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Kolmogorov and topology

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At the beginning of the nineteen thirties, Andreï Nikolaevich Kolmogorov published a few papers on topology, totaling roughly thirty pages. These works immediately made him one of the main creators of modern algebraic topology. In this short survey, I have tried to present Kolmogorov's remarkable results (not only for a public of topologists), and to explain the source of some of the ideas leading to those results. This survey is based on a lecture by the author, entitled *Kolmogorov, cohomology and cobordisms*, which was presented during the conference *Kolmogorov and contemporary mathematics* in commemoration of the centennial of Kolmogorov, held in Moscow from June 16 to June 21, 2003.

7.1 Prelude

Pavel Samuilovich Urysohn (1898–1924) and Pavel Sergeevich Aleksandrov (1896–1982) are recognized as the founders of the Soviet school of topology. In the years 1920–1924, Urysohn's enthusiasm and the remarkable results he had obtained played a great role in bringing talented young researchers to this area of mathematical research. In [AK53], Aleksandrov remembers that Kolmogorov, still a student, first attracted Urysohn's attention when, during a lecture, he pointed out a mistake in the complicated constructions Urysohn was using to prove his theorem concerning the topological dimension of Euclidean space of dimension three[1]:

[1] Urysohn is the creator of a famous theory of topological dimension, and it was natural to check that this notion was consistent with the usual dimension in the case of the Euclidean spaces $\mathbb{R}^n$. (Editor's note.)

> "P. S. Urysohn corrected this mistake a few days later, but the mathematical acuity of the eighteen year old student Kolmogorov had made a great impression on him."

(Something similar happened during N. N. Lusin's special course. Kolmogorov himself described it in this manner: "Although my contribution was rather childish, it helped me to become known in the lusitanian[2] circle" [Kol72].)

Fifty years later, in his article "A scientific teacher" [Kol72] devoted to Urysohn, Kolmogorov describes his first scientific steps and his conversations with Urysohn. He describes his hesitations concerning the choice of a research subject, due among other things

> "to a vague desire to do mathematics, but with strong applications to physics and natural sciences ... As far as possible, Pavel Samuilovich tried to enlist me in his researches concerning Poincaré's problem about closed geodesics on surfaces ... All these questions appealed to me in themselves, they corresponded to the idea I had of what a mathematician should most occupy himself with ... The internal logic of my personal investigations brought me to topology only much later, after mathematical passions for logic and probability theory."

In November 1925, at Moscow University, the famous *Moscow topological circle* was created, Aleksandrov being its permanent leader. Until the beginning of the nineteen-sixties, this was the center of the Moscow school of topology.

The programme of the first ten years of the circle is described in the paper [Nem36], in which one finds in particular a very impressive *list of lectures given at the topological circle between 1925 and 1935*, in which Kolmogorov's lectures appear:

N° 52. The group of homeomorphisms in metric spaces, April 1929.
N° 53. The group of homeomorphisms in metric spaces, May 1929.
N° 55. The group of homeomorphisms in a topological space, February 1930.
N° 69. The topological axiomatics of projective space, June 1931.
N° 83. On the theory of continuous repartitions, May 1933.

This list indicates that Kolmogorov, who already enjoyed a world-wide reputation, was participating actively in the topology seminar, but had not yet found his own topological subject. The ensuing events are all the more surprising.

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[2] At the end of the 1910s, Lusin, with Egorov, had created a brilliant research group at Moscow University, which was called "Lusitania" by its members, as a pun on Lusin's name. *Lusitania* is the name of an antique Roman province which corresponds roughly to present-day Portugal; it is also the name of a British ocean liner which had been much in the news in 1915 and the following years: after it was sunk by a German submarine, with the loss of more than a thousand civilian lives, international public opinion was scandalized, so that when American troops landed in Europe two years later, a cartoon showed a picture of Kaiser Wilhelm II asking one of his officers: "How many ships did it take to bring them here?", and the officer answering: "Only one, the *Lusitania*"! (Editor's note.)

From September 4 to September 10, 1935, at the Institute of Mathematics of Moscow University, the first international topology conference was held. It was an extraordinary moment in the history of topology. The results which were presented at this conference delineated the main directions of research for many years.

Here is a selection of the lectures:

- J. W. Alexander (Princeton), *On the rings of complexes and the combinatorial theory of integration*;
- A. N. Kolmogorov (Moscow), *On the homology rings of closed sets*;
- E. Čech (Brno), *On the Betti groups with coefficients in an arbitrary field*;
- H. Freudenthal (Amsterdam), *On topological approximations of spaces*;
- A. W. Tucker (Princeton), *On discrete spaces*;
- M. H. Stone (Cambridge, Mass.), *The theory of maps in general topology*;
- P. S. Aleksandrov (Moscow), *Some solved and unsolved problems in general topology*;
- W. Hurewicz (Amsterdam), *Homology and homotopy*;
- K. Borsuk (Warsaw), *On spherical spaces*;
- S. Lefschetz (Princeton), *On locally-connected manifolds*;
- H. Hopf (Zurich), *New investigations on* n*-dimensional manifolds*;
- G. Nöbeling (Erlangen), *On triangulation of manifolds and the Main Hypothesis of combinatorial topology*;
- H. Whitney (Cambridge, Mass.), *Topological properties of differentiable manifolds*;
- P. Smith (New York), *On 2-periodic maps*;
- P. Heegaard (Oslo), *On the four-color problem*;
- G. de Rham (Lausanne), (1) *On Reidemeister's new topological invariants*; (2) *Topological aspects of the theory of multiple integrals*;
- A. A. Markov (Leningrad), *On equivalence of closed braids*;
- L. S. Pontriaguin (Moscow), *Topological properties of compact Lie groups*;
- E. R. van Kampen (New Heaven), *The structure of compact groups*;
- J. von Neumann (Princeton), *The theory of integration in continuous groups*;
- A. Weil (Paris), (1) *A topological proof of a theorem of Cartan*; (2) *The systems of curves on a torus.*

A complete list of lectures is found in [Ale36a], and the programme of the conference is in [Shi03, n° 2, pp. 590–593].

Among all those wonderful subjects and all those mathematicians – all internationally-renowned leaders of topology – the lectures of Kolmogorov and Alexander on the construction of dual complexes for quite general spaces, the homology of which has a natural structure of ring, attracted widespread attention. In modern terminology, this was the construction of the cohomology ring of topological spaces.

All ulterior developments of algebraic topology have confirmed the extraordinary importance of the results of those two authors.

7.2 The main topological results of A. N. Kolmogorov

7.2.1 Algebraic topology

During the nineteen-thirties, a branch of topology, called *algebraic topology* by Solomon Lefschetz (1884–1972), emerged. Lefschetz himself created the deep theory of homology of projective complex algebraic varieties, including the fundamental intersection theory of algebraic cycles; he then carried his ideas over to topology. Another approach to homology theory was developed by Élie Cartan (1869–1951), based on Poincaré's ideas and on Riemannian geometry. Cartan constructed the "tensor theory of homology", and conjectured that it leads to ordinary homology theory. Cartan's programme was implemented by de Rham (see, e.g. [Nov04], [Die89], about the history and the development of the main ideas of algebraic topology).

The results obtained both by Kolmogorov and Alexander are considered as a complete solution of this fundamental problem, with a global mathematical reach. In the monography [Lef42] which introduced the name "algebraic topology", the theory of Kolmogorov and Alexander plays a prominent role. Concerning the origins of the ideas of this theory, Lefschetz wrote:

> "Chiefly for purposes of extending the concepts of *differential* and *integral* to general topological spaces Alexander [Ale36b, Ale38], and later Kolmogorov [K4, K5, K6, K7], have developed a type of theory based directly upon chains and cochains."

Kolmogorov himself explained the main idea in [K9]

> "The author's goal is to construct a particular difference calculus which, on the one hand, leads to differential operators acting on anti-symmetric tensors (multivectors) by a limit process, and on the other hand is closely related to the concepts of combinatorial topology.
> In particular, it is possible to define new invariants of complexes and closed sets using this difference calculus, and to prove some generalizations of the known duality theorems."

Kolmogorov wrote the following comments concerning his work of 1936–37 on homology theory in the edition of his selected works published in honor of his eightieth birthday [Kol85]:

> "The initial impulse for these works was reading the thesis of Georges de Rham [DeR31] (1931), in which the duality of Betti groups of differentiable manifolds and Betti groups generated by currents was established. After the 1930's, I did not work on those subjects anymore; yet, the idea presented in the four notes in Comptes Rendus de l'Académie des Sciences de Paris, which is to exploit the duality between the groups of antisymmetric functions of n points and of additive antisymmetric functions of n sets, still seems to me to have some pedagogical interest."

Except for those notes and the articles [K2, K3, K9], Kolmogorov produced other works of algebraic topology which he did not publish, despite having a great opinion of this subject:

> "... Homologic topology interested me a lot, and during the years 1934–36, I should have worked on more on this ..."

writes Kolmorogov in a letter to N. N. Lusin about his research plans, dated October 7, 1945 [Shi03, 1, p. 227].

The theory of Kolmogorov and Alexander on a space X relates the set of chains with the boundary operator Δ (which is called the chain complex) and its dual, the set of cochains with the coboundary operator ∇ (called the cochain complex).

Fix an abelian group G. Let $\{A\}$ be the family of closed sets of the space X. A p-chain (or chain of dimension p) on G is an antisymmetric function φ^p with values in G, depending on $p+1$ sets $(A_0, \dots, A_p)$, with the following properties:

(1) it is equal to zero whenever $\bigcap A_i = \varnothing$ or $A_i = A_j$ for some $i \neq j$;
(2) it is multilinear, in the sense that if the interiors of A_i and A'_i are disjoint, we have

$$\varphi^p(\dots, A_i \cup A'_i, \dots) = \varphi^p(\dots, A_i, \dots) + \varphi^p(\dots, A'_i, \dots).$$

The boundary of the chain φ^p is the $(p-1)$-chain

$$(\Delta\varphi^p)(A_0, \dots, A_{p-1}) = \varphi^p(X, A_0, \dots, A_{p-1}).$$

Obviously, we have $\Delta\Delta\varphi^p = 0$. A p-chain φ^p is called a p-boundary if it is of the form $\Delta\varphi^{p+1}$, where φ^{p+1} is a $(p+1)$-chain. And a p-chain φ^p is a p-cycle if $\Delta\varphi^p = 0$. The relation $\Delta\Delta\varphi^{p+1} = 0$ guarantees that p-boundaries are p-cycles. The p-dimensional homology (*in degree* p) of X with values in G is then the set of p-cycles modulo p-boundaries.

The dual complex, formed with cochains, is described in detail in [Lef42], for instance. Here, we only need to know that p-cochains are constructed from antisymmetric functions ψ_p with values in the *Pontryagin dual* H of G (i.e. the character group of G), depending on $p+1$ points $(x_0, \dots, x_p)$, $x_i \in X$, and that the coboundary operator[3] is given by the formula

$$(\nabla\psi_p)(x_0, \dots, x_{p+1}) = \sum_q (-1)^q \psi_p(\dots, x_{q-1}, x_{q+1}, \dots),$$

and obviously $\nabla\nabla\psi^p = 0$. In the dual complex formed with cochains, a p-cochain ψ^p is called a p-coboundary if it is of the form $\nabla\psi^{p-1}$, where ψ^{p-1} is a $(p-1)$-cochain. A p-cochain ψ^p is called a p-cocycle if $\nabla\varphi^p = 0$. The

[3] In [K9], Kolmogorov, who was writing for the public of the *Seminar on vector and tensor analysis and its applications to mechanics and physics*, used the notation rot for this operator

relation $\nabla\nabla\psi^{p-1} = 0$ guarantees that p-coboundaries are p-cocycles. The p-dimensional cohomology (*in degree* p) of X with values in H is the set of p-cocycles modulo p-coboundaries.

The applications of Kolmogorov's approach to the construction of homology and cohomology theory with duality theorem for homology and cohomology are discussed in the comments of G.S. Chogoshvili "Homology Theory" in [Kol85].

In their lectures at the first international topology conference in 1935, Kolmogorov and Alexander gave an explicit multiplication formula, which associates a $(p+q)$-cochain to a p-cochain and a q-cochain. This operator, like Cartan's exterior product of differential forms, is anticommutative: Kolmogorov uses the notation $[\cdot, \cdot]$ for this operation, and we have then

$$[\psi_p, \psi_q] = (-1)^{pq}[\psi_q, \psi_p].$$

Kolmogorov and Alexander proved that this operation induces an associative and anticommutative product on cocyles, and in this manner, they introduced a ring structure on the cohomology.

Shortly after the conference, Čech [Čec36], Whitney [Whi37] and Alexander [Ale36b] showed the existence of another operation on cochains, denoted $\smile$, given by a formula which is close to the original formula of Kolmogorov and Alexander. This operation induces also an associative and anticommutative product on cocyles. In cohomology, there is a relation

$$[\psi_p, \psi_q] = \frac{(p+q)!}{p!q!}\, \psi_p \smile \psi_q. \qquad (*)$$

Hence, in cohomology with coefficients in $\mathbb{Q}$, both products are equivalent, but already in integral cohomology, if there are cocycles of finite order, one may have cocycles, say a and b, such that (cohomologically) we have $[a, b] = 0$ and at the same time $a \smile b \neq 0$. In textbooks of algebraic topology, when the Kolmogorov-Alexander product for cohomology is mentioned, what is meant is the operation $\smile$. In what follows, we will say that $\smile$ is the *standard* operation.

The operation $\smile$ in combinatorial topology leads to a combinatorial analogue of Lefschetz's intersection theory, whereas Cartan's theory of products of differential forms corresponds to Kolmogorov's $[\cdot, \cdot]$ operation.

At the level of cochains, in contrast with the cocycles, the formula for the standard operation $\smile$ *depends of the choice of an ordering of the vertices* in a simplicial complex. The operation $[\cdot, \cdot]$ is deduced from the operation $\smile$ by averaging

$$[\psi_p, \psi_q](x_0, \ldots, x_{p+q}) = \sum_{\sigma} \psi_p \smile \psi_q\, (x_{\sigma(0)}, \ldots, x_{\sigma(p+q)}) \qquad (**)$$

where the sum is over all permutations σ of $(0, \ldots, p+q)$. Thus, the formula $(**)$ expresses the relation between the two products of combinatorial topology, which are philosophically related with the theories of Cartan and Lefschetz.

The important distinction between integral cohomology (with coefficients in $\mathbb{Z}$) and rational cohomology (with coefficients in $\mathbb{Q}$) is expressed in the fundamental rule of combinatorial topology:

> *It is impossible to obtain the standard multiplication using anticommutative operations on cochains with integral coefficients.*

We must emphasize that *at the level of cochains* the operation $[\cdot, \cdot]$ *is anticommutative, but not associative*, whereas the operation $\smile$ *is not anticommutative, but is associative.*

The defect of anticommutativity of the standard operation at the level of cochains was used by Steenrod [Ste62] to construct "cohomological operations", and also to define structures of modules over the algebra of cohomological operations on the cohomology ring.

A question is raised: *what additional structure may be introduced in cohomology, using the fact that Kolmogorov's operation is not associative at the level of cochains?*

The ulterior progress of algebraic topology has led to extraordinary cohomology theories, the best known of which (in terms of their numerous applications) are K-theory and cobordism theory. Almost all those applications use in an essential way the product structure on the cohomology ring and the powerful algebras of cohomological operations (see [Nov04]).

The transformation of the standard product to the Kolmogorov product by means of the formula $(*)$ is the unique non-trival (non-invertible[4]) transformation of the product in the case of classical cohomology. As shown in [BBNY00], there is a great variety of transformations of products in the case of complex cobordism theory. Those transformations are related to important structures in analysis, representation theory, and commutative and non-commutative algebra.

7.2.2 General topology

A number of fundamental notions are also due to Kolmogorov, who introduced them in order to solve some important problems of general topology. These notions are commonly used today, appearing constantly in new applications.

- Kolmogorov's name is attached to the separation Axiom $\mathcal{T}_0$ – the weakest separation axiom in use. A topological space X is called a $\mathcal{T}_0$-space, or a *Kolmogorov space* if, for any pair of distinct points $x, y \in X$, $x \neq y$, there exists an open set of X containing one of the points but not the

[4] The formula $(*)$ is not invertible in cohomology with *integral* coefficients because $\frac{p!q!}{(p+q)!}$ is not an integer (except when it is equal to 1)

other[5]. An important non-trivial example of Kolmogorov space is the set of simplexes of a simplicial complex, with the topology in which the closure of a point (i.e. of a simplex) is the set of all its "faces" of arbitrary dimension (including itself, of course): see [AH35] (for instance, p. 132 of the 1974 edition) or [Kur66] (Chap. 1, Sect. 5, IX: $\mathcal{T}_0$-spaces).

- Kolmogorov found a necessary and sufficient condition for a general topological vector space to be normable [K1][6]. In the course of solving this problem, he introduced the notion of *bounded* subset in topological vector spaces. This concept turned out to be of fundamental importance in their duality theory, and in the applications of topological vector spaces to analysis.
- Recall that a map from a topological space X to a topological space Y is *open* if the image of any open set in X is open in Y. Kolmogorov constructed a surprising example of an open map from a topological space of dimension 1 onto a space of dimension 2. This had a great importance in general topology. It shows, in particular, that the notion of "dimension of a topological space" is not trivial. At the root of the construction of this map is an explicit open map from the 2-dimensional torus onto the Möbius strip, such that the equator of the torus is mapped to the boundary of the strip. Later on, Kolmogorov's example found an application in the theory of group actions on topological spaces: it was remarked that it describes an effective action of a 0-dimensional abelian group (namely, the compact group of 2-adic integers) on a 1-dimensional compact space (the Menger's curve), such that the space of orbits is of dimension 2 (the Pontryagin 2-dimensional surface) (see [Wil63]: Kolmogorov's example is one of the "three famous examples in topology" mentioned in the title).

  Kolmogorov wrote the following comments concerning the origins of this result in [Kol85] (see p. 476):

  > "The possibility of an increase in dimension under open mappings ([K8]) interested P. S. Aleksandrov very much. For some time we together tried to prove that increase in dimension is impossible. In these attempts we gradually understood why we failed. An analysis of the failure led us to a counterexample."

  Further investigations in this direction are described in the recently survey "Problems of dimension raising" (see [Che05], §2).
- A. N. Kolmogorov in collaboration with I. M. Gel'fand published the paper [K10]. In introduction to this paper we read:

[5] Recall that a topological space is *separated*, or *Hausdorff*, if it satisfies the (stronger) Axiom $\mathcal{T}_2$: *any two distinct points $x, y \in X$ have disjoint neighborhoods.* There is also an Axiom $\mathcal{T}_1$: *any singleton is a closed set.* These axioms express properties of separation of varying strength: it is obvious that $\mathcal{T}_2$ implies $\mathcal{T}_1$, and that $\mathcal{T}_1$ implies $\mathcal{T}_0$. (Editor's note.)

[6] See Theorem 4 of Chap. 8 (by Vladimir Tikhomirov) in this volume. (Editor's note.)

> "... we consider the ring of continuous functions on a topological space as a purely algebraic object without defining any topological relations in it. It turns out that in the case of bicompact spaces, considered by M. H. Stone, and also in some much more general cases, even the purely algebraic structure of the ring of continuous functions determines the topological space to within a homeomorphism."

Many monographs and textbooks containing basic general topology and functional analysis include this result. Further investigations in this direction are described in the recently survey [BR04].

7.3 A topological idea of Kolmogorov

In May 1929, having defended his thesis four years before, Kolmogorov had already published 18 works, which brought him worldwide renown. In the paragraph concerning this period in his *remembrances of P. S. Aleksandrov*, we read [Kol86]:

> "Our personal contacts with Pavel Sergeevich were very limited at that time, although we often met at the concerts in the Small Room of the conservatory ... In 1929, we met again during a trip on the Volga. I do not remember very well how I had decided to suggest to Pavel Sergeevich that he be the third companion[7]. However, he immediately accepted ... One may consider the day of departure – June 16 – as the starting point of our friendship."

To this friendship, mathematics owes the correspondence between Kolmogorov and Aleksandrov. A selection of this correspondence is now accessible in the commemorative edition of Kolmogorov's works [Shi03]. In Kolmogorov's letters, one finds many remarks on mathematical results, which give insights into his ideas and scientific projects. Some of these ideas and projects were much in advance on their time.

In the letter to Aleksandrov dated September 22, 1932, "from Dnepropetrovsk to Zurich, Switzerland" (see [Shi03], Vol. 2, p. 439), Kolmogorov wrote:

> "Questions of topology. It seems to me that it is not difficult to prove that an n-dimensional closed set may be embdedded in a space of sufficiently large dimension, and in only one way. I know how to prove this for polyhedra, which embed in (Euclidean) space of dimension $4n+2$. It seems that the properties that can be expressed in terms of the complement of the set in this space should be called homologic?

[7] It is a rowing-boat trip, which Kolmogorov and his *gymnasium* friend Nikolaï Njuberg had first thought of doing together only

In other words, F and F_1 are homologically equivalent if, being embedded in a space of sufficiently large dimension, their complements are homeomorphic. Then, the question of determining whether homologic invariants are complete will have a well-defined meaning – namely, to characterize completely, or not, the complement in a space of sufficiently large dimension."

Almost fourty years later, Borsuk developed a new direction in homotopic topology – shape theory [Bor71]. Using the results of this theory [Cha72, GS73], one can find the answers to the questions raised in Kolmogorov's letter.

For instance, let F and F_1 be compact polyhedra (or even compact absolute neighborhood retracts[8]) of dimension n. Then, under some restrictions on their position in Euclidean space $\mathbb{R}^N$ with sufficiently large dimension N, they are homotopy-equivalent if and only if their complements $\mathbb{R}^N \setminus F$ and $\mathbb{R}^N \setminus F_1$ are homeomorphic.

References

[AH35] Alexandroff, P. (Aleksandrov, P.S.), Hopf, H.: Topologie, I (in German), vol. 45 of Grundlehren der Math. Wiss. Springer, Berlin (1935) Corrected printing: Springer, Berlin, New York (1974)

[AK53] Aleksandrov, P.S., Khinchin, A.J.: Andreï Nikolaevich Kolmogorov: in honor of his 50th birthday (Russian). Uspekhi Mat. Nauk, **8**, 178–200 (1953)

[Ale36a] Aleksandrov, P.S.: First international congress on topology, Moscow (Russian). Uspekhi Mat. Nauk, **1**, 260–262 (1936)

[Ale36b] Alexander, J.W.: On the connectivity ring of an abstract space. Annals of Math., **37**, 698–708 (1936)

[Ale38] Alexander, J.W.: A theory of connectivity in terms of gratings. Annals of Math., **39**, 883–912 (1938)

[BBNY00] Botvinnik, B.I., Buchstaber, V.M., Novikov, S.P., Yuzvinsky, S.A.: Algebraic aspects of the theory of multiplications in complex cobordism theory. Uspekhi Mat. Nauk, **55:4**, 5–24 (2000). Russian Math. Surv., **55**:4, 613–633 (2000)

[Bor71] Borsuk, K.: Theory of shape. Aarhus, Lecture Notes Series vol. 28 (1971)

[BR04] Buchstaber, V.M., Rees, E.G.: Rings of continuous functions, symmetric products, and Frobenius algebras. Uspekhi Mat. Nauk, **59**:1, 125–144 (2004). Russian Math. Surv. **59**:1, 125–146 (2004)

[Čec36] Čech, E.: Multiplication on a complex. Annals of Math., **37**, 681–697 (1936)

[Cha72] Chapman, T.A.: On some applications of infinite-dimensional manifolds to the theory of shape. Fund. Math., **76**, 181–193 (1972)

[8] Absolute neighborhood retract: an important notion invented by Borsuk in 1931. In [Bor71], he explains his motivation as follows: the use of absolute neighborhood retracts helps to bring topology closer to geometric intuition. See [Mar99] for the history of this notion, and of shape theory (and their relations)

[Che05] Chernavskii, A.V.: On the work of L.V. Keldysh and her seminar. Uspekhi Mat. Nauk, **60**:4, 11–36 (2005). Russian Math. Surv., **60**:4, 589–614 (2005)

[DeR31] de Rham, G.: Sur l'Analysis situs des variétés à n-dimensions. J. Math pures et appl., **10**, 115–200 (1931)

[Die89] Dieudonné, J.: *A history of algebraic and differential topology: 1900–1960*. Birkhäuser (1989)

[GS73] Geoghegan, R., Summerhill, R.R.: Concerning the shapes of finite-dimensional compacta. Trans. Amer. Math. Soc., **179**, 281–292 (1973)

[Kol85] Kolmogorov, A.N.: Selected works, Volume I. Mathematics and Mechanics, Tikhomirov, V.M. (ed). Kluwer, Boston, Mathematics and its Applications, vol. 25 (1991). Original Russian edition: Nauka, Moscow (1985)

[Kol86] Kolmogorov, A.N.: Remembering P.S. Aleksandrov (Russian). Uspekhi Mat. Nauk, **41**, 187–203 (1986). Russian Math. Surveys, **41**, 225–246 (1986)

[Kol72] Kolmogorov, A.N.: A scientific tutor (remembrances of P.S. Urysohn) (Russian). In: Neiman, L.: The joy of discovery. Literature for children, Moscow, 160–164 (1972). Nauka, collection “Quantum”, vol. 64, 10–14 (1988)

[Kur66] Kuratowski, K.: Topology, vol. I. Academic Press, New York, London (1966)

[Lef42] Lefschetz, S.: Algebraic Topology. New York (1942)

[Mar99] Mardešić, S.: Absolute neighborhood retracts and shape theory. In: James, I.M. (editor): *History of topology*, 241–269. North-Holland, Amsterdam (1999)

[Nem36] Nemytskij, V.V.: The Moscow topological circle after 10 years (Russian). Uspekhi Mat. Nauk, **2**, 279–285 (1936)

[Nov04] Novikov, S.P.: Algebraic topology. Steklov Mathematical Institute, Russian Academy of Sciences, ser. Contemporary problems in Mathematics, No. 4 (2004). Completed version: Topology in the 20th century: a view from the inside. Uspekhi Mat. Nauk (Russian). Russian Math. Surveys, vol. **59**:5, 803–829 (2004)

[Shi03] Shiryaev, A.N. (editor): A.N. Kolmogorov, Commemorative edition (Russian), vol. 1–3, Moscow, Physmatlit (2003)

[Ste62] Steenrod, N.E.: Cohomology operations (lectures written and revised by D.B.A. Epstein). Princeton Univ. Press, Princeton, New Jersey (1962)

[Whi37] Whitney, H.: On products in a complex. Proc. Nat. Acad. Sci. USA, **23**, 285–291 (1937)

[Wil63] Williams, R.F.: A useful functor and three famous examples in topology. Trans. Amer. Math. Soc., **106**, 319–329 (1963)

Topological articles by A.N. Kolmogorov

[K1] Kolmogorov, A.N.: Zur normierbarkeit eines allgemeinen topologischen linearen Raumes. Stud. math., **5**, 29–33 (1934). (Art. n° 23 in [Kol85].)

[K2] Kolmogorov, A.N.: Über die dualität im Aufbau der kombinatorischen Topologie. Math. Sb., **1**, 97–102 (1936). (Art. n° 29 in [Kol85].)

[K3] Kolmogorov, A.N.: Homologierung des Komplexes und des lokalbikompakten Raumes. Math. Sb., **1**, 701–706 (1936). (Art. n° 30 in [Kol85].)

[K4] Kolmogorov, A.N.: Les groupes de Betti des espaces localement bicompacts. C.R. Acad. Sci. Paris, **202**, 1144–1147 (1936). (Art. n° 32 in [Kol85].)

[K5] Kolmogorov, A.N.: Propriétés des groupes de Betti des espaces localement bicompacts. C.R. Acad. Sci. Paris, **202**, 1325–1327 (1936). (Art. n° 33 in [Kol85].)

[K6] Kolmogorov, A.N.: Les groupes de Betti des espaces métriques. C.R. Acad. Sci. Paris, **202**, 1558–1560 (1936). (Art. n° 34 in [Kol85].)

[K7] Kolmogorov, A.N.: Cycles relatifs. Théorème de dualité de M. Alexander. C.R. Acad. Sci. Paris, **202**, 1641–1643 (1936). (Art. n° 35 in [Kol85].)

[K8] Kolmogorov, A.N.: Über offene Abblidungen. Ann. Math., **38**, 36–38 (1937). (Art. n° 36 in [Kol85].)

[K9] Kolmogorov, A.N.: Skew-symmetric forms and topological invariants (Russian). Proc. Seminar on Vector and Tensor Analysis and Applications in Geometry, Mechanics and Physics, Moscow, Leningrad: GONTI, vol. 1, 345–347 (1937). (Art. n° 37 in [Kol85].)

[K10] Kolmogorov, A.N., Gel'fand, I.M.: On rings of continuous functions on topological spaces (in Russian). Dokl. Akad. Nauk SSSR **22**:1, 11–15 (1939). (Art. n° 41 in [Kol85].)