 at the position (ij) and 0 at the other positions.

Remark. The matrix K is a commutation (2×2) matrix; i.e., for any two (2×2) matrices A and $\boldsymbol{B}$, we have $K(A\cdot B)K = (B\cdot A)$.

Let t be a (2×2) matrix, and let $t=\frac{1}{2\pi i}\tau$. We set

$$\nabla_t=\begin{pmatrix}\frac{\partial}{\partial t_{1,1}} & \frac{1}{2}\frac{\partial}{\partial t_{1,3}}\\ \frac{1}{2}\frac{\partial}{\partial t_{1,3}} & \frac{\partial}{\partial t_{3,3}}\end{pmatrix}.$$

Then $\nabla_t=2\pi i\nabla_\tau$.

Theorem 6. *If a function $\varphi(\mathbf{z},t)$ is related to $\sigma(\mathbf{u},\lambda)$ by the substitution*

$$\rho^3\varphi(\mathbf{z},t)=e^{\mathbf{z}^\top G\mathbf{z}-r}\sigma(\mathbf{u},\lambda), \tag{17}$$

where $\mathbf{u}=RA\mathbf{z}$,

$$A=\begin{pmatrix}A_{1,1} & A_{1,3}\\ A_{3,1} & A_{3,3}\end{pmatrix},\qquad G=\begin{pmatrix}G_{1,1} & G_{1,3}\\ G_{1,3} & G_{3,3}\end{pmatrix},\qquad R=\begin{pmatrix}\rho & 0\\ 0 & \rho^3\end{pmatrix},$$

$$\det A=1,\qquad A_{i,j}=A_{i,j}(t),\qquad G_{i,j}=G_{i,j}(t),\qquad r=r(t),\qquad \lambda_i=\rho^{-i}\widetilde{\lambda}_i(t),$$

and satisfies the system of equations

$$\nabla_t\varphi(\mathbf{z},t)=\frac{1}{2}\nabla_z\nabla_z^\top\varphi(\mathbf{z},t),$$

then $\widetilde{\lambda}_i=\rho^i\lambda_i$, A, and G satisfy the system of equations

$$\begin{aligned}
&\nabla_t\widetilde{\lambda}_j=A^\top M_jA+\frac{j}{2}\widetilde{\lambda}_jG,\\
&(\nabla_t\cdot A)=(1_2\cdot A)(K+N)(1_2\cdot G)-\frac{1}{2}(G\cdot FA)+(A\cdot 1_2)^\top S(A\cdot A),\\
&(\nabla_t\cdot G)=(1_2\cdot G)(K+N)(1_2\cdot G)+\frac{1}{10}(A\cdot A)^\top P(A\cdot A),
\end{aligned}$$

and the formula

$$\nabla_t r=-\frac{5}{2}G+A^\top\mathcal{R}A$$

holds, where

$$M_j=\rho^jR\left[\begin{pmatrix}T_{2,j} & \frac{1}{2}T_{4,j}\\ \frac{1}{2}T_{4,j} & T_{6,j}\end{pmatrix}-\frac{1}{4}T_{0,j}\begin{pmatrix}\operatorname{tr}(\beta_2) & \frac{1}{2}\operatorname{tr}(\beta_4)\\ \frac{1}{2}\operatorname{tr}(\beta_4) & \operatorname{tr}(\beta_6)\end{pmatrix}\right]R,$$

$$S=(R\cdot R^{-1})\left[\begin{pmatrix}\beta_2^\top & \frac{1}{2}\beta_4^\top\\ \frac{1}{2}\beta_4^\top & \beta_6^\top\end{pmatrix}-\frac{1}{4}\begin{pmatrix}\operatorname{tr}(\beta_2) & \frac{1}{2}\operatorname{tr}(\beta_4)\\ \frac{1}{2}\operatorname{tr}(\beta_4) & \operatorname{tr}(\beta_6)\end{pmatrix}\cdot F\right](R\cdot R),$$

$$P=-5(R\cdot R)\begin{pmatrix}\gamma_2 & \frac{1}{2}\gamma_4\\ \frac{1}{2}\gamma_4 & \gamma_6\end{pmatrix}(R\cdot R),$$

$$\mathcal{R}=R\left[\frac{3}{4}\begin{pmatrix}\operatorname{tr}(\beta_2) & \frac{1}{2}\operatorname{tr}(\beta_4)\\ \frac{1}{2}\operatorname{tr}(\beta_4) & \operatorname{tr}(\beta_6)\end{pmatrix}-\begin{pmatrix}\delta_2 & \frac{1}{2}\delta_4\\ \frac{1}{2}\delta_4 & \delta_6\end{pmatrix}\right]R;$$

the matrices T, β_i, and γ_i and the quantities δ_i are defined in (14) *and* (15).

The following result is similar to Lemma 2:

Lemma 3. *Let $\sigma(\mathbf{u},\lambda)$ be the sigma function with matrix parameters ω, ω', η, and η'. Then the function $\varphi(\mathbf{z},t)$, defined by* (17) *with* $2\omega = RA$ *satisfies the quasi periodicity condition* $\varphi(\mathbf{z}+\mathbf{m},t) = e^{2\pi i \mathbf{m}^\top \hat{\varepsilon}'}\varphi(\mathbf{z},t)$, *where* $t = \frac{1}{2\pi i}\tau = \frac{1}{2\pi i}\omega^{-1}\omega'$, *if and only if* $G = -2\omega^\top\eta = -(2\omega)^\top \varkappa(2\omega)$.

The rational limit of the sigma function in genus g is a polynomial in the variables $\mathbf{u}$. Such polynomials have remarkable properties (see [24]) inherited from the function $\sigma(\mathbf{u},\lambda)$, although they correspond to the case $\lambda = 0$.

In the case of genus 2, $\lambda_i = 0$ for $i = 4,6,8,10$ in the rational limit. Thus, $S_2 = -E_{1,3}$, $S_4 = S_6 = 0$, $\gamma_i = 0$, $\mathrm{tr}(\beta_i) = 0$, and $T_{i,j} = 0$, and the formulas from Theorem 6 take the form

$$\nabla_t r = -\frac{5}{2}G,$$
$$\nabla_t \cdot A = -\frac{1}{2}(G\cdot FA) + (1_2\cdot A)(K+N)(1_2\cdot G) - (A^\top \cdot 1_2)(E_{1,1}\cdot E_{3,1})(A\cdot A),$$
$$\nabla_t \cdot G = (1_2\cdot G)(K+N)(1_2\cdot G).$$

We set $g(t) = c_0 + \mathrm{tr}(qt) + c_2 \det t$, where c_0, c_2, and $q = \begin{pmatrix} q_{11} & q_{13} \\ q_{13} & q_{33}\end{pmatrix}$ are parameters independent of t.

Theorem 7. *A general solution of the equation*

$$\nabla_t \cdot G = (1_2\cdot G)(K+N)(1_2\cdot G)$$

has the form

$$G = -\frac{1}{2}\nabla_t \ln g(t),$$

where the parameters of the function $g(t)$ are related by

$$c_0 c_2 = \det q.$$

Let $c_2 \neq 0$, then $g(t) = \det(m+\alpha t)$, where $\alpha^2 = c_2$, $\alpha m' = q$, $\det m = c_0$, and m' is adjoint matrix for m.

Corollary 2. *In this case*

$$G = -\frac{1}{2}\alpha(m+\alpha t)^{-1}.$$

REFERENCES

1. V. Buchstaber, and D. Leikin, *Funct. Anal. Appl.* **38**, 88–101 (2004).
2. V. Buchstaber, and D. Leikin, *Functional Anal. Appl.* **36**, 267–280 (2002).
3. V. Buchstaber, and D. Leikin, *Proceedings of the Steklov Math. Inst.* **251**, 49–120 (2005).
4. V. Buchstaber, V. Enolskii, and D. Leikin, *Functional Anal. Appl.* **34**, 159–171 (2000).
5. V. Buchstaber, V. Enolskii, J. Eilbeck, D. Leikin, and M. Salerno, *J. Math. Phys.* **43**, 2858–2881 (2002).

6. J. Eilbeck, V. Enolskii, and E. Previato, *Lett. Math. Phys.* **63**, 5–17 (2003).
7. V. Buchstaber, and S. Shorina, "w-Function of the KdV Hierarchy," in *Geometry, Topology, and Mathematical Physics, S.P.Novikov's seminar: 2002–2003*, edited by V.M. Buchstaber and I.M. Krichever, Amer. Math. Soc. Transl., Providence, RI, 2004, vol. 212 of 2, pp. 41–46.
8. H. Braden, V. Enolskii, and A. Hone, *J. Nonlin. Math. Phys* **12**, 46–62 (2005), supplement 2, `arXiv:math.NT/0501162`.
9. J. Eilbeck, V. Enolskii, S. Matsutani, Y. Ônishi, and E. Previato, *Int. Math. Res. Notices* **2007**, rnm140–38 (2007), `arXiv:math.AG/0610019`.
10. V. Enolskii, S. Matsutani, and Y. Ônishi, *Tokyo J. Math.* **31**, 27–38 (2008), `arXiv:math.AG/0508366`.
11. A. Nakayashiki, Sigma function as a tau function (2009), preprint, `arXiv:0904.0846`.
12. Y. Ônishi, Determinant expressions for hyperelliptic abelian functions (2005), with an Appendix by S. Matsutani: *Connection of the formula of Cantor and Brioshi-Kiepert type*, Proc. Edinburgh Math.Soc, Preprint NT/0105189.
13. V. Buchstaber, and V. Enolskii, *Russian Math. Surveys* **50**, 195–197 (1995).
14. V. Buchstaber, V. Enolskii, and D. Leikin, *Russian Math. Surveys* **54**, 628–629 (1999).
15. V. Buchstaber, and D. Leikin, *Russian Math. Surveys* **56**, 1155–1157 (2001).
16. V. Buchstaber, and D. Leikin, *Doklady Math. Sci.* **66**, 87–90 (2003).
17. V. Buchstaber, D. Leikin, and M. Pavlov, *Funct. Anal. Appl.* **37**, 251–262 (2003).
18. V. Arnold, *Singularities of Caustics and Wave Fronts*, Kluwer, Dordrecht, 1990, Fazis, Moscow,1996.
19. P. Clarkson, and P. Olver, *J. Diff. Eq.* **124**, 225–246 (1996).
20. B. Dubrovin, "Geometry of 2D topological field theories," in *Integrable systems and quantum groups*, Springer, Berlin, 1996, vol. 1620 of *Lect. Notes Math.*, pp. 120–348, `hep-th/9407018`.
21. V. Buchstaber, and D. Leikin, *Funct. Anal. Appl.* **42**, 268–278 (2008).
22. V. Buchstaber, and E. Bunkova, *Proceedings of the Steklov Institute of Mathematics* **266**, 1–28 (2009).
23. V. Buchstaber, V. Enolskii, and D. Leikin, "Kleinian functions, hyperelliptic Jacobians and applications," in *Reviews in Mathematics and Math. Physics*, edited by I.M. Krichever and S.P. Novikov, Gordon and Breach, London, 1997, vol. 10, pp. part 2, 3–120, see improved version in math-ph.
24. V. Buchstaber, V. Enolskii, and D. Leikin, *Functional Anal. Appl.* **33**, 83–94 (1999).

GEOMETRIC METHODS IN PHYSICS

Proceedings of the XXVIII Workshop on Geometric Methods in Physics

Białowieża, Poland *28 June – 4 July 2009*

EDITORS

Piotr Kielanowski
University of Białystok, Poland
and
Cinvestav, México City, México

S. Twareque Ali
Concordia University, Montréal, Quebec, Canada

Anatol Odzijewicz
University of Białystok, Poland

Martin Schlichenmaier
University of Luxembourg, Luxembourg

Theodore Voronov
University of Manchester, United Kingdom

All papers have been peer reviewed

SPONSORING ORGANIZATION
University of Białystok, Poland

AMERICAN INSTITUTE OF PHYSICS

Melville, New York, 2009
AIP CONFERENCE PROCEEDINGS ▌ 1191

Heat Equations and Sigma Functions

V. M. Buchstaber

Steklov Mathematical Institute, Russian Academy of Sciences
buchstab@mi.ras.ru
School of Mathematics, University of Manchester, Manchester
Victor.Buchstaber@manchester.ac.uk

Abstract. A review of solutions of some important differential equations in terms of sigma function is given. In particular we discuss the heat equation and the Chazy type equations.

Keywords: heat equation, Chazy equation, sigma functions
PACS: 02.30.Jr, 02.30Gp

INTRODUCTION

The theory of Abelian functions was a central topic of the 19th century mathematics. In mid-seventies of the last century a new wave of investigation arose in this field in response to the discovery that Abelian functions provide solutions to a number of challenging problems of modern Theoretical and Mathematical Physics.

In the cycle of our joint papers with V. Enolski and D. Leykin we have developed a theory of multivariate sigma function, an analogue of the classical Weierstrass sigma function.

Considerable progress in the development of the theory of sigma functions was made with the construction of operators annihilating sigma functions and the calculation of Lie algebras of such operators (see [1]). In the hyperelliptic case, this calculation is very effective (see [2, 3]).

A number of applications of the theory of multivariate sigma functions have been found (see [4–17]).

This survey is based on the joint papers with D. Leykin and E. Bunkova. We will describe in terms of sigma functions solutions of some important differential equations related to the classical heat equations.

HEAT EQUATION AND CHAZY EQUATION

The elliptic theta function $\theta(z,\tau)$ is an entire function of z defined by the series

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z},$$

where $\tau \in \mathbb{C}$ is a parameter with $\operatorname{Im}\tau > 0$.

CP1191, *Geometric Methods in Physics—Proceedings of the XXVIII Workshop on Geometric Methods in Physics*
edited by P. Kielanowski, S. Twareque Ali, A. Odzijewicz, M. Schlichenmaier, and T. Voronov

We need the following properties of the theta function.

1. This function is quasi periodic in z with respect to the lattice generated by $(1,\tau)$:

$$\theta(z+1,\tau)=\theta(z,\tau), \quad \theta(z+\tau,\tau)=e^{-2\pi i(z+\frac{1}{2}\tau)}\theta(z,\tau).$$

2. The theta function satisfies the heat equation

$$\frac{\partial}{\partial\tau}\theta(z,\tau)=\frac{1}{4\pi i}\frac{\partial^2}{\partial z^2}\theta(z,\tau). \tag{1}$$

We also need the function

$$\theta_1(z,\tau)=e^{\frac{\pi i}{4}\tau+\pi i(z+\frac{1}{2})}\theta\left(z+\frac{1}{2}(1+\tau),\tau\right)=\sum_{n\in\mathbb{Z}}e^{\pi i(n+\frac{1}{2})^2\tau+2\pi i(n+\frac{1}{2})(z+\frac{1}{2})},$$

which satisfies the heat equation (1) as well as the quasi periodicity condition $\theta_1(z+1,\tau)=-\theta_1(z,\tau)$.

Each elliptic curve whose affine part has the Weierstrass form

$$V_{(g_2,g_3)}=\{(x,y)\in\mathbb{C}^2 \mid y^2=4x^3-g_2x-g_3\}$$

determines a sigma function $\sigma(u,g_2,g_3)$. This function is an entire function of $u\in\mathbb{C}$ with parameters $(g_2,g_3)\in\mathbb{C}^2$; it has a series expansion in powers of u over the polynomial ring $\mathbb{Q}[g_2,g_3]$ in the vicinity of $u=0$ (see, e.g., [3]). An initial segment of the series has the form

$$\sigma(u)=u-\frac{g_2u^5}{2\cdot 5!}-\frac{6g_3u^7}{7!}-\frac{g_2^2u^9}{4\cdot 8!}-\frac{18g_2g_3u^{11}}{11!}+(u^{13}). \tag{2}$$

We need the following properties of the sigma function.
Set

$$\begin{pmatrix} l_0 \\ l_2 \end{pmatrix}=T\begin{pmatrix} \frac{\partial}{\partial g_2} \\ \frac{\partial}{\partial g_3} \end{pmatrix}, \quad T=\begin{pmatrix} 4g_2 & 6g_3 \\ 6g_3 & \frac{1}{3}g_2^2 \end{pmatrix}, \tag{3}$$

$$H_0=u\frac{\partial}{\partial u}-1, \quad H_2=\frac{1}{2}\frac{\partial^2}{\partial u^2}+\frac{1}{24}g_2u^2. \tag{4}$$

1. The elliptic sigma-function satisfies the system of equations

$$Q_0\sigma=0, \quad Q_2\sigma=0, \tag{5}$$

   where $Q_0=l_0-H_0, \quad Q_2=l_2-H_2$.
2. The equation $Q_0\sigma=0$ implies that the function σ is homogeneous of degree 1 in the variables (u,g_2,g_3) with respect to the grading $\deg u=1$, $\deg g_2=-4$, $\deg g_3=-6$.
3. The discriminant of the elliptic curve $V_{(g_2,g_3)}$ equals $\Delta=g_2^3-27g_3^2$. Let $\mathcal{B}=\{(g_2,g_3)\in\mathbb{C}^2 \mid \Delta\neq 0\}$; then the curve $V_{(g_2,g_3)}$, where $(g_2,g_3)\in\mathcal{B}$, is non degenerate.

The function $\sigma(u)$ is quasi periodic in u with respect to the lattice generated by $(2\omega, 2\omega')$:

$$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u), \quad \sigma(u+2\omega') = -e^{2\eta'(u+\omega')}\sigma(u).$$

The parameters ω, ω', η and η' are determined by the relations

$$2\omega = \oint_a \frac{dx}{y}, \quad 2\omega' = \oint_b \frac{dx}{y}, \quad 2\eta = -\oint_a \frac{xdx}{y}, \quad 2\eta' = -\oint_b \frac{xdx}{y},$$

where $\frac{dx}{y}$ and $\frac{xdx}{y}$ form a basis of holomorphic differentials on $V_{(g_2,g_3)}$ and a and b are basis cycles on the curve such that the integrals satisfy the Legendre identity $\eta\omega' - \omega\eta' = \frac{\pi i}{2}$.

In this notation, we have

$$\sigma(u, g_2, g_3) = \sqrt{\frac{\pi}{\omega}} \frac{1}{\sqrt[8]{\Delta}} \exp(2\omega\eta z^2)\theta_1(z, \tau), \tag{6}$$

where $u = 2\omega z$, $\omega' = \omega\tau$, and $\eta = 2\varkappa\omega$.

Theorem 1 (uniqueness conditions for the sigma function)**.**
The entire function $\sigma(u, g_2, g_3)$ is uniquely determined by the conditions

$$Q_0\sigma = 0, \quad Q_2\sigma = 0, \quad \sigma(0, g_2, g_3) = 0, \quad \left(\frac{\partial}{\partial u}\sigma(u, g_2, g_3)\right)\bigg|_{u=0} = 1.$$

Proof. Consider the series expansion of the sigma function $\sigma(u, g_2, g_3)$ in powers of u in the form

$$\sigma(u, g_2, g_3) = \sum_k \gamma_k(g_2, g_3)u^k.$$

The conditions of the theorem give us a recursion for the coefficients $\gamma_k(g_2, g_3)$ as polynomials of g_2 and g_3. The explicit form of this recursion see in [3]. □

The vector fields l_0, l_2 are tangent to the discriminant variety

$$\{(g_2, g_3) \in \mathbb{C}^2 \mid \Delta(g_2, g_3) = 0\}$$

(see, e.g., [18]) because

$$l_0\Delta = 12\Delta, \quad l_2\Delta = 0.$$

Let $(g_2(t), g_3(t))$ be the integral curve of the field

$$\frac{\partial}{\partial t} = \dot{g}_2\frac{\partial}{\partial g_2} + \dot{g}_3\frac{\partial}{\partial g_3} = \gamma(t)l_0 + l_2$$

on $\mathcal{B}$. Then along the curve $(g_2(t), g_3(t))$ we have

$$0 = (\gamma Q_0 + Q_2)\sigma = (\gamma l_0 + l_2)\sigma + (-\gamma u\partial_u + \gamma - \frac{1}{2}\partial_u^2 - \frac{g_2}{24}u^2)\sigma.$$

Therefore

$$\frac{\partial}{\partial t}\sigma = \frac{1}{2}\partial_u^2\sigma + \gamma(t)u\partial_u\sigma + \frac{g_2}{24}u^2\sigma - \gamma(t)\sigma.$$

and

$$\dot{g_2} = 4\gamma g_2 + 6g_3, \qquad \dot{g_3} = 6\gamma g_3 + \frac{1}{3}g_2^2. \tag{7}$$

Let $\gamma(t)$ be a solution of the Chazy equation (the theory of this equation see e.g. in [19]).

$$\dddot{\gamma}(t) = 12\gamma(t)\ddot{\gamma}(t) - 18\dot{\gamma}(t)^2. \tag{8}$$

Lemma 1. *The system* (7) *has the solution*

$$g_2(t) = 12\gamma(t)^2 - 12\dot{\gamma}(t), \qquad g_3(t) = 12\gamma(t)\dot{\gamma}(t) - 2\ddot{\gamma}(t) - 8\gamma(t)^3.$$

Set

$$\sigma(u, g_2(t), g_3(t)) = e^{s(u,t)}\phi(u,t), \tag{9}$$

where $s(u,t) = -\frac{\gamma(t)}{8\omega^2}u^2 + b(t)$ and $\dot{b}(t) = -\frac{3}{2}\gamma(t)$.

Lemma 2. *Let* $\sigma(u, g_2, g_3)$ *be the sigma function with parameters* ω, ω', η, *and* η'. *Then the function* $\phi(u,t)$ *defined by* (9), *satisfies the periodicity condition* $\phi(u+2\omega,t) = -\phi(u,t)$ *if and only if* $\gamma(t) = -4\omega\eta$.

Proof. We have

$$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u) = -e^{2\eta(u+\omega)}e^{s(u,t)}\phi(u).$$

On the other hand

$$\sigma(u+2\omega) = e^{s(u+2\omega,t)}\phi(u+2\omega) = -e^{s(u+2\omega,t)}\phi(u).$$

Therefore, $-\frac{\gamma}{2\omega}(u+\omega) = 2\eta(u+\omega)$ and $\gamma = -4\omega\eta$. □

Theorem 2. *The function* $\phi(u,t)$ *defined by* (9), *is a solution of the heat equation*

$$\frac{\partial}{\partial t}\phi = \frac{1}{2}\partial_u^2\phi \tag{10}$$

if an only if $\gamma(t)$ *is a solution of the Chazy equation* (8).

Example (a nonquasipereodic solution $\phi(u,t)$ to equation (10), defined by (9)).
Set $\tilde{g_2} = \rho^4 g_2$, $\tilde{g_3} = \rho^6 g_3$, $\rho = 2\omega$ and $u = \rho z$.

Then a constant solution $\gamma(t) \equiv \beta$, $\beta \neq 0$, of the Chazy equation (8) gives $\tilde{g_2}(t) \equiv 48\beta^2$, $\tilde{g_3}(t) \equiv -64\beta^3$, and $r(t) \equiv -3\beta t + r_0$. In this case, the curve V degenerates ($\Delta = 0$), and we have

$$\varphi(z,t) = e^{3\beta t - r_0}e^{\beta z^2}\sigma(z, 48\beta^2, -64\beta^3).$$

Set $\varphi_1(t) = e^{3\beta t - r_0}$, $\varphi_2(z) = e^{\beta z^2}\sigma(z, 48\beta^2, -64\beta^3)$. Equation (10) takes the form $\varphi_1'\varphi_2 = \frac{1}{2}\varphi_1\varphi_2''$. This equation is equivalent to the system

$$\frac{\varphi_1'}{\varphi_1} = \frac{1}{2}\kappa, \qquad \frac{\varphi_2''}{\varphi_2} = \kappa.$$

We obtain $\kappa = 6\beta$ and $\varphi_2(z) = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$, where $\lambda^2 = 6\beta$. It follows from the conditions $\sigma(0) = 0$ and $\sigma'(0) = 1$ that

$$\sigma\left(z, \frac{4}{3}\lambda^4, -\frac{8}{27}\lambda^6\right) = \frac{1}{2\lambda} e^{3/2}\left(e^{-(1/6)(\lambda z - 3)^2} - e^{-(1/6)(\lambda z + 3)^2}\right).$$

In his study of third order ordinary differential equations having the Painlevé property, Chazy (1911) was led to the remarkable family of equations

$$y_{xxx} = 2yy_{xx} - 3y_x^2 + \alpha(6y_x - y^2)^2.$$

He showed that when $\alpha = 0$ or $\alpha = \frac{4}{36-k^2}$ where $6 < k \in \mathbb{N}$, then the nontrivial solutions $y = f(x)$ to this equations have a movable circular natural boundary.

The Chazy equation also arises as a reduction of the stationary, incompressible Prandtl boundary layer equations

$$\psi_{\eta\eta\eta} = \psi_\eta \psi_{\xi\eta} - \psi_\xi \psi_{\xi\eta}.$$

The Chazy equation is a reduction of the self-dual Yang–Mills equations when the Yang–Mills potentials take values in the infinite-dimensional Lie algebra $\mathfrak{sdiff}(SU(2))$ of all "divergence-free" vector fields on $SU(2)$ (see details in [19]).

The problem of differentiation of elliptic functions over parameters ω, ω' was solved by Frobenius and Stickelberger (1882). An affine connection on the modular curve, named the FS-connection, was introduced in terms of basic operators giving this differentiation. The coefficient of this connection satisfies the Chazy equation. Solutions of the Chazy equation are used to construct Frobenius structures (see [20] for details).

Weierstrass (1894) found the operators l_0, l_2 considered above solving the problem of differentiation of elliptic functions over parameters g_2, g_3. See above how solutions of the Chazy equation appear in this case.

Our works are devoted to the construction of operators of differentiation of Abelian functions of higher genus over parameters (see [21]) and their applications to the construction of an analogues of the FS connection (see [22]).

TWO-DIMENSIONAL CHAZY TYPE EQUATIONS

The theta function $\theta(\mathbf{z}, \tau)$ of genus 2 is an entire function of $\mathbf{z} = (z_1, z_3)^\top$ determined by the series

$$\theta(\mathbf{z}, \tau) = \sum_{\mathbf{n} \in \mathbb{Z}^2} e^{\pi i \mathbf{n}^\top \tau \mathbf{n} + 2\pi i \mathbf{n}^\top \mathbf{z}},$$

where $\tau = \begin{pmatrix} \tau_{1,1} & \tau_{1,3} \\ \tau_{1,3} & \tau_{3,3} \end{pmatrix}$ is a symmetric matrix, $\tau_{i,j} \in \mathbb{C}$, and $\mathrm{Im}\,\tau$ is a positive definite matrix. We need the following properties of the two-dimensional theta function.

1. This function is quasi periodic in $\mathbf{z}$ with respect to the lattice generated by $(1_2, \tau)$:

$$\theta(\mathbf{z}+\mathbf{m},\tau) = \theta(\mathbf{z},\tau), \quad \theta(\mathbf{z}+\tau\mathbf{m},\tau) = e^{-\pi i \mathbf{m}^\top \tau \mathbf{m} - 2\pi i \mathbf{m}^\top \mathbf{z}} \theta(\mathbf{z},\tau),$$

where $\mathbf{m} \in \mathbb{Z}^2$ and 1_2 is the identity (2×2) matrix.

2. The function $\theta(\mathbf{z},\tau)$ satisfies the system of equations

$$\nabla_\tau \theta(\mathbf{z},\tau) = \frac{1}{4\pi i} \nabla_z \nabla_z^\top \theta(\mathbf{z},\tau), \tag{11}$$

where

$$\nabla_\tau = \begin{pmatrix} \frac{\partial}{\partial \tau_{1,1}} & \frac{1}{2}\frac{\partial}{\partial \tau_{1,3}} \\ \frac{1}{2}\frac{\partial}{\partial \tau_{1,3}} & \frac{\partial}{\partial \tau_{3,3}} \end{pmatrix}, \quad \nabla_z = \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_3} \end{pmatrix}.$$

The function $\theta[\hat{\varepsilon}](\mathbf{z},\tau)$ is defined by the series

$$\theta[\hat{\varepsilon}](\mathbf{z},\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^2} e^{\pi i (\mathbf{n}+\varepsilon')^\top \tau (\mathbf{n}+\varepsilon') + 2\pi i (\mathbf{n}+\varepsilon')^\top (\mathbf{z}+\varepsilon)}$$

and satisfies system (11). Here, $\hat{\varepsilon} = (\varepsilon,\varepsilon') \in \frac{1}{2}\mathbb{Z}^2 \times \frac{1}{2}\mathbb{Z}^2$ is the vector of half-integer characteristics.

Any hyperelliptic curve with affine part of the form

$$V_\lambda = \left\{ (x,y) \in \mathbb{C}^2 \,|\, y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10} \right\}$$

determines a sigma function $\sigma(\mathbf{u},\lambda)$ (see [23]). This function is an entire function of $\mathbf{u} = (u_1,u_3)^\top \in \mathbb{C}^2$ with parameters $\lambda = (\lambda_4,\lambda_6,\lambda_8,\lambda_{10})^\top \in \mathbb{C}^4$; it has a series expansion in powers of $\mathbf{u}$ over the polynomial ring $\mathbb{Q}[\lambda_4,\lambda_6,\lambda_8,\lambda_{10}]$ in the vicinity of $\mathbf{u}=0$. The initial segment of the series has the form

$$\begin{aligned}\sigma(\mathbf{u},\lambda) = u_3 - \frac{1}{3}u_1^3 + \frac{1}{6}\lambda_6 u_3^3 - \frac{1}{12}\lambda_4 u_1^4 u_3 - \frac{1}{6}\lambda_6 u_1^3 u_3^2 - \\ - \frac{1}{6}\lambda_8 u_1^2 u_3^3 - \frac{1}{3}\lambda_{10} u_1 u_3^4 + \left(\frac{1}{60}\lambda_4\lambda_8 + \frac{1}{120}\lambda_6^2\right) u_3^5 + (\mathbf{u}^7).\end{aligned} \tag{12}$$

Here and in what follows, $(\mathbf{u}^k)$ denotes the ideal generated by the monomials $u_1^i u_3^j$ with $i+j=k$. The sigma function is odd with respect to $\mathbf{u}$.

We set $\nabla_\lambda = \left(\frac{\partial}{\partial\lambda_4}\ \frac{\partial}{\partial\lambda_6}\ \frac{\partial}{\partial\lambda_8}\ \frac{\partial}{\partial\lambda_{10}}\right)^\top$, $\nabla_u = \left(\frac{\partial}{\partial u_1}\ \frac{\partial}{\partial u_3}\right)^\top$, and $F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Let $E_{i,j}$ denote the (2×2) matrix in which the position (i,j) is occupied by 1 and all the other entries are zero. We need the following properties of the two-dimensional sigma function (see [23], [3] for details).

1. The following system of equations holds:

$$Q_i \sigma = 0, \quad \text{where} \quad Q_i = l_i - H_i, \quad i = 0,2,4,6, \tag{13}$$

$$(l_0\, l_2\, l_4\, l_6)^\top = T\nabla_\lambda, \tag{14}$$

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 & 4\lambda_6\lambda_{10} - \frac{8}{5}\lambda_8^2 \end{pmatrix},$$

$$H_0 = \mathbf{u}^\top F\nabla_u - 3, \tag{15}$$

$$H_2 = \frac{1}{2}\nabla_u^\top E_{1,1}\nabla_u + \mathbf{u}^\top \beta_2\nabla_u + \frac{1}{2}\mathbf{u}^\top\gamma_2\mathbf{u} - \delta_2,$$
$$H_4 = \nabla_u^\top E_{1,3}\nabla_u + \mathbf{u}^\top \beta_4\nabla_u + \frac{1}{2}\mathbf{u}^\top\gamma_4\mathbf{u} - \delta_4,$$
$$H_6 = \frac{1}{2}\nabla_u^\top E_{3,3}\nabla_u + \mathbf{u}^\top \beta_6\nabla_u + \frac{1}{2}\mathbf{u}^\top\gamma_6\mathbf{u} - \delta_6,$$

$$\beta_2 = \begin{pmatrix} 0 & 1 \\ -\frac{4}{5}\lambda_4 & 0 \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} 0 & 0 \\ -\frac{6}{5}\lambda_6 & \lambda_4 \end{pmatrix}, \quad \beta_6 = \begin{pmatrix} 0 & 0 \\ -\frac{3}{5}\lambda_8 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} -\frac{3}{5}\lambda_4 & 0 \\ 0 & -\frac{4}{5}\lambda_4^2 + 3\lambda_8 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} -\frac{2}{5}\lambda_6 & \lambda_8 \\ \lambda_8 & -\frac{6}{5}\lambda_4\lambda_6 + 6\lambda_{10} \end{pmatrix},$$

$$\gamma_6 = \begin{pmatrix} -\frac{1}{5}\lambda_8 & 2\lambda_{10} \\ 2\lambda_{10} & -\frac{3}{5}\lambda_4\lambda_8 \end{pmatrix}, \quad \delta_2 = 0, \quad \delta_4 = \lambda_4, \quad \delta_6 = \frac{1}{2}\lambda_6.$$

2. The equation $Q_0\sigma = 0$ implies that the function σ is homogeneous of degree 3 in variables $(\mathbf{u},\lambda)$ with respect to the grading $\deg u_i = i$, $i = 1,3$, $\deg\lambda_j = -j$, $j = 4,6,8,10$.
3. The discriminant Δ of the curve V_λ equals $\frac{16}{5}\det T$. Let $\mathcal{B} = \{\lambda \in \mathbb{C}^4 | \Delta(\lambda) \neq 0\}$. Then the curve V_λ, where $\lambda \in \mathcal{B}$, is non degenerate.
   The matrices of the parameters ω, ω', η and η' are determined by the relations (see [23, pp. 8-9])

$$2\omega = \left(\oint_{a_i} du_j\right)_{i,j=1,3}, \qquad 2\omega' = \left(\oint_{b_i} du_j\right)_{i,j=1,3},$$

$$2\eta = -\left(\oint_{a_i} dr_j\right)_{i,j=1,3}, \qquad 2\eta' = -\left(\oint_{b_i} dr_j\right)_{i,j=1,3},$$

where $du_1 = \frac{xdx}{y}$, $du_3 = \frac{dx}{y}$, $dr_1 = \frac{x^2dx}{y}$, and $dr_3 = \frac{(3x^3+\lambda_4 x)dx}{y}$ form a basis of holomorphic differentials on V_λ and a_i and b_i are basis cycles on the curve chosen so as to satisfy the Legendre identity

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^T \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = \frac{\pi i}{2}.$$

The Legendre identity is equivalent to the existence of symmetric matrices τ and $\varkappa$ such that $\omega' = \omega\tau$, $\eta = 2\varkappa\omega$, and $\eta' = 2\varkappa\omega' - \frac{\pi i}{2}(\omega^\top)^{-1}$. In this notation, we have

$$\sigma(\mathbf{u},\lambda) = \frac{\pi}{\sqrt{\det(2\omega)}}\frac{1}{\sqrt[8]{\Delta}}\exp\left(\mathbf{z}^\top(2\omega)^\top\eta\mathbf{z}\right)\theta[\hat{\varepsilon}](\mathbf{z},\tau), \tag{16}$$

where $\mathbf{u} = (2\omega)\mathbf{z}$ and $\hat{\varepsilon}$ is the vector of half-integer characteristics corresponding to the vector of Riemann constants.

4. The function $\sigma(\mathbf{u},\lambda)$ is quasi periodic in $\mathbf{u}$ with respect to the lattice generated by the (2×4) matrix of periods $(2\omega, 2\omega')$.
   Set $\Omega_1(\mathbf{m},\mathbf{m}') = \omega\mathbf{m} + \omega'\mathbf{m}'$ and $\Omega_2(\mathbf{m},\mathbf{m}') = \eta\mathbf{m} + \eta'\mathbf{m}'$, where $\mathbf{m},\mathbf{m}' \in \mathbb{Z}^2$. Then

$$\sigma\left(\mathbf{u}+2\Omega_1(\mathbf{m},\mathbf{m}'),\lambda\right)$$
$$= \exp\left\{2\Omega_2^\top(\mathbf{m},\mathbf{m}')\left(\mathbf{u}+\Omega_1(\mathbf{m},\mathbf{m}')\right)+2\pi i\left(\mathbf{m}^\top\hat{\varepsilon}-\hat{\varepsilon}^\top\mathbf{m}'+\frac{1}{2}\mathbf{m}^\top\mathbf{m}'\right)\right\}\sigma(\mathbf{u},\lambda).$$

In particular, for $\mathbf{m}' = 0$, we obtain

$$\sigma(\mathbf{u}+2\omega\mathbf{m},\lambda) = \exp\left\{2\mathbf{m}^\top\eta^\top(\mathbf{u}+\omega\mathbf{m})+2\pi i\mathbf{m}^\top\hat{\varepsilon}'\right\}\sigma(\mathbf{u},\lambda).$$

Theorem 3 (uniqueness conditions for the two-dimensional sigma function).
The entire function $\sigma(\mathbf{u},\lambda)$ is uniquely determined by the system of equations (13) *with the initial condition* $\sigma(\mathbf{u},0) = u_3 - \frac{1}{3}u_1^3$.

The linear recursion equations for the coefficients of the series expansion of $\sigma(\mathbf{u},\lambda)$ see in [3].

The vector fields l_0, l_2, l_4, l_6 are tangent to the discriminant variety

$$\{(\lambda) \in \mathbb{C}^4 \,|\, \Delta(\lambda) = 0\}$$

because

$$l_0\Delta = 40\Delta, \qquad l_2\Delta = 0, \qquad l_4\Delta = 12\lambda_4\Delta, \qquad l_6\Delta = 4\lambda_6\Delta.$$

We have

$$[l_0,l_2] = 2l_2, \qquad [l_0,l_4] = 4l_4, \qquad [l_0,l_6] = 6l_6, \qquad [l_2,l_4] = \frac{8}{5}\lambda_6 l_0 - \frac{8}{5}\lambda_4 l_2 + 2l_6,$$

$$[l_2,l_6] = \frac{4}{5}\lambda_8 l_0 - \frac{4}{5}\lambda_4 l_4, \qquad [l_4,l_6] = -2\lambda_{10}l_0 + \frac{6}{5}\lambda_8 l_2 - \frac{6}{5}\lambda_6 l_4 + 2\lambda_4 l_6.$$

Consider the integral curve $\lambda(t) = \left(\lambda_4(t),\lambda_6(t),\lambda_8(t),\lambda_{10}(t)\right)$ of the vector field $\gamma(t)l_0 + l_4$. Then along the curve $\lambda(t)$ we have

$$\frac{\partial}{\partial t} = \gamma(t)l_0 + l_4$$

and

$$\dot{\lambda}_4 = 4\gamma(t)\lambda_4 + 8\lambda_8$$
$$\dot{\lambda}_6 = 6\gamma(t)\lambda_6 + 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6$$
$$\dot{\lambda}_8 = 8\gamma(t)\lambda_8 + 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2$$
$$\dot{\lambda}_{10} = 10\gamma(t)\lambda_{10} + 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8$$

We obtain
$$\frac{\partial}{\partial t}\sigma = \partial_{u_1}\partial_{u_3}\sigma + l_u\sigma + V(u,t)\sigma - E(t)\sigma,$$
where $l_u = (3\gamma+\lambda_4)\,u_1\partial_{u_1} + (\gamma u_3 - \frac{6}{5}\lambda_6 u_1)\,\partial_{u_3}$, $V(u,t) = 3\,(\lambda_{10} - \frac{1}{5}\lambda_6\lambda_4)\,u_1^2 + \lambda_8 u_1 u_3 - \frac{1}{5}\lambda_6 u_3^2$ and $E(t) = 3\gamma + \lambda_4$.

Set
$$\sigma\,(\mathbf{u},\lambda\,(t)) = e^{s(\mathbf{u},t)}\phi\,(\mathbf{u},t),$$
where $s(\mathbf{u},t) = \frac{3}{5}\lambda_6 u_1^2 - \gamma(t)u_1u_3 + r(t)$ and $\lambda_4 = -2\gamma$.

Theorem 4. *The function $\phi\,(\mathbf{u},t)$ is a solution of the equation*
$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1\partial u_3} + \sum_{i,j}\chi_{i,j}(t)u_iu_j\phi + \tau(t)\phi,$$
where
$$\chi_{3,3}(t) = -\frac{1}{5}\lambda_6;\quad \chi_{3,1}(t) = \lambda_8 + \dot{\gamma}(t) - \gamma(t)^2;\quad \chi_{1,1}(t) = \frac{3}{5}(-\dot{\lambda}_6 + 4\gamma(t)\lambda_6 + 5\lambda_{10});$$
and $\tau(t) = -2\gamma(t) - \dot{r}(t)$.

Here $r(t)$ is an additional function. If $r(t) \equiv 0$, then $\tau(t) = -2\gamma(t)$. Thus we obtain for λ_i:
$$\lambda_8 = -\frac{1}{4}\dot{\gamma}(t) + \gamma(t)^2;\qquad \lambda_6^2 = -\frac{5}{12}\dot{\lambda}_8;\qquad \lambda_{10} = \frac{1}{10}\dot{\lambda}_6 - \frac{23}{25}\gamma(t)\lambda_6.$$

Theorem 5. *The function $\gamma(t)$ is a solution of the equation $\dot{\gamma}(t) - 4\gamma(t)^2 = const$ if $\dot{\lambda}_8(t) \equiv 0$, and a solution of the equation*
$$\Big(\rho\,(t) + 130\gamma(t)\Big)\Big(\rho\,(t) + 42\gamma(t)\Big) + 10\dot{\rho}\,(t) = 0,$$
if $\dot{\lambda}_8(t) \neq 0$, where $\rho\,(t) = 5\nu\,(t) - 122\gamma(t)$, $\nu\,(t) = \frac{\dot{\varepsilon}(t)}{\varepsilon(t)}$, $\varepsilon\,(t) = \dot{\lambda}_8$.

If $\lambda_6 \equiv 0$ we have $\lambda_{10} = 0$. Then $\Delta = 2^{16}\cdot\frac{1}{5}\lambda_8^3(4\lambda_8 - \lambda_4^2)^2$. Thus if $\lambda_8 \neq 0$ and $4\lambda_8 \neq \lambda_4^2$ then our curve is non singular.

Let $\lambda_6 \equiv 0$ then $\lambda_{10} = 0$, $\lambda_4 = -2\gamma(t)$ and $-\frac{1}{4}\dot{\gamma}(t) + \gamma(t)^2 = \lambda_8 = const$. In this case

$$\chi_{3,3}(t) = 0; \quad \chi_{3,1}(t) = \frac{3}{4}\dot{\gamma}(t); \quad \chi_{1,1}(t) = 0; \quad \tau(t) = -2\gamma(t) - \dot{r}(t).$$

Corollary 1. *Let* $\lambda_8(t) = a^2 = const$, $\lambda_{10}(t) = \lambda_6(t) = 0$ *and* $\lambda_4(t) = -2\gamma(t)$, *where*

$$\gamma(t) = a\frac{1+ce^{8at}}{1-ce^{8at}}.$$

Set $\dot{r}(t) = -2\gamma(t)$. *Then the equation*

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1 \partial u_3} + 3\left(\gamma(t)^2 - a^2\right) u_1 u_3 \phi$$

has a solution

$$\phi(\mathbf{u}, t) = e^{\gamma(t)u_1u_3 - r(t)}\sigma(\mathbf{u}, \lambda(t)).$$

In the case $a = ib$, $b \in \mathbb{R}$, $c = -1$ we have $\lambda_8(t) = -b^2 = const$,

$$\gamma(t) = ib\frac{1-e^{8ibt}}{1+e^{8ibt}} = b\,\mathrm{tg}(4bt),$$

and $r(t) = \frac{1}{2}\ln\cos(4bt) + c_1$. Then the equation is

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1 \partial u_3} + 3\left(\frac{b}{\cos(4bt)}\right)^2 u_1 u_3 \phi.$$

In the case $a \in \mathbb{R}$, $c = -1$ we have $\lambda_8(t) = a^2 = const$,

$$\gamma(t) = a\frac{1-e^{8at}}{1+e^{8at}} = -a\,th(4at).$$

and $r(t) = \frac{1}{2}\ln\left(\mathrm{ch}(4at)\right) + c_2$. Then the equation is

$$\frac{\partial}{\partial t}\phi = \frac{\partial^2\phi}{\partial u_1 \partial u_3} - 3\left(\frac{a}{\mathrm{ch}(4at)}\right)^2 u_1 u_3 \phi.$$

Let $A \cdot B$ denote the tensor product of matrices A and B; i.e.,

$$\begin{pmatrix} A_{1,1} & A_{1,3} \\ A_{3,1} & A_{3,3} \end{pmatrix} \cdot \begin{pmatrix} B_{1,1} & B_{1,3} \\ B_{3,1} & B_{3,3} \end{pmatrix} = \begin{pmatrix} A_{1,1}B & A_{1,3}B \\ A_{3,1}B & A_{3,3}B \end{pmatrix}.$$

We set

$$K = \begin{pmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{pmatrix} \text{ and } N = \begin{pmatrix} E_{11} & E_{21}^{\top} \\ E_{12}^{\top} & E_{22} \end{pmatrix},$$

where $\boldsymbol{E_{ij}}$ as above, is the (2×2) matrix with 1 at the position (ij) and 0 at the other positions.

Remark. The matrix K is a commutation (2×2) matrix; i.e., for any two (2×2) matrices A and $\boldsymbol{B}$, we have $K(A \cdot B)K = (B \cdot A)$.

Let t be a (2×2) matrix, and let $t=\frac{1}{2\pi i}\tau$. We set

$$\nabla_t=\begin{pmatrix}\dfrac{\partial}{\partial t_{1,1}} & \dfrac{1}{2}\dfrac{\partial}{\partial t_{1,3}}\\ \dfrac{1}{2}\dfrac{\partial}{\partial t_{1,3}} & \dfrac{\partial}{\partial t_{3,3}}\end{pmatrix}.$$

Then $\nabla_t=2\pi i\nabla_\tau$.

Theorem 6. *If a function $\varphi(\mathbf{z},t)$ is related to $\sigma(\mathbf{u},\lambda)$ by the substitution*

$$\rho^3\varphi(\mathbf{z},t)=e^{\mathbf{z}^\top G\mathbf{z}-r}\sigma(\mathbf{u},\lambda), \tag{17}$$

where $\mathbf{u}=RA\mathbf{z}$,

$$A=\begin{pmatrix}A_{1,1} & A_{1,3}\\ A_{3,1} & A_{3,3}\end{pmatrix},\qquad G=\begin{pmatrix}G_{1,1} & G_{1,3}\\ G_{1,3} & G_{3,3}\end{pmatrix},\qquad R=\begin{pmatrix}\rho & 0\\ 0 & \rho^3\end{pmatrix},$$

$$\det A=1,\qquad A_{i,j}=A_{i,j}(t),\qquad G_{i,j}=G_{i,j}(t),\qquad r=r(t),\qquad \lambda_i=\rho^{-i}\widetilde{\lambda}_i(t),$$

and satisfies the system of equations

$$\nabla_t\varphi(\mathbf{z},t)=\frac{1}{2}\nabla_z\nabla_z^\top\varphi(\mathbf{z},t),$$

then $\widetilde{\lambda}_i=\rho^i\lambda_i$, A, and G satisfy the system of equations

$$\nabla_t\widetilde{\lambda}_j=A^\top M_jA+\frac{j}{2}\widetilde{\lambda}_jG,$$

$$(\nabla_t\cdot A)=(1_2\cdot A)(K+N)(1_2\cdot G)-\frac{1}{2}(G\cdot FA)+(A\cdot 1_2)^\top S(A\cdot A),$$

$$(\nabla_t\cdot G)=(1_2\cdot G)(K+N)(1_2\cdot G)+\frac{1}{10}(A\cdot A)^\top P(A\cdot A),$$

and the formula

$$\nabla_t r=-\frac{5}{2}G+A^\top\mathcal{R}A$$

holds, where

$$M_j=\rho^jR\left[\begin{pmatrix}T_{2,j} & \frac{1}{2}T_{4,j}\\ \frac{1}{2}T_{4,j} & T_{6,j}\end{pmatrix}-\frac{1}{4}T_{0,j}\begin{pmatrix}\operatorname{tr}(\beta_2) & \frac{1}{2}\operatorname{tr}(\beta_4)\\ \frac{1}{2}\operatorname{tr}(\beta_4) & \operatorname{tr}(\beta_6)\end{pmatrix}\right]R,$$

$$S=(R\cdot R^{-1})\left[\begin{pmatrix}\beta_2^\top & \frac{1}{2}\beta_4^\top\\ \frac{1}{2}\beta_4^\top & \beta_6^\top\end{pmatrix}-\frac{1}{4}\begin{pmatrix}\operatorname{tr}(\beta_2) & \frac{1}{2}\operatorname{tr}(\beta_4)\\ \frac{1}{2}\operatorname{tr}(\beta_4) & \operatorname{tr}(\beta_6)\end{pmatrix}\cdot F\right](R\cdot R),$$

$$P=-5(R\cdot R)\begin{pmatrix}\gamma_2 & \frac{1}{2}\gamma_4\\ \frac{1}{2}\gamma_4 & \gamma_6\end{pmatrix}(R\cdot R),$$

$$\mathcal{R}=R\left[\frac{3}{4}\begin{pmatrix}\operatorname{tr}(\beta_2) & \frac{1}{2}\operatorname{tr}(\beta_4)\\ \frac{1}{2}\operatorname{tr}(\beta_4) & \operatorname{tr}(\beta_6)\end{pmatrix}-\begin{pmatrix}\delta_2 & \frac{1}{2}\delta_4\\ \frac{1}{2}\delta_4 & \delta_6\end{pmatrix}\right]R;$$

the matrices T, β_i, and γ_i and the quantities δ_i are defined in (14) *and* (15).

The following result is similar to Lemma 2:

Lemma 3. *Let $\sigma(\mathbf{u},\lambda)$ be the sigma function with matrix parameters ω, ω', η, and η'. Then the function $\varphi(\mathbf{z},t)$, defined by* (17) *with* $2\omega = RA$ *satisfies the quasi periodicity condition* $\varphi(\mathbf{z}+\mathbf{m},t) = e^{2\pi i \mathbf{m}^\top \hat{\varepsilon}'}\varphi(\mathbf{z},t)$, *where* $t = \frac{1}{2\pi i}\tau = \frac{1}{2\pi i}\omega^{-1}\omega'$, *if and only if* $G = -2\omega^\top\eta = -(2\omega)^\top \varkappa(2\omega)$.

The rational limit of the sigma function in genus g is a polynomial in the variables $\mathbf{u}$. Such polynomials have remarkable properties (see [24]) inherited from the function $\sigma(\mathbf{u},\lambda)$, although they correspond to the case $\lambda = 0$.

In the case of genus 2, $\lambda_i = 0$ for $i = 4,6,8,10$ in the rational limit. Thus, $S_2 = -E_{1,3}$, $S_4 = S_6 = 0$, $\gamma_i = 0$, $\mathrm{tr}(\beta_i) = 0$, and $T_{i,j} = 0$, and the formulas from Theorem 6 take the form

$$\nabla_t r = -\frac{5}{2}G,$$
$$\nabla_t \cdot A = -\frac{1}{2}(G\cdot FA) + (1_2\cdot A)(K+N)(1_2\cdot G) - (A^\top\cdot 1_2)(E_{1,1}\cdot E_{3,1})(A\cdot A),$$
$$\nabla_t \cdot G = (1_2\cdot G)(K+N)(1_2\cdot G).$$

We set $g(t) = c_0 + \mathrm{tr}(qt) + c_2\det t$, where c_0, c_2, and $q = \begin{pmatrix} q_{11} & q_{13} \\ q_{13} & q_{33}\end{pmatrix}$ are parameters independent of t.

Theorem 7. *A general solution of the equation*

$$\nabla_t \cdot G = (1_2\cdot G)(K+N)(1_2\cdot G)$$

has the form

$$G = -\frac{1}{2}\nabla_t \ln g(t),$$

where the parameters of the function $g(t)$ are related by

$$c_0c_2 = \det q.$$

Let $c_2 \neq 0$, then $g(t) = \det(m+\alpha t)$, where $\alpha^2 = c_2$, $\alpha m' = q$, $\det m = c_0$, and m' is adjoint matrix for m.

Corollary 2. *In this case*

$$G = -\frac{1}{2}\alpha(m+\alpha t)^{-1}.$$

REFERENCES

1. V. Buchstaber, and D. Leikin, *Funct. Anal. Appl.* **38**, 88–101 (2004).
2. V. Buchstaber, and D. Leikin, *Functional Anal. Appl.* **36**, 267–280 (2002).
3. V. Buchstaber, and D. Leikin, *Proceedings of the Steklov Math. Inst.* **251**, 49–120 (2005).
4. V. Buchstaber, V. Enolskii, and D. Leikin, *Functional Anal. Appl.* **34**, 159–171 (2000).
5. V. Buchstaber, V. Enolskii, J. Eilbeck, D. Leikin, and M. Salerno, *J. Math. Phys.* **43**, 2858–2881 (2002).

6. J. Eilbeck, V. Enolskii, and E. Previato, *Lett. Math. Phys.* **63**, 5–17 (2003).
7. V. Buchstaber, and S. Shorina, "w-Function of the KdV Hierarchy," in *Geometry, Topology, and Mathematical Physics, S.P.Novikov's seminar: 2002–2003*, edited by V.M. Buchstaber and I.M. Krichever, Amer. Math. Soc. Transl., Providence, RI, 2004, vol. 212 of 2, pp. 41–46.
8. H. Braden, V. Enolskii, and A. Hone, *J. Nonlin. Math. Phys* **12**, 46–62 (2005), supplement 2, arXiv:math.NT/0501162.
9. J. Eilbeck, V. Enolskii, S. Matsutani, Y. Ônishi, and E. Previato, *Int. Math. Res. Notices* **2007**, rnm140–38 (2007), arXiv:math.AG/0610019.
10. V. Enolskii, S. Matsutani, and Y. Ônishi, *Tokyo J. Math.* **31**, 27–38 (2008), arXiv:math.AG/0508366.
11. A. Nakayashiki, Sigma function as a tau function (2009), preprint, arXiv:0904.0846.
12. Y. Ônishi, Determinant expressions for hyperelliptic abelian functions (2005), with an Appendix by S. Matsutani: *Connection of the formula of Cantor and Brioshi-Kiepert type*, Proc. Edinburgh Math.Soc, Preprint NT/0105189.
13. V. Buchstaber, and V. Enolskii, *Russian Math. Surveys* **50**, 195–197 (1995).
14. V. Buchstaber, V. Enolskii, and D. Leikin, *Russian Math. Surveys* **54**, 628–629 (1999).
15. V. Buchstaber, and D. Leikin, *Russian Math. Surveys* **56**, 1155–1157 (2001).
16. V. Buchstaber, and D. Leikin, *Doklady Math. Sci.* **66**, 87–90 (2003).
17. V. Buchstaber, D. Leikin, and M. Pavlov, *Funct. Anal. Appl.* **37**, 251–262 (2003).
18. V. Arnold, *Singularities of Caustics and Wave Fronts*, Kluwer, Dordrecht, 1990, Fazis, Moscow,1996.
19. P. Clarkson, and P. Olver, *J. Diff. Eq.* **124**, 225–246 (1996).
20. B. Dubrovin, "Geometry of 2D topological field theories," in *Integrable systems and quantum groups*, Springer, Berlin, 1996, vol. 1620 of *Lect. Notes Math.*, pp. 120–348, hep-th/9407018.
21. V. Buchstaber, and D. Leikin, *Funct. Anal. Appl.* **42**, 268–278 (2008).
22. V. Buchstaber, and E. Bunkova, *Proceedings of the Steklov Institute of Mathematics* **266**, 1–28 (2009).
23. V. Buchstaber, V. Enolskii, and D. Leikin, "Kleinian functions, hyperelliptic Jacobians and applications," in *Reviews in Mathematics and Math. Physics*, edited by I.M. Krichever and S.P. Novikov, Gordon and Breach, London, 1997, vol. 10, pp. part 2, 3–120, see improved version in math-ph.
24. V. Buchstaber, V. Enolskii, and D. Leikin, *Functional Anal. Appl.* **33**, 83–94 (1999).