VLADIMIR ABRAMOVICH ROKHLIN AND ALGEBRAIC TOPOLOGY

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Abstract. The article considers the scientific heritage of V. A. Rokhlin in algebraic topology from the point of view of the modern development of mathematics and shows the influence of its results on the development of algebraic topology up to the present. The second part of the article contains new results with fairly detailed sketches of their proofs. There we introduce the notion of partially framed manifolds, which naturally arise in the study of the characteristic classes of vector bundles over the loop space $\Omega SU(2) = \Omega SP(1)$. We obtain theorems on the divisibility of the signature of such manifolds as a result of calculations of characteristic classes with values in complex and quaternionic cobordism.

1. Introduction.

This article is based on the author's talk at the conference "Topology, Geometry, and Dynamics" (August 19–23, 2019, Euler Mathematical Institute, Saint Petersburg), dedicated to the Centennial of Vladimir Abramovich Rokhlin, website: http://www.pdmi.ras.ru/EIMI/2019/tgd/index.html

The book [88] incorporates 12 papers on algebraic topology written by V. A. Rokhlin. Add to them the survey [84], one of the first in the world surveys on bordism groups, and the paper [66], where joint results of S.P. Novikov and V.A. Rokhlin are presented.

We also mention the extensive Rokhlin's survey "Homotopy Groups" (see [78]), devoted to a topic that subsequently became one of the central topics of algebraic topology. This survey was published in the journal "Uspekhi Matematicheskikh Nauk" in 1946 with a note: "The article is printed in the form in which it was submitted by the author in 1940". Thus, this survey was submitted to publication by a 19-year-old author! Later (1950), his survey [79] was also devoted to the same topic.

The subject of the survey article [84] are "internal (intrinsic) homology groups" (see [81]), which V. A. Rokhlin studied following the approach of L. S. Pontryagin. In foreign literature, these groups were called "bordism groups" and "cobordism groups" (from English "boundary" and French "bord"), which became widely used after the work of R. Thom [95].

Topics of Rokhlin's papers include:

- The mappings of the $(n+3)$-dimensional sphere into the n-dimensional sphere.
- The 2-torsion in bordism groups of oriented manifolds.
- The embeddings of 3-dimensional manifolds in $\mathbb{R}^5$.
- The necessary and sufficient condition for an orientable closed manifold M^4 to be the boundary of an oriented manifold W^5.
- The formula $3\tau(M^4) = (p_1(M^4), \langle M^4 \rangle)$ connecting the signature $\tau(M^4)$ of a manifold M^4 with its first Pontryagin class $p_1(M^4)$.
- The divisibility by 16 of the signature for any closed almost parallelizable manifold M^4.
- The extrinsic and intrinsic definitions of the characteristic Pontryagin cycles and the characteristic Pontryagin classes.
- The combinatorial invariance of the rational Pontryagin classes.
- Problems of homotopical and topological invariance of the rational Pontryagin classes.
- The integer Pontryagin classes and smoothing problems for combinatorial manifolds.
- The additivity of the signature with respect to the connected sum of manifolds M_1^{4k} and M_2^{4k} along a common connected component of their boundaries.

- Problems of realization of 2-dimensional cycles in 4-dimensional manifolds.

All these papers were published in the period of explosive development of mathematics (1951 – 1971). During these years, many fundamental problems of algebraic topology were solved. It became possible due to the discovery of implicit connections of algebraic topology with real and complex algebraic geometry, functional analysis, the theory of differential equations, commutative and homological algebra. Moreover, these connections opened new areas of research in mathematics and physics. Rokhlin's papers contributed significantly to the success of algebraic topology (see [67] – [70], [98]).

Many papers of V.A. Rokhlin were devoted to the theory of 4-dimensional manifolds. Over the years, it became clear that this theory is fundamentally different from the theory of manifolds of other dimensions. The role of V. A. Rokhlin's results in creating the rich and "wild world" of 4-dimensional manifolds is amply presented in [90].

Monographs by P. E. Conner, E. E. Floyd [22], [23] and R. E. Stong [94] played an important role in developing the theory of cobordism as a new direction of algebraic topology. These monographs referenced the survey by V.A. Rokhlin (see above), and quite fully reflected his contribution to the theory of cobordisms.

This article is focused on the role of V. A. Rokhlin's results in developing algebraic topology right up to the present moment.

In his works, V. A. Rokhlin used geometry of bordism groups and the methods of homology theories. Starting from the paper of S. P. Novikov [65], cohomological operations and formal group laws in the cobordism theory began to be successfully applied to problems of algebraic topology. The cobordism theories, and especially the complex cobordism theory, opened up new approaches to well-known problems and led to the formulation of new problems.

Our article is naturally divided into two parts.

The first part (Sections 2–5) is devoted to the results of V. A. Rokhlin and their influence on the development of algebraic topology up to the present time. In this part, we discuss problems whose solutions use and develop the results of V. A. Rokhlin. For example, V. A. Rokhlin's theorem (1952) stating that the signature of any almost parallelizable 4-dimensional manifold is divisible by 16, turned out to be one of his most famous and fruitful results in the algebraic topology. In 1958, at the International Mathematical Congress in Edinburgh, M. Kervaire and J. Milnor give a talk "Bernoulli numbers, homotopy groups and a theorem of Rokhlin", see [47]. This talk was focused on their result on the divisibility of the signature of any almost parallelizable $(8k + 4)$-dimensional manifold by 16 and on applications of this result to the problem of the homotopy groups of spheres.

The second part of this article (Sections 6–9) is devoted to characteristic classes and cohomological operations in cobordisms, as well as transformations of cobordism theories. There we discuss problems whose statements are close to the statements of problems from the first part of our article. For example, we introduce the notion of partially framed manifolds. This notion naturally arises in the study of characteristic classes for vector bundles over the loop space $\Omega SU(2) = \Omega SP(1)$. Using the Chern classes in complex cobordism and the Pontryagin–Borel classes in symplectic (quaternionic) cobordism, we obtain theorems on the divisibility of the signature of manifolds with the corresponding partial framing.

The first part of the article is intended for a wide readership, and therefore includes definitions and basic concepts. The second part of the article contains new results with fairly detailed sketches of their proofs.

2. Bordism groups.

An n-dimensional cycle in a space X can be defined as the image of a map $f\colon M^n \to X$, where M^n is a smooth closed n-dimensional manifold. This notion goes back to H. Poincaré (1895), see [74]. It led to homology theories (bordism theories) and dual cohomology theories (cobordism theories), whose constructions essentially use the fundamental notion of a transversally regular mapping $f\colon M_1 \to M_2$ along V, where M_1, M_2 are manifolds and V is a submanifold of M_2.

L. S. Pontryagin proposed a construction that allowed to exploit methods of the smooth manifold theory to calculate stable homotopy groups of spheres using transversely regular mappings. In this approach, he applied manifolds with stably trivial tangent bundles.

R. Thom associated with each vector bundle the so-called Thom space. He showed that the Pontryagin construction can be generalized to the case of manifolds with any *fixed* stable structure on the tangent bundle and, as a result, he reduced the calculation of bordism groups to the calculation of stable homotopy groups of Thom spaces.

An n-dimensional cycle is homologous to zero if the map f can be extended to a map $F\colon W^{n+1} \to X$, where W^{n+1} is a smooth $(n+1)$-dimensional manifold with boundary $\partial W^{n+1} = M^n$. Let (M_1^n, f_1) and (M_2^n, f_2) be two cycles. Set

$$(M_1^n, f_1) \oplus (M_2^n, f_2) = (M_1^n \sqcup M_2^n, f_1 \sqcup f_2),$$

where "$\sqcup$" denotes the disjoint union. We say that $(M_1^n, f_1) \sim (M_2^n, f_2)$ if the map $f = f_1 \sqcup f_2\colon M_1^n \sqcup M_2^n \to X$ of the disjoint union of manifolds is homologous to zero.

2.1. Pontryagin's construction and Pontryagin's Theorem.

A framing of a manifold is a choice of trivialization of its stable tangent bundle.

Theorem 1 (L. S. Pontryagin, [75])**.** *The framed bordism group Ω_n^{fr} of n-dimensional manifolds with a framing of the stable normal bundle (or equivalently of the stable tangent bundle) is isomorphic to the n-th stable homotopy group of spheres π_n^s.*

The proof of this theorem uses the following construction:

Let M^n be a smooth framed manifold. Consider its smooth embedding in Euclidean space R^{n+k}, $k > n$, and fix the corresponding trivialization of the normal bundle $\nu(M^n)$. We consider spheres S^{n+k} and S^k with base points as one-point compactifications of the spaces $\mathbb{R}^{n+k}$ and $\mathbb{R}^k$. The Pontryagin construction associates with a framed manifold M^n a map of spheres $S^{n+k} \to S^k$, which takes the complement to the space $\nu(M^n)$ to the base point of S^k and takes the space $\nu(M^n)$ to $\mathbb{R}^k$ according to the trivialization of the normal bundle $\nu(M^n)$.

Note that the pre-image of $0 \in \mathbb{R}^k$ under this map is the manifold M^n. On the other hand, if we have a smooth map $f\colon S^{n+k} \to S^k$ that is regular at $0 \in \mathbb{R}^k$, then $f^{-1}(0)$ is a smooth manifold M^n with a framing defined by a choice of basis of $\mathbb{R}^k$ using the theorem on regular values of a smooth map.

2.2. Thom's construction and Thom's theorem.

From now on, by a bundle we will mean a locally trivial bundle over a CW-complex, and by a k-dimensional vector G-bundle we will mean a bundle $E \to X$ with fiber $\mathbb{R}^k$ and with a representation $\varrho\colon G \to O(k)$ of the structure group G of the bundle E.

One of the first results of the theory of vector bundles is that there always exists a scalar product in a fiber over a point x that continuously depends on x. Thus, with each bundle $E \to X$ one can associate bundles $D(E) \to X$ and $S(E) \to X$ whose fibers are a disk D^k and a sphere $S^{k-1} \subset D^k$ of radius 1,respectively. The space $D(E)$ has the structure of a CW-complex, with respect to which the space $S(E)$ is a CW-subcomplex, and therefore we obtain a pair of CW-complexes $(D(E), S(E))$. The Thom space $Th(E)$ of a vector bundle $E \to X$ is the CW-complex $D(E)/S(E)$ with the canonical base point to which $S(E)$ is mapped under the canonical projection $D(E) \to Th(E)$. If X is a point, then $E = \mathbb{R}^k$, and the Thom space $Th(\mathbb{R}^k)$ is denoted by $Th(k)$. By the construction, $Th(k)$ is a sphere S^k with a base point. Therefore, for any point $x \in X$, there is an embedding $S^k \subset Th(E)$. Let $X = M$ be a closed smooth n-dimensional manifold. Then the complement to the base point in $Th(E)$ is a smooth manifold of dimension $n + k$. The zero section of the bundle $E \to M$ defines an embedding $M \subset Th(E)$ whose normal bundle is identified with $E \to M$.

The homotopy realization of bordism groups, obtained by Thom in [95], is a consequence of the following general result. For each homotopy class of maps $f\colon S^N \to Th(E)$, the following holds:

1. There exists a smooth map $\varphi\colon S^N \to Th(E)$ transversely regular along M.

2. The set $V = \varphi^{-1}(M)$ is a smooth $(N-k)$-dimensional submanifold in $\mathbb{R}^N$ whose normal bundle $\nu \to V$ is a k-dimensional vector G-bundle. The mapping φ is the composition $S^N \to Th(\varphi^*\nu) \to Th(E)$.
3. If φ_1 and φ_2 are two smooth homotopic mappings $S^N \to Th(E)$ transversely regular along M, then in the bordism group of G-manifolds (i.e., manifolds whose stable normal bundle has the structure group G), the bordism classes $[V_1]$ and $[V_2]$ coincide. Here $V_i = \varphi_i^{-1}(M)$, $i = 1, 2$.

Let us consider the classifying space $B(G; k)$ of k-dimensional vector G-bundles. A representation $\varrho\colon G \to O(k)$ defines a map $B\varrho\colon B(G; k) \to BO(k)$. In the case when G is a compact Lie group, the space $B(G; k)$ is realized as a direct limit $\lim\limits_{m\to\infty} M(G; m, k)$, where $M(G; m, k)$ are smooth compact manifolds with k-dimensional vector G-bundles. We denote by $Th(G; m, k)$ the Thom spaces of these bundles and by Ω_n^G the bordism group of smooth n-dimensional closed manifolds whose stable tangent bundle has the structure of a vector G-bundle.

Theorem 2 (R. Thom, [95])**.** *There exists an isomorphism* $\Omega_n^G \cong \pi_{n+k}(Th(G; n+k, k))$, *where* $k > n$.

To study the entire set of groups Ω_n^G, $n = 0, 1, \ldots$, it is necessary to consider the structure groups $G = \lim\limits_{q\to\infty} G_q$, where $e = G_0 \subset G_1 \subset \ldots$ is a series of groups.

In the case when G is the identity group e, the manifold $M(e; m, k) = S^k$ does not depend on m, and Theorem 2 gives Theorem 1.

In the case $G = O = \lim\limits_{k\to\infty} O(k)$, the manifold $M(O(k); m, k)$ is the Grassmann manifold $Gr(m, k)$ of k-dimensional planes in $\mathbb{R}^m$.

In [5], M. F. Atiyah proposed a general construction of stable Thom spectra corresponding to the classical series of compact Lie groups and defined theories of bordism $G_*(X)$ and cobordism $G^*(X)$ by methods of homotopy topology (for details see, for example, [94]).

In Section 6, we will give the explicit construction of the Thom spectrum for a series of unitary groups $e = U(0) \subset U(1) \subset \ldots \subset U(q) \subset \ldots$

2.3. Unoriented bordism $O_*(X)$, the groups $\varOmega_n^O$ and the ring $\varOmega_*^O$.

The bordism group of smooth n-dimensional manifolds is denoted by $O_n(X)$. The group $O_*(X) = \sum\limits_{n\geqslant 0} O_n(X)$ corresponds to the series of groups $e = O(0) \subset O(1) \subset \ldots \subset O(q) \subset \ldots$ The operation $[M_1 \times M_2] = [M_1][M_2]$ is well defined and turns the group $\varOmega_*^O = O_*(\mathrm{pt})$ into a commutative graded ring $\varOmega^O = \varOmega_*^O = \sum\limits_{n\geqslant 0} \varOmega_n^O$.

Theorem 3 (L. S. Pontryagin, R. Thom)**.**

(a) *Two manifolds are unoriented bordant if and only if they have identical sets of Stiefel–Whitney characteristic numbers.*
(b) $\varOmega^O$ *is a polynomial ring over* $\mathbb{Z}_2$ *with one generator* a_i *in every positive dimension* $i \neq 2^k - 1$.
(c) *For every CW-complex* X *the group* $O_*(X)$ *is a free graded* $\varOmega^O$*-module isomorphic to* $H_*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \varOmega^O = H_*(X; \varOmega^O)$.

Corollary 4. *Let* M_1 *and* M_2 *be homotopy equivalent* n*-dimensional manifolds. Then their unoriented bordism classes coincide.*

2.4. Bordism groups of manifolds with additional structures.

Additional structures on manifolds lead naturally to the bordism groups $G_*(X) = \sum_{n\geqslant 0} G_n(X)$, where G is a series of structure groups of the stable tangent bundle of M. The classical series of groups give the following groups $G = \lim\limits_{q\to\infty} G_q$:

$$\text{unit } e, \quad O, \quad SO, \quad Spin, \quad U, \quad SU, \quad Sp.$$

Set $G_n(\mathrm{pt}) = \Omega_n^G$, $G_*(\mathrm{pt}) = \Omega_*^G$ and $\Omega_n^e = \Omega_n^{fr}$. Operation $[M_1 \times M_2] = [M_1][M_2]$ turns the groups Ω_*^G into commutative rings. Using the canonical embeddings, we obtain

$$SO(n) \subset O(n) \Longrightarrow SO_*(X) \longrightarrow O_*(X); \qquad U(n) \subset SO(2n) \Longrightarrow U_*(X) \longrightarrow SO_*(X);$$

$$SU(n) \subset U(n) \Longrightarrow SU_*(X) \longrightarrow U_*(X); \qquad Sp(n) \subset U(2n) \Longrightarrow Sp_*(X) \longrightarrow U_*(X).$$

The development of the bordism theory led to the consideration of additional structures on manifolds that are defined by the conditions on the characteristic classes of tangent bundles (see the groups $W_n(2,\mathbb{R})$ and $W_n(2,\mathbb{C})$ below) or by operators on these bundles (see the groups Ω^{SC} below).

Numerous results on the images of homomorphisms between the scalar rings for different bordism theories are presented in chapter IV of monograph [94].

2.5. Oriented bordism $SO_*(X)$, the groups Ω_n^{SO} and the ring Ω_*^{SO}.

The simplest additional structure is an orientation. Two oriented closed n-dim manifolds M_1 and M_2 are *oriented bordant* if there is an oriented $(n+1)$-dim manifold W with boundary such that $\partial W = M_1 \sqcup \overline{M}_2$, where $\overline{M}_2$ denotes M_2 with the orientation reversed.

Let $I = [0,1]$. Given an oriented manifold M, the manifold $M \times I$ has the canonical orientation such that $\partial(M \times I) = M \sqcup \overline{M}$. Hence, $-[M] = [\overline{M}]$ in Ω_n^{SO}.

Theorem 5 (L. S. Pontryagin, R. Thom, J. Milnor, V. A. Rokhlin, B. G. Averbukh, C. T. C. Wall)**.**

(a) $\Omega^{SO} \otimes \mathbb{Q}$ *is a polynomial ring over* $\mathbb{Q}$ *generated by the bordism classes of complex projective spaces* $\mathbb{C}P^{2i}$, $i \geqslant 1$.
(b) *The subring* Tors $\subset \Omega^{SO}$ *of torsion elements contains only elements of order* 2. *The quotient* $\Omega^{SO}/\mathrm{Tors}$ *is a polynomial ring over* $\mathbb{Z}$ *with one generator* a_i *in every dimension* $4i$, *where* $i \geqslant 1$.
(c) *Two oriented manifolds are bordant if and only if they have identical sets of Pontryagin and Stiefel–Whitney characteristic numbers.*

In [81], [83], V.A. Rokhlin proved the exactness of the sequence

$$\Omega_n^{SO} \xrightarrow{\times 2} \Omega_n^{SO} \longrightarrow \Omega_n^{O}. \tag{1}$$

Analyzing this result, C.T.C. Wall introduced the bordism groups $W_n(2;\mathbb{R})$ for the smooth manifolds M^n with the condition $w_1^2 = 0$, where $w_1 = w_1(\mathcal{T}M^n)$ is the first Stiefel-Whitney class of the tangent bundle of M^n (see [99]).

As mentioned above, in [5] (1961), M. Atiyah proposed a construction of bordism theories based on the spectra of Thom spaces. Using stable homotopy theory techniques he introduced the dual cohomology theories, which he called cobordism. As an application of the results of [5] on the cobordism theories $O^*(X)$ and $SO^*(X)$, it was proved that the Rokhlin exact sequence can be interpreted as the exact $SO^*(X)$-sequence of the triple $(\mathbb{R}P^2, \mathbb{R}P^1, \mathbb{R}P^0)$, where $\mathbb{R}P^k$ is the real k-dimensional projective space.

P. E. Conner and E. E. Floyd (1964–1966) obtained analogues of the results of Rokhlin, Wall, and Atiyah in case of the forgetful homomorphism $SU_*(X) \longrightarrow U_*(X)$. To do this, they introduced the bordism groups $W_n(2;\mathbb{C})$ for smooth manifolds M^n with the condition $c_1^2 = 0$, where $c_1 = c_1(\tau(M^n))$ is the first Chern class of the stable complex tangent bundle of M^n. They obtained, using the cobordism theories $U^*(X)$ and $SU^*(X)$, the result in the form of the corresponding exact $SU^*(X)$-sequence of the triple $(\mathbb{C}P^2, \mathbb{C}P^1, \mathbb{C}P^0)$, where $\mathbb{C}P^k$ is the complex k-dimensional projective space. See [54] for a modern survey of the results in this direction.

Let (X,Y) be a pair of CW-complexes. The oriented bordism group $SO_n(X,Y)$ is constructed using the SO-bordism classes of maps $f\colon (M^n, \partial M^n) \to (X,Y)$, where M^n is an oriented n-dimensional manifold with boundary ∂M^n. Such a map f defines a homomorphism $f_*\colon H_n(M^n, \partial M^n;\mathbb{Z}) \to H_n(X,Y;\mathbb{Z})$. Therefore, the homomorphism of realization of cycles

$$\mu_{SO}^H\colon SO_n(X,Y) \to H_n(X,Y;\mathbb{Z}), \quad \mu_{SO}^H([M^n, \partial M^n; f]) = f_*([M^n, \partial M^n])$$

is defined, where $[M^n, \partial M^n]$ is the fundamental cycle generating the group $H_n(M^n, \partial M^n; \mathbb{Z}) = \mathbb{Z}$.

Let $\mathcal{C}$ be the class (in the sense of Serre) of finite groups of odd order.

Theorem 6 (P. E. Conner, E. E. Floyd, [22]).
For any pair (X, Y) of CW-complexes,

1. *The homomorphism μ^H_{SO} defines a $\mod \mathcal{C}$-isomorphism*
$$SO_n(X, Y) \cong \sum_{p+q=n} H_n(X, Y; \Omega^{SO}_q).$$
2. *There is the followng SO-analogue of the Rokhlin exact sequence:*
$$SO_n(X, Y) \xrightarrow{\times 2} SO_n(X, Y) \to O_n(X, Y).$$

Corollary 7. *If the group $H_*(X, Y; \mathbb{Z})$ is torsion-free, then the homomorphism μ^H_{SO} defines an isomorphism*
$$SO_n(X, Y) \cong \sum_{p+q=n} H_n(X, Y; \Omega^{SO}_q).$$

3. The signature and its applications.

Let M^n be an oriented manifold, possibly with a nonempty boundary ∂M^n. The signature $\tau(M^{4k})$ is defined as the signature of the quadratic form $(x^2, [M^{4k}])$, where $x \in H^{2k}(M^{4k}; \mathbb{R})$ and $[M^{4k}] \in H_{4k}(M^{4k}, \partial M^{4k}; \mathbb{R})$ is the fundamental cycle. This form is non-degenerate only for closed manifolds. The signature $\tau(M^n)$ equals to zero if $n \not\equiv 0 \mod 4$. The signature is an invariant of homotopy equivalence of oriented closed manifolds that preserves orientation.

3.1. The properties of the signature.

- If $M^n = \partial W$, where W is closed and oriented, then $\tau(M^n) = 0$.
- If $M^n = M^n_1 \sqcup M^n_2$ then $\tau(M^n) = \tau(M^n_1) + \tau(M^n_2)$.
- $\tau(\overline{M}^n) = -\tau(M^n)$.
- $\tau(M^n_1 \times M^n_2) = \tau(M^n_1)\tau(M^n_2)$.
- $\tau(\mathbb{C}P^{2k}) = 1$.

Therefore, we have a *ring homomorphism* $\tau\colon \Omega^{SO}_* \longrightarrow \mathbb{Z}$. Using assertion (a) of Theorem 5, we obtain that the condition $\tau(\mathbb{C}P^{2k}) = 1$ determines the ring homomorphism τ uniquely.

Theorem 8 (V. A. Rokhlin, R. Thom, 1952–1953).
For any 4-dimensional oriented manifold, there is the formula $3\tau(M^4) = (p_1(M^4), \langle M^4 \rangle)$, where $p_1(M^4)$ is the first Pontryagin class of the tangent bundle $\mathcal{T}M^4$.

A generalization of this formula to all oriented $4n$-dimensional manifolds played a fundamental role in the theory of rational Pontryagin classes, see below.

Theorem 9 (S. P. Novikov, V. A. Rokhlin, 1966, see [66]).
Let M^{4k}_1, M^{4k}_2 be oriented manifolds with boundaries $\partial M^{4k}_1, \partial M^{4k}_2$ and let V^{4k-1} be a closed manifold that is a connected component of both ∂M^{4k}_1 and ∂M^{4k}_1. Then $\tau(M_1 \bigcup_V M_2) = \tau(M_1) + \tau(M_2)$.

In the monograph [68] (2002), S.P. Novikov noted that this additivity property corresponds to the axioms of a *topological quantum field theory* (see [8], [89], [91], [101]), and that in the classical topology only the signature $\tau(M^{4k})$ and the Euler characteristic $\chi(M^{2k})$ have this property.

Excerpt from the paper of S. P. Novikov: "Rokhlin in 1965 drew my attention repeatedly to the fact that for prime p (large enough for a given dimension), the definition of combinatorial Pontryagin–Hirzebruch classes $\mod p$ is unknown, and this issue is not trivial. I thought about it and found an interesting "additive" property of the signature of manifolds with boundary by gluing along connected boundary components. From this we with Rokhlin extracted right definition of the classes mod p. This additivity has been used by Yanih and others after the Moscow Congress, where I talked about it much, for the axiomatization of the signature. We now know that this property of the signature is equivalent in modern terminology to building an "abelian" nontrivial topological quantum field theory."

These results of S.P. Novikov and V.A. Rokhlin led V.M. Buchstaber to the formulation of the following problem: *Assume given two smooth manifolds $M_1^{n_1}$ and $M_2^{n_2}$ with a fixed structure on their tangent bundles. Suppose that the boundaries $\partial M_1^{n_1}$ and $\partial M_2^{n_2}$ are disconnected unions of p copies of closed manifolds $M_1^{n_1-1}$ and $M_2^{n_2-1}$, respectively. Give a construction that turns the direct product $M_1^{n_1} \times M_2^{n_2}$ into a smooth manifold $W^{n_1+n_2}$ with the corresponding structure on the tangent bundle and with a similar boundary, i.e. $\partial W^{n_1+n_2}$ is a disconnected union of p copies of a closed manifold $W^{n_1+n_2-1}$.*

In paper [60], O. K. Mironov described an obstruction to the surgery of the direct product $M_1^{n_1} \times M_2^{n_2}$ to the required manifold $W^{n_1+n_2}$ and showed that in some important cases this obstruction is trivial.

3.2. **Hirzebruch characteristic classes.**

A characteristic class $\pi(\xi) \in H^*(X;\mathbb{Q})$ of a real vector bundle $\xi \to X$ is called multiplicative if $\pi(1) = 1$ and $\pi(\xi_1 \oplus \xi_2) = \pi(\xi_1)\pi(\xi_2)$. An example of such a class is the full Pontryagin class

$$\mathcal{P}(\xi) = \sum_{k\geqslant 0} p_k(\xi) = \sum_{k\geqslant 0} (-1)^k c_{2k}(c\xi).$$

According to the splitting principle, any multiplicative characteristic class $\pi(\xi)$ is completely determined by its value on the universal two-dimensional oriented bundle $\xi(2) \to BSO(2)$, which can be identified with the bundle $r\eta$, where $\eta \to \mathbb{C}P^\infty = BU(1)$ is the universal one-dimensional complex bundle. We have

$$\pi(\xi(2)) = 1 + \sum \alpha_i x^i \in H^*(BSO(2);\mathbb{Q}),$$

where $x = p_1(\xi(2)) = -c_2(c\xi(2)) = -c_2(\eta + \bar\eta) = t^2$ and $t = c_1(\eta) \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$. Moreover, each series $\alpha(x) = 1 + \sum \alpha_i x^i$ defines a multiplicative characteristic class $\pi(\xi) = \pi(\xi;\alpha)$.

For any oriented vector bundle $\xi \to X$ we get

$$\pi(\xi;\alpha) = 1 + \sum_{n>0} L_n(\xi;\alpha), \qquad L_n(\xi;\alpha) = \sum_{|\omega|=n} \alpha_\omega p_\omega(\xi),$$

where $L_n(\xi;\alpha)$ are called characteristic Hirzebruch classes. Here $\omega = (i_1,\dots,i_k) \in \mathbb{Z}^k$, $|\omega| = i_1 + \ldots + i_k$, $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_k$, $\alpha_\omega = \alpha_{i_1}\ldots\alpha_{i_k}$ and $p_\omega(\xi) \in H^{4|\omega|}(X;\mathbb{Q})$ is a polynomial in the Pontryagin classes $p_i(\xi)$, $i \in N$, corresponding to the symmetrization of the monomial $x_1^{i_1}\ldots x_k^{i_k}$.

The sequence ω, $|\omega| = n$, is considered as a partition of the number n, i. e. sequences ω_1 and ω_2 are identified if they differ only by sequences of zeros at their ends. The maximal number k such that $i_k > 0$ is called the length of the partition ω. For the partition ω of length n with $i_1 = 1$ we obtain the Pontryagin class $p_n(\xi)$. For the partition ω of length 1 with $|\omega| = n$ we obtain the class $p_{(n)}(\xi)$ corresponding to the Newton polynomial $\sum x_i^n$. In the expansion of the characteristic class $L_n(\xi;\alpha)$, we use the lexicographic ordering of the partitions ω, $|\omega| = n$, with respect to which the class $p_n(\xi)$ is minimal and the class $p_{(n)}(\xi)$ is maximal.

Let us denote by $L_n^{\mathcal{T}}(M;\alpha)$ and $L_n^{\nu}(M;\alpha)$, $n \in N$, the characteristic classes of the tangent $\mathcal{T}(M)$ and normal $\nu(M)$ bundles of an oriented smooth manifold M. Since the bundle $\mathcal{T}(M) \oplus \nu(M)$ is trivial, we get $\pi(\mathcal{T}(M);\alpha)\pi(\nu(M);\alpha) = 1$, i.e. $\pi(\nu(M);\alpha) = \pi(\mathcal{T}(M);\beta)$, where $\alpha(x)\beta(x) = 1$. Therefore, it is sufficient to use only the classes $L_n^{\mathcal{T}}(M;\alpha)$, which are called the characteristic Hirzebruch classes of the manifold M.

In the case $\dim M = 4n$, we obtain the characteristic Hirzebruch number $L(M;\alpha) = \big(L_n^{\mathcal{T}}(M;\alpha), \langle M\rangle\big)$, where $\langle M\rangle$ is the fundamental cycle. This number is called the Hirzebruch genus (corresponding to the series α) of the manifold M.

Theorem 10 (F. Hirzebruch, 1956)**.**

(a) *Each series* $\alpha(x) = 1 + \ldots$ *defines a ring homomorphism*

$$L(\alpha)\colon \Omega_{SO} \to \mathbb{Q} \ :\ L(\alpha)[M^{4n}] = L(M^{4n};\alpha),\ L(\alpha)[M^k] = 0,\ k \neq 4n,$$

called the Hirzebruch genus corresponding to the series α*.*

(b) *Each ring homomorphism* $\Omega_{SO} \to \mathbb{Q}$ *defines a series* $\alpha(x) = 1 + \ldots$

(c) *The signature of oriented manifolds defines a ring homomorphism* $\tau\colon \Omega_{SO} \to \mathbb{Z}$*, which corresponds to the series* $\alpha(x) = \frac{\sqrt{x}}{\tanh\sqrt{x}}$*, where* tanh *is the hyperbolic tangent. That is, the following formula holds*

$$L\left(M^{4n}; \frac{\sqrt{x}}{\tanh\sqrt{x}}\right) = \tau(M^{4n}).$$

This theorem is one of the most famous theorems of modern mathematics.

The formula

$$\alpha(x) = \frac{\sqrt{x}}{\tanh\sqrt{x}} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + \ldots + (-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k x^k + \ldots$$

where B_k is the k-th Bernoulli number, gives the generating series $\mathcal{L}(\xi) = 1 + \sum\limits_{n>0} \mathcal{L}_n(\xi)$ of characteristic classes. Following the paper [85] of V. A. Rokhlin, we call the classes $\mathcal{L}_n(\xi)$ the Pontryagin–Hirzebruch classes. We have: $\mathcal{L}_1 = \frac{1}{3}p_1,\ \mathcal{L}_2 = \frac{1}{45}(5p_{(1,1)} - p_{(2)}) = \frac{1}{45}(7p_2 - p_1^2)$. Therefore, the following formulas hold

$(p_1(M^4), \langle M^4\rangle) = 3\tau(M^4)$ (Rokhlin–Thom); $(7p_2(M^8) - p_1^2(M^8), \langle M^8\rangle) = 45\tau(M^8)$ (Hirzebruch).

3.3. Pontryagin classes of PL-manifolds.

The result that the signature of an oriented $4n$-dimensional manifold can be expressed as a polynomial in Pontryagin classes follows from Thom's calculation of the ring $\Omega_{SO}\otimes\mathbb{Q}$. According to Hirzebruch's Theorem 10, this polynomial is the Pontryagin–Hirzebruch class $\mathcal{L}_n$.

Theorem 11 (V. A. Rokhlin, A. S. Schwarz [82] and R. Thom [96])**.**

(a) *There are defined characteristic classes for oriented piecewise linear manifolds (PL-manifolds), which for smooth oriented manifolds coincide with the Pontryagin–Hirzebruch classes* $\mathcal{L}_n$*,* $n \in N$*.*

(b) *The characteristic classes* $\mathcal{L}_n$ *of smooth manifolds* M*, and hence the rational characteristic Pontryagin classes* $p_n(M)$*,* $n \in N$*, are combinatorially invariant.*

In the famous work [57], J. Milnor proved the existence of 28 different (only 27 of them are 'non-standard') smooth structures on a 7-dimensional sphere. This proof uses the result of R. Thom (see [95]) that every closed smooth 7-dimensional manifold is the boundary of a smooth 8-dimensional manifold. As a result, J. Milnor obtained closed 8-dimensional PL-manifolds that, as shown in [82], do not admit smooth structures.

Theorem 11 leads naturally to a problem: *Find a formula for calculating the rational Pontryagin classes of a* PL*-manifold from a triangulation of this manifold.*

Many publications was devoted to this problem, starting with the papers of A. M. Gabriélov, I. M. Gel'fand and M. V. Losik [28]–[30]. To date, remarkable results in this direction have been obtained by A. A. Gaifullin, see [31], [32] and his subsequent works, including [34]. Recently, based on the formula of A. A. Gaifullin for the first combinatorial Pontryagin class, his student D. Gorodkov showed that the 8-dimensional simplicial manifold with 15 vertices constructed by U. Brehm and W. Kühnel [11] is homeomorphic to the quaternionic plane. Thus, he completed the solution of the long-standing problem of minimal triangulation of the quaternionic plane.

3.4. **Pontryagin classes and the fundamental group.**

The Hirzebruch formula shows that the class $\mathcal{L}_n(M^{4n})$ is a homotopy invariant of oriented manifolds M^{4n}. Until 1964, no other rational homotopy invariant characteristic classes of manifolds were known. Moreover, according to the Browder–Novikov theorem, there are no other homotopy invariant classes in the case of simply connected manifolds.

In 1965, S. P. Novikov published the papers [62] and [63] in which he proved that the characteristic class $\mathcal{L}_n(M^{4n+1})$ is also homotopy invariant. In 1966, V. A. Rokhlin proved in [85] that for classes $z \in H_{4n}(M^{4n+2}, \mathbb{Q})$, the scalar product $(\mathcal{L}_n(M^{4n+2}), z)$ is homotopy invariant if $Dz = y_1 y_2$, where D denotes the Poincare duality operator. In this paper, V. A. Rokhlin thanks S. P. Novikov for sending him the paper [63] which was not yet published at that time. The fact is that in papers [62] and [63] S.P. Novikov introduced the non-simply connected signature formula for classes $z \in H_{4n}(M^{4n+k}, \mathbb{Q})$, where $Dz = y_1 \cdots y_k$. This formula formed the basis of the famous conjecture of S. P. Novikov on the homotopy invariance of "higher signatures" for non-simply connected manifolds. At the same time, it played a key role in proving the topological invariance of the rational Pontryagin classes, which S. P. Novikov obtained in [64]. In this paper, he thanks V. A. Rokhlin "for numerous useful discussions and advices".

3.5. **Signature of manifolds with additional structure.**

Following [52], we give results on the divisibility of the signature of $4n$-dimensional oriented manifolds with an additional structure. These results give obstruction to the existence in the bordism class $[M] \in \Omega^{SO}_{4n}$ of a representative with this additional structure. A famous example of such a result for $n = 1$ is given by Theorem 16 (see below).

Theorem 12 (S. D. Ochanine, [71])**.** *The signature of an SU-manifold M^{4n} is divisible by* 16 *if n is odd, and is divisible by* 2 *if n is even.*

In [93], R. E. Stong proved that the homomorphism $\Omega^{SU}_{8k+4} \to \Omega^{Spin}_{8k+4}/Tors$ is surjective.

Corollary 13. *The signature of a Spin-manifold M^{8k+4} is divisible by* 16*.*

A self-adjoint complex bundle (in short, an SC-bundle) over X is a pair (η, T), where $\eta \to X$ is a complex bundle and $T\colon \eta \to \eta$ is an antilinear isomorphism. Let (η_1, T_1) and (η_2, T_2) be two SC-bundles. Then their sum $(\eta_1 \oplus \eta_2, T_1 \oplus T_2)$ and tensor product $(\eta_1 \otimes \eta_2, T_1 \otimes T_2)$ are defined. Let (η, T) be a self-adjoint complex bundle. If $T^2 = 1$, then the bundle η is isomorphic to the bundle $\zeta \otimes \mathbb{C}$, where ζ is the real subbundle of η consisting of the fixed points of T. A self-adjoint complex bundle (η, T) with $T^2 = 1$ is called a CO-bundle. If $T^2 = -1$, then the bundle η has the structure of a symplectic (quaternionic) bundle (in short, an Sp-bundle) in which T plays the role of the operator of multiplication by the quaternionic unit j.

In [16] (see also [17]), V. M. Buchstaber constructed characteristic classes of SC- and CO-bundles with values in the cobordism theories $SC^*(\cdot)$ and $CO^*(\cdot)$. These classes satisfy the identities

$$\mu^{SC}_{Sp} p^{Sp}_k(\zeta) = p^{SC}_k(\zeta), \quad \mu^{SC}_{CO} p^{CO}_k(\zeta) = p^{SC}_k(\zeta),$$

where

$$\mu^{SC}_{Sp}\colon Sp^*(\cdot) \to SC^*(\cdot), \quad \mu^{SC}_{CO}\colon CO^*(\cdot) \to SC^*(\cdot)$$

are the canonical transformations and $p^{Sp}_k(\zeta)$ are the characteristic Borel–Conner–Floyd classes in the theory $Sp^*(\cdot)$. Results on the divisibility of characteristic numbers of SC-, Sp-, and CO-manifolds were also obtained in [16], [17].

Theorem 14 (P. S. Landweber, [52])**.** *The signature of an SC-manifold M^{4n} is divisible by* 16 *if $n = 4k + 3$, divisible by* 8 *if $n = 4k + 1$, and divisible by* 2 *if $n = 4k + 2$, where $k \geqslant 0$.*

The proof of this theorem is based on the results of [16], [17].

Theorem 15 (L. P. Jones, [46])**.** *The signature of an Sp-manifold M^{4n} is divisible by* 16 *if $n = 2k + 1$, divisible by* 4 *if $n = 8k + 2$ or $n = 8k + 6$, and divisible by* 2 *if $n = 8k + 4$, where $k \geqslant 0$.*

4. The signature of 4-dimensional manifolds.

In this section we show the connection of V. A. Rokhlin's result on the signature of closed 4-dimensional *Spin*-manifolds with various fundamental questions of algebraic topology.

4.1. **The Rokhlin invariant.**

Theorem 16 (V. A. Rokhlin, 1952). *Let M^4 be an oriented smooth closed 4-dimensional Spin-manifold, i. e. $w_2(\mathcal{T}M^4) = 0$. Then the signature $\tau(M^4)$ is divisible by 16.*

As was proved in the 1960s, a PL-manifold of dimension $n < 7$ has a unique compatible smooth structure (see [59]). Therefore, Theorem 16 is also valid for PL-manifolds.

By a theorem of C. Arf, the signature of an even unimodular quadratic form is divisible by 8, so Rokhlin's theorem adds one extra factor 2 to the divisibility of the signature. Under the conditions of Rokhlin's theorem, we obtain $p_1(M^4) \equiv 0 \mod 48$.

The classical Kummer's surface M^4 is an oriented smooth closed 4-dimensional manifold with $w_2(M^4) = 0$ and $\tau(M^4) = -16$. So, the condition of divisibility by 16 in the Rokhlin theorem cannot be improved. An Enriques surface M^4 is an oriented smooth closed 4-dimensional manifold with even intersection form of signature -8, but the class $w_2(M)$ does not vanish.

This result led to the well-known Rokhlin invariant of 3-dimensional *Spin*-manifolds.

Let $V = V^3$ be a closed 3-dimensional manifold and let s be a Spin-structure on V. The Rokhlin invariant $\mu(V, s) \in \mathbb{Z}_{16}$ is defined to be the signature of any smooth compact 4-dimensional Spin-manifold M with a Spin-boundary (V, s).

4.2. **The Manolescu Theorem.**

Let Θ_3^H be the group of homology bordisms of oriented 3-dimensional homology spheres V (the operation in this group is induced by the connected sum of spheres). Take a smooth *Spin*-manifold M with a given homology sphere V as the boundary. Then the Rokhlin invariant of the homological sphere V is defined as the signature of the *Spin*-manifold M divided by 8 and reduced $\mod 2$. This invariant defines an epimorphism $\mu\colon \Theta_3^H \to \mathbb{Z}/2$.

The following question has long been asked: *Is there a 3-dimensional homology sphere V such that $[V]$ is an element of order 2 in the group Θ_3^H and $\mu(V) = 1$?*

This question is interesting because in 1980 Galewski, Stern, and, independently, Matumoto proved:

1) If the answer is positive, then any topological manifold of dimension at least 5 can be triangulated (i. e., is homeomorphic to a simplicial complex).
2) If the answer is negative, then in each dimension $n > 4$ there exists a non-triangulable topological manifold.

In 2013, C. Manolescu proved that there is no 3-dimensional homological sphere V such that $[V]$ is an element of order 2 in the group Θ_3^H and $\mu(V) = 1$, i. e. the answer is negative! (See [55].) This result was widely recognized, and the plenary talk of C. Manolescu at the conference "Rokhlin-100" was devoted to this result and its consequences.

4.3. **The Birman–Craggs homomorphisms.**

Let $\mathcal{W}_g$ be the mapping class group of an orientable closed surface W_g of genus g.

The Torelli group $\mathcal{I}_g$ is the subgroup in $\mathcal{W}_g$ consisting of all classes of maps that act trivially on the group $H_1(S_g; \mathbb{Z})$. In [9], J. Birman and R. Craggs used the Rokhlin invariant of 3-dimensional homology spheres to define the following family of homomorphisms $\mathcal{I}_g \to \mathbb{Z}/2\mathbb{Z}$, which are now called the Birman–Craggs homomorphisms. Let M be a 3-dimensional homology sphere with a chosen Heegaard splitting D of genus g. By twisting this Heegaard splitting by an element h of the Torelli group, we obtain a new homology sphere M_h. For each pair (M, D), the homomorphism $\mu_{M,D}\colon \mathcal{I}_g \to \mathbb{Z}/2\mathbb{Z}$ is defined by the formula $\mu_{M,D}(h) = \mu(M_h) - \mu(M)$, where μ is the Rokhlin invariant.

In [44], D. Johnson proved that the Birman–Craggs homomorphisms are enumerated by quadratic functions on $H_1(S_g, \mathbb{Z}/2\mathbb{Z})$ with zero Arf invariant. He also described the vector space generated by these quadratic functions. In [45], D. Johnson proved that any homomorphism from the Torelli group to $\mathbb{Z}/2\mathbb{Z}$ is a linear combination of Birman–Craggs homomorphisms.

4.4. **The Korkine–Zolotarev form and the Freedman manifold.**

V. A. Rokhlin noted that his signature divisibility theorem has "an interesting application" (see [80] and [88, p. 41]):

A. N. Korkine and E. I. Zolotarev (1873) constructed a positively defined unimodular quadratic form of rank 8 taking only even values. They found an explicit shape of their form when solving the problem of tightest packing by balls of the space $\mathbb{R}^8$. The lattice in $\mathbb{R}^8$ with the norm given by this quadratic form is now known as E_8. V. A. Rokhlin showed in his 1952 work [80] that this form cannot be realized as the intersection form of a 4-dimensional smooth (or, equivalently, PL) manifold.

In 1978, M. Freedman and R. Kirby [26] gave a new geometric proof of Rokhlin's Theorem 16 and noticed that Rokhlin already had a short sketch of their proof in 1971 [87]. They also mentioned in [26] that "Rokhlin's Theorem is an anomaly in this sense: in dimensions $4k$, $k > 1$ there are closed, orientable, almost parallelizable (non-smooth) PL-manifolds P^{4k} with $\sigma(P^{4k}) = 8$." M. Freedman's E_8-manifold [27] is a simply connected closed topological manifold with $w_2(M) = 0$ and intersection form E_8 of signature 8. The Rokhlin theorem implies that this manifold has no smooth or PL structure.

4.5. **The Rokhlin–Ochanine invariant.**

Theorem 17 (S. D. Ochanine, 1981, see [72] and Corollary 13)**.**
Let M be an oriented closed smooth Spin-manifold with $\dim M \equiv 4 \mod 8$. *Then*

$$\tau(M) \equiv 0 \mod 16,$$

Let V be an $(8k+3)$-dimensional spin manifold (with a spin structure δ_V), such that (V, δ_V) is the boundary of (W, δ_W) for some compact $(8k+4)$-dimensional spin manifold W. The generalized Rokhlin invariant $R(V, \delta_V)$ (the Rokhlin–Ochanine invariant) is defined to be the signature of $W \mod 16$. For applications of this invariant, see [56].

For $k \geqslant 2$ we obtain by the plumbing construction a topological manifold M^{4k} with intersection form E_8. Therefore, the manifold has signature 8. In addition it is $(2k-1)$connected, hence all its cohomology groups vanish in dimensions below $2k$. Following Thom and Wu, the Stiefel–Whitney classes can be defined for topological manifolds. In our case, we have $w_1(M) = w_2(M) = 0$ since there is no cohomology in dimensions 1 and 2. If M had a smooth structure, all the conditions of the theorem above would be satisfied, and hence the signature would be at least 16. Therefore, M *has no* smooth structure.

4.6. **Two-dimensional submanifolds in 4-dimensional manifolds.**

It is known that 2-dimensional integer homology classes of any topological space X are realized by images of 2-dimensional oriented surfaces. Let M^4 be a connected closed 4-dimensional manifold and let $x \in H_2(M^4; \mathbb{Z})$ be an element of the 2-dimensional integer homology group.

The well-known work of V.A. Rokhlin [86] is devoted to the problem: *What is the minimal genus of an oriented connected 2-dimensional submanifold in M^4 realizing a class x?*

Theorem 18 (V. A. Rokhlin, [86])**.** *Let M^4 be a connected closed manifold with $H_1(M^4; \mathbb{Z}) = 0$, and let V^2 be an oriented connected closed submanifold of genus g realizing a class $x \in H_2(M^4; \mathbb{Z})$.*

1. *If the class x is divisible by 2, then*

$$g \geqslant \left| \frac{xx}{4} - \frac{\tau(M^4)}{2} \right| - \frac{b_2(M^4)}{2},$$

where $b_2(M^4)$ is the 2-dimensional Betti number of M^4.

2. *If the class x is divisible by an odd number $h = p^k$, where p is a prime, then*

$$g \geqslant \left| \frac{(h^2-1)xx}{4h^2} - \frac{\tau(M^4)}{2} \right| - \frac{b_2(M^4)}{2}.$$

Moreover, if the group $\pi_1(M^4 \setminus V^2)$ is finite, then statement 2 *holds for any odd h.*

Consider the case when $M^4 = \mathbb{C}P^2$ is a complex projective plane, and $x = nx_0$, where x_0 is the generator of the group $H_2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}$. Then, according to Theorem 18, for even n we have $g \geqslant \frac{1}{4}n^2-1$, and for $n \equiv 0 \mod h$, where h is a power of an odd prime p, we have $g \geqslant \frac{h^2-1}{4h^2}n^2-1$. On the other hand, according to the classical results of algebraic geometry (see [42], Adjunction formula), a class $x \neq 0$ is realized by an algebraic curve of genus $g = \frac{1}{2}(|n|-1)(|n|-2)$. V. A. Rokhlin observes that for $|n| < 5$ his theorem gives a solution to the above problem, and for $n = 4$ (that is, $g = 3$) it is a new result.

Thom's conjecture: *The genus of an algebraic smooth curve in $\mathbb{C}P^2$ is equal to the lower boundary of the genera of* 2*-dimensional manifolds realizing the same homology class.*

P. Kronheimer and T. Mrowka proved Thom's conjecture using the monopole equation and the Seiberg–Witten invariants (see [48], [49], [50]). They noted that in the case of genus 3 the result follows from V. A. Rokhlin's paper [86].

In [86] V. A. Rokhlin also considered the case $M^4 = S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$. In this case, $x = x(n,m) = nx_1 + mx_2$, where $n, m \in N$ and x_1, x_2 are fixed generators of the group $H_2(M^4;\mathbb{Z})$. It follows from Theorem 18 that $g \geqslant \frac{1}{2}|nm|-1$ for even n, m, and $g \geqslant \frac{h^2-1}{2h^2}|nm|-1$ for $n, m \equiv 0 \mod h$, where $h = p^k$ is a power of an odd prime p. On the other hand, according to the classical results of algebraic geometry (see [42], Adjunction formula), the class x is realized for $n \neq 0, m \neq 0$ by a nonsingular algebraic curve of genus $g = (|n|-1)(|m|-1)$. Thus, Theorem 18 gives the exact values for the minimal genus when both n, m are even and one of them is equal to 2 or -2, and also for the obvious case when $|n| \leqslant 1$ or $|m| \leqslant 1$.

In 2000, P. Ozsváth and Z. Szabó [73] obtained complete results for 4-dimensional symplectic surfaces:

Theorem 19. *A embedded symplectic surface in a closed symplectic* 4*-dimensional manifold is genus-minimizing in its homology class.*

Corollary 20. *An embedded holomorphic curve in a Kähler surface is genus-minimizing in its homology class.*

5. Framed bordism, and theorems of Rokhlin and Milnor–Kervaire.

The J-homomorphism $J_{n,q}\colon \pi_n(SO(q)) \to \pi_{n+q}(S^q)$ was defined by G. W. Whitehead (1942), extending a construction of H. Hopf (1935). When $q \to \infty$ we obtain $J_n\colon \pi_n(SO) \to \pi_n^s = \Omega_n^{fr}$. The J-homomorphism plays an important role in the study of smooth manifolds.

Let B_k denote the k-th Bernoulli number.

Theorem 21 (J. F. Adams, D. Quillen, D. Sullivan, 1963–1971)**.** *The image of J_n is a direct summand in Ω_n^{fr}. The order $|J_n|$ is equal to: the denominator of $\left(\frac{B_k}{4k}\right)$ for $n = 4k-1$;* 2 *for $n \equiv 0, 1 \mod 8$; and* 1 *for $n \equiv 2, 4, 5, 6 \mod 8$.*

5.1. Theorems of Rokhlin and Milnor–Kervaire.

A connected smooth manifold M^k with base point x_0 is *almost parallelizable* if $M^k - x_0$ is parallelizable. If M^k is embedded in Euclidean space $\mathbb{R}^{m+k}$, $m \geqslant k+1$, then M^k is almost parallelizable if and only if the restriction of the normal bundle ν to $M^k - x_0$ is trivial.

Lemma 22. *Let M^4 be an oriented closed smooth* 4*-manifold with $w_2 = 0$. Then M^4 is almost parallelizable.*

Theorem 23 (M. Kervaire, J. Milnor, 1958–1960, see[47])**.**
1. The Pontryagin number $p_n[M^{4n}]$ *of an almost parallelizable* $4n$*-manifold is divisible by* $(2n-1)!a_n|J_n|$*, where* $a_n = 2$ *for odd* n*, and* $a_n = 1$ *for even* n*.*
2. There exists an almost parallelizable $4n$*-manifold* M_0^{4n} *with* $p_n[M_0^{4n}] = (2n-1)!a_n|J_n|$*.*

For $n = 1$, this gives Rokhllin's result, since $|J_1| = 24$, $a_1 = 2$.

6. Thom spaces and Atiyah duality.

From the beginning of the 60s of the last century, a stable homotopy topology began to develop intensively. The attention was directed to the theory of spectra of CW-complexes and to constructing generalized homology and cohomology theories (see [100]). The construction of Thom spaces (see Subsection 2.2) led to the spectra of Thom spaces, which attracted much attention due to the papers of M. Atiyah [5], [6] and S. P. Novikov [61]. Then the generalized homology and cohomology theories based on Thom spectra was called the bordism and cobordism theories.

6.1. Properties of Thom spaces.

We give the properties of Thom spaces, which are essentially used in the methods and applications of the bordism and cobordism theories.

1. **Functoriality.** Let $f\colon X \to Y$ be a map of CW-complexes and let $E \to Y$ be a vector bundle over Y. Then the map $f\colon X \to Y$ extends to the map of Thom spaces $Th(f)\colon Th(f^*E) \to Th(E)$.

2. **Multiplicativity.** For the vector bundle $E_1 \times E_2 \to X_1 \times X_2$, the formula

$$Th(E_1 \times E_2) = Th(E_1) \wedge Th(E_2)$$

holds, where $X \wedge Y = (X \times Y)/(X \vee Y)$ and $X \vee Y = (X \times y_0) \cup (x_0 \times Y)$, $x_0 \in X$, $y_0 \in Y$ are the base points. In the case when $X_1 = x_0$ and $\dim_{\mathbb{R}} E_1 = k$, we obtain $Th(E_1 \times E_2) = \Sigma^k Th(E_2)$.

3. **Atiyah duality.** Recall that CW-complexes X and Y are called dual in the sphere S^N if there are embeddings $X \subset S^N$, $Y \subset S^N$ such that $X \cap Y = \emptyset$ and $S^N \setminus X$, $S^N \setminus Y$ are deformation retracts of the complexes $Y \subset S^N \setminus X$ and $X \subset S^N \setminus Y$, respectively.

Let $M = M^n$ be a smooth manifold with boundary ∂M^n. Let $\nu \to M$ be the normal bundle of an embedding $M \subset \mathbb{R}^{n+k}$ and ν' is its restriction to the boundary $\partial M \subset (\mathbb{R}^{n+k-1} \times 0) \subset \mathbb{R}^{n+k}$. Embed $\mathbb{R}^{n+k}$ in $\mathbb{R}^{n+k+1}$ in the standard way. Then the CW-complex $M/\partial M$ is dual to the complex $Th(\nu)$, and the CW-complex $M/\emptyset$ is dual to the complex $Th(\nu)/Th(\nu')$ in the sphere S^{n+k+1}, where $k > n$ and $M/\emptyset = M \sqcup \mathrm{pt}$.

6.2. Orientable bundles and Thom isomorphism.

Let $h^*(\cdot)$ be a multiplicative cohomology theory on the category of CW-complexes, and let $\Omega_h = h^*(pt)$ be the graded ring of scalars of this theory with unit $1 \in \Omega_h^0$. A k-dimensional vector bundle $E \to X$ is called h^*-orientable if the canonical embedding $i\colon S^k \to Th(E)$ induces an epimorphism $i^*\colon h^*(Th(E)) \to h^*(S^k) \simeq \Omega_h^{*-k}$.

An element $a \in h^k(Th(E))$ such that the element $i^*a \in h^k(S^k) \simeq \Omega_h^0$ is equal to $1 \in \Omega_h^0$ is called an h^*-orientation of the bundle E. The zero section of the bundle E defines an embedding $s\colon X \to Th(E)$ and an orientation a of this bundle defines the element s^*a, which is called the Euler class of the bundle E with orientation a. By construction, if the bundle E has an everywhere nonzero section, then the map s is null-homotopic and the Euler class of E is zero.

Theorem 24. *Let* $E \to X$ *be a* k*-dimensional* h^**-orientable vector bundle with an* h^**-orientation* $a \in h^k(Th(E))$*. Then the homomorphism*

$$Th(a)\colon h^*(X) \to h^*(Th(E), *) = \widetilde{h}^*(Th(E)), \quad Th(a)b = a \smile b,$$

is an isomorphism.

In the case of Thom spaces, the $\smile$-product is defined by the $h^*(\cdot)$-homomorphism that is induced by the map $\widetilde{\Delta}\colon X \to X \wedge Th(E)$. The map $\widetilde{\Delta}$ is constructed using the composition of the diagonal map $\Delta\colon D(E) \to D(E) \times D(E)$ with the canonical projection $D(E) \times D(E) \to D(E) \wedge Th(E)$ and the projection $D(E) \to X$, which is a homotopy equivalence.

7. The theories of complex bordism $U_*(X)$ and cobordism $U^*(X)$.

In his talk at the International Mathematical Congress (Moscow, 1966), S.P. Novikov put forward a new approach to the problems of algebraic topology, based on the theory of complex cobordism. A detailed exposition of this approach, including the algebra $\mathcal{A}_U$ of cohomological operations in the theory of complex cobordism and its Landweber–Novikov subalgebra S, the Adams–Novikov spectral sequence, and the Mishchenko–Novikov formal group law of geometric cobordisms, was published in 1967 in article [65].

In 1969, D. Quillen (see [77]) published an article in which he proved that the formal group law of geometric cobordisms can be identified with the universal one-dimensional commutative formal group introduced by M. Lazard.

In 1970, V. M. Buchstaber (see [12]) showed that the Chern-Dold character $ch_U\colon U^*(X) \to H^*(X; \Omega_U \otimes \mathbb{Q})$ in complex cobordism is defined by the exponential of the formal group law of geometric cobordisms and commutes with the canonical action of the algebra $\mathcal{A}_U$.

7.1. Scalar rings $\Omega^U_* \simeq \Omega^*_U$.

Complex structure gives one of the most important example of an additional structure on manifolds. However, a direct attempt to define the bordism relation on complex manifolds fails because the manifold W must be odd-dimensional and therefore cannot be complex. This can be remedied by considering *stably complex* structures (also known as *stably almost complex* or *quasicomplex* structures).

A stable complex structure (in short, an U-structure) on an n-dimensional smooth manifold M is defined by a complex vector bundle $\xi \to M$ and the isomorphism of real vector bundles $c_{\mathcal{T}}\colon \mathcal{T}M \oplus R^{2m-n} \to \xi$, where ξ is considered as the real bundle $r\xi$ and m is the dimension of the complex bundle ξ. A manifold with a fixed U-structure is called an U-manifold.

The bordism relation can be defined between stably complex manifolds M_1^n and M_2^n using the stably complex structure on $\partial W^{n+1} = M_1^n \sqcup M_2^n$.

As in the case of unoriented bordism, the set of bordism classes $[M, c_{\mathcal{T}}]$ of n-dim stably complex manifolds is an abelian group with respect to disjoint union of manifolds. This group is called the *n-dimensional complex bordism group* and denoted by Ω^U_n. The opposite element to the bordism class $[M, c_{\mathcal{T}}]$ in the group Ω^U_n may be represented by $\overline{M}$, that is by the same manifold M with the U-structure determined by the isomorphism

$$\mathcal{T}M \oplus \mathbb{R}^{2m-n} \oplus \mathbb{C} \xrightarrow{c_{\mathcal{T}} \oplus l} \xi \oplus \mathbb{C}$$

where $l\colon \mathbb{C} \to \mathbb{C}$ is the complex conjugation. Let M be a U-manifold, then there exists a U-structure on $M \times I$ such that $\partial(M \times I) = M \sqcup \overline{M}$. The direct product of U-manifolds turns $\Omega^U = \bigoplus_{n \geqslant 0} \Omega^U_n$ into a graded ring called the *complex bordism ring.*

Theorem 25 (J. Milnor [58], S. P. Novikov [61], for details see [94])**.**

(a) *$\Omega^U \otimes \mathbb{Q}$ is a polynomial $\mathbb{Q}$-algebra generated by the bordism classes of complex projective spaces $\mathbb{C}P^i$, $i \geqslant 1$.*

(b) *Two stably complex manifolds are bordant if and only if they have identical sets of Chern characteristic numbers c_ω.*

(c) *$\Omega^U \cong \mathbb{Z}[m_2, m_4, \ldots, m_{2i}, \ldots]$ is a polynomial ring over $\mathbb{Z}$ with one generator m_{2i} in every even dimension $2i \geqslant 2$.*

(d) *A bordism class $[M^{2n}] \in \Omega^U_{2n}$ gives a multiplicative generator of the ring Ω^U if and only if $c_{(n)}(M^{2n}) = \pm p$, when $n = p^i - 1$, p is a prime, and $c_{(n)}(M^{2n}) = \pm 1$ in other cases.*

(e) *Consider the canonical forgetful homomorphism* $\mu_U^{SO}\colon \Omega^U \to \Omega^{SO}$ *and identify* Ω^U *with* $\mathbb{Z}[m_2, m_4, \dots, m_{2i}, \dots]$. *Then*
   (i) *the composition* $\mathbb{Z}[m_4, \dots, m_{4i}, \dots] \to \Omega^U \to \Omega^{SO}/\mathrm{Tors}$ *is an isomorphism;*
   (ii) *the kernel of* $\mu_U^{SO}\colon \Omega^U \to \Omega^{SO}$ *is an free* Ω^U*-module with generators* $m_{2(2^s-1)}$ *and* $2m_{4k+2}$*, where* $2k+1 \neq 2^s-1$*,* $s = 1, 2, \dots$

(f) *Consider the canonical forgetful homomorphism* $\mu_U^O\colon \Omega^U \to \Omega^O$ *and identify* Ω^O *with* $\mathbb{Z}_2[a_2, \dots, a_i, \dots,]$*, where* $i \neq 2^k-1$*, see Theorem* 3*. Then the image of* μ_U^O *is isomorphic to the subring* $\mathbb{Z}_2[(a_2)^2, \dots, (a_i)^2, \dots] \subset \Omega^O$.

7.2. **The Thom's spectrum MU.**

The theories of complex bordism $U_*(\cdot)$ and complex cobordism $U^*(\cdot)$ are defined via the *Thom spectrum*

$$MU = \{T_n;\ \varepsilon_n\colon \Sigma T_n \to T_{n+1},\ n \geqslant 0\},$$

where $T_{2n} = Th\xi_n = MU(n)$ is the Thom space of the universal n-dimensional complex bundle $\xi_n \to BU(n)$, $BU(n) = \lim\limits_{N\to\infty} \mathbb{C}G_{N,n}$, and $\mathbb{C}G_{N,n}$ is the Grassmann manifoids of complex n-dimensional subspaces in $\mathbb{C}^N$. By definition, $T_{2n+1} = \Sigma T_{2n}$, the map ε_{2n} is an identity, and $\varepsilon_{2n+1}\colon \Sigma T_{2n+1} \to T_{2n+2}$ corresponds to the map

$$\varepsilon_{2n+1}\colon \Sigma^2 MU(n) = Th(\xi_n \oplus 1) \to Th(\xi_{n+1}) = MU(n+1),$$

where 1 is a one-dimensional trivial complex bundle. Note that $BU(0) = \mathrm{pt}$, $\xi_0 = \mathrm{pt}$, $T_0 = MU(0) = S^0$ is a zero-dimensional sphere, $T_2 = MU(1) = \mathbb{C}P^\infty$, and $\varepsilon_1\colon \Sigma^2 S^0 = S^2 \to \mathbb{C}P^\infty = MU(1)$ is the standard embedding.

In [100], G. W. Whitehead proposed a general approach to constructing homology theories from spectra of spaces. In the case of the Thom spectrum MU, for a pair of CW-complexes (X, Y) we obtain

$$U_k(X,Y) = \lim_{n\to\infty} [S^{k+n}, (X/Y) \wedge T_n], \quad U^k(X,Y) = \lim_{n\to\infty} [\Sigma^n(X/Y), T_{n+k}],$$

where $[\cdot,\cdot]$ is the set of homotopy classes of based maps of CW-complexes.

Let by definition $U_k(X) = U_k(X, \emptyset)$, $U^k(X) = U^k(X, \emptyset)$ and $\widetilde{U}_k(X) = U_k(X, \mathrm{pt})$, $\widetilde{U}^k(X) = U^k(X, \mathrm{pt})$. For $X = \mathrm{pt}$ we obtain $U_k(\mathrm{pt}) \simeq U^{-k}(\mathrm{pt})$.

Set $U_*(\mathrm{pt}) = \Omega_*^U$ and $U^*(\mathrm{pt}) = \Omega_U^*$. Further on, we will use the general notation Ω_U for these rings, if it is clear from the context which one is implied.

7.3. **Poincare duality, cap-product and Gysin homomorphisms.**

Let M be a U-manifold with a fixed U-orientation $a(\nu)$ of the normal bundle ν. Then, according to the Atiyah duality theorem and the Thom theorem about isomorphism defined by the class $a(\nu)$, we obtain the Poincare duality isomorphisms

$$U_k(M^n) \to U^{n-k}(M^n, \partial M^n), \quad U^k(M^n) \to U_{n-k}(M^n, \partial M^n).$$

We have $\Omega_U = \pi_*(MU)$. The unit $1 \in \Omega_U$ corresponds to the identity map $T_0 \to T_0$ and gives rise to the canonical map of spectra $i_a^U\colon S \to MU$, where $S = \{S^n, n \geqslant 0\}$ is the sphere spectrum, $i^U = \{i_n^U = \Sigma^n \varepsilon_0\colon S^n \to T_n\}$, and $i_2^U = \varepsilon_2$.

In the case $M^n = \mathrm{pt}$, the Poincare duality isomorphism coincides with the isomorphism $U_k(\mathrm{pt}) \simeq U^{-k}(\mathrm{pt})$.

In the classical case, there is the well-known cap-product

$$\frown\colon H_n(X) \otimes H^m(X) \to H_{n-m}(X).$$

For further purposes, we describe explicitly the cap-product

$$\frown\colon U_n(X) \otimes U^m(X) \to U_{n-m}(X).$$

Let M^n be a closed U-manifold with a fixed U-orientation $\alpha(\nu)$ of the normal bundle. Consider the Poincare duality isomorphism $D = D(\alpha(\nu))\colon U^k(M^n) \to U_{n-k}(M^n)$. Set $a \in U_n(X)$ and $b \in U^m(X)$. Choose a representative $f\colon M^n \to X$ of the bordism class a and set by definition

$a \frown b = f_* D f^* b \in U_{n-m}(X)$. The fundamental class $\langle M^n \rangle \in U_n(M^n)$ is defined by the identity map $M^n \to M^n$. We obtain $\langle M^n \rangle \frown b = Db$ for any $b \in U^m(M^n)$.

Consider the homomorphism $\varepsilon\colon U_*(X) \to \Omega_U$ induced by the map $X \to \mathrm{pt}$. We obtain the Ω_U-bilinear scalar product $(\cdot,\cdot)\colon U_*(X) \otimes U^*(X) \to \Omega_U$ by the formula $(a,b) = \varepsilon(a \frown b)$.

Consider a map $f\colon M_1^{n_1} \to M_2^{n_2}$ of two closed U-manifolds with fixed Poincare duality isomorphisms $D_1\colon U^k(M_1^{n_1}) \to U_{n_1-k}(M_1^n)$ and $D_2\colon U_k(M_2^{n_2}) \to U^{n_2-k}(M_2^{n_2})$. Then the Gyzin homomorphisms

$$f_!^*\colon U^k(M_1^{n_1}) \to U^{n_2-n_1+k}(M_2^{n_2}), \quad f_*^!\colon U_k(M_2^{n_2}) \to U_{n_1-n_2+k}(M_1^{n_1})$$

are defined by the formulas $f_!^* b_1 = D_2 f_* D_1 b_1$ and $f_*^! a_2 = D_1 f^* D_2 a_2$.

7.4. **U-orientations of complex vector bundles.**

A complex n-dimensional bundle $\zeta \to X$ over a CW-complex X is classified by a map $f_\zeta\colon X \to BU(n)$. The corresponding map of Thom spaces $Th_{f_\zeta}\colon Th\zeta \to MU(n)$ defines a cobordism class $u(\zeta) \in U^{2n}(Th\zeta)$. By construction, the class $u(\zeta)$ is the universal U-orientation (the universal Thom class) of the bundle ζ. (For a general definition of orientation, see Subsection 6.2.)

The orientation $u(\zeta)$ is functorial, that is, if $f\colon X \to Y$ is a continuous map and $\zeta \to Y$ is a complex vector bundle over Y, then $(Thf)^* u(\zeta) = u(f^*\zeta)$. Therefore, the Euler class, defined by the orientation $u(\zeta)$, defines a characteristic class of complex n-dimensional bundles that is equal to the n-th Chern–Conner–Floyd class.

7.5. **$\mathbb{C}$-oriented cohomology theories.**

Let $h^*(\cdot)$ be a multiplicative cohomology theory on the category of CW-complexes and let $\Omega_h = h^*(pt)$ be the graded ring of scalars of this theory, with unit $1 \in \Omega_h^0$. A theory $h^*(\cdot)$ is called $\mathbb{C}$-*orientable* if the universal complex line bundle $\eta \to \mathbb{C}P^\infty$ is orientable in the theory $h^*(\cdot)$ (see Subsection 6.2). The homotopy equivalence $\mathbb{C}P^\infty \cong T\eta$ lead us to the following definition:

A $\mathbb{C}$-*oriented* theory is a pair $(h^*(\cdot), u_h)$, where $h^*(\cdot)$ is a $\mathbb{C}$-orientable theory and $u_h \in \widetilde{h}^2(\mathbb{C}P^\infty)$ is a fixed orientation.

Lemma 26. *Let $(h^*(\cdot), u_h)$ be a $\mathbb{C}$-oriented multiplicative cohomology theory. Then there exist an isomorphism $h^*(\mathbb{C}P^\infty) \simeq \Omega_h[[u_h]]$.*

In any $\mathbb{C}$-oriented theory $(h^*(\cdot), u)$, the Chern–Conner–Floyd characteristic classes $cf_k^h(\zeta)$, $k = 0, 1, \ldots,$ of complex vector bundles ζ are defined (see [23]). These classes are uniquely determined by the following conditions: $cf_1^h(\xi) = u$, $cf_k^h(\zeta) = 0$ if $k > \dim_{\mathbb{C}} \zeta$, and

$$cf_k^h(\zeta_1 \oplus \zeta_2) = \sum_{k_1+k_2=k} cf_{k_1}^h(\zeta_1)\, cf_{k_2}^h(\zeta_2).$$

In what follows we write $cf_k(\zeta)$ instead of $cf_k^h(\zeta)$ whenever the theory $h^*(\cdot)$ is clear from the context.

The main examples of $\mathbb{C}$-orientable theories in this paper are the following: the classical cohomology $H^*(\cdot,\mathbb{Z})$, the complex K-theory $K^*(\cdot)$, the complex cobordism theory $U^*(\cdot)$, and the oriented cobordism theory $SO^*(\cdot)$.

In the theory $U^*(\cdot)$, the $\mathbb{C}$-orientation is determined by the aforementioned canonical U^*-orientation of the bundle $\eta = \eta_1$.

In the theory $H^*(\cdot,\mathbb{Z})$, the canonical orientation is the $\mathbb{C}$-orientation determined by the class $u_H \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ such that $i_N^* u_H = \mathcal{D}\langle \mathbb{C}P^{N-1} \rangle$. Here $\langle \mathbb{C}P^{N-1} \rangle \in H_{2N-2}(\mathbb{C}P^N)$ is the homology class represented by the embedding $\mathbb{C}P^{N-1} \subset \mathbb{C}P^N$, and $\mathcal{D}$ is the classical Poincaré duality operator. In the theory $SO^*(\cdot)$ the canonical orientation is determined similarly, using the operator $\mathcal{D}$ in this theory.

In the theory $K^*(\cdot)$ with the ring of scalars $K^*(pt) = \mathbb{Z}[\beta^{-1}, \beta]$, $\deg \beta = -2$, the canonical $\mathbb{C}$-orientation is defined by the class $u_K = (1 - [\bar{\xi}])\beta^{-1} \in \widetilde{K}^2(\mathbb{C}P^\infty)$, where $\bar{\xi}$ is the conjugate of the universal bundle ξ. This choice of the class u_K will be explained below.

Not all multiplicative cohomology theories are $\mathbb{C}$-orientable. For example, the following theories do not admit a $\mathbb{C}$-orientation: the real K-theory, the cobordism theories $SU^*(\cdot)$ and $Sp^*(\cdot)$, and the cohomotopy theory $S^*(\cdot)$ (defined by the sphere spectrum $S = \{S^n\}$).

Theorem 27. *The theory $(U^*(\cdot), u)$ is the universal $\mathbb{C}$-oriented cohomology theory, in the sense that for any $\mathbb{C}$-oriented theory $(h^*(\cdot), u_h)$ there exists a unique multiplicative transformation $\mu_U^h\colon U^*(\cdot) \to h^*(\cdot)$ such that $\mu_U^h u = u_h$.*

Given a $\mathbb{C}$-oriented theory $(h^*(\cdot), u_h)$, the set of all multiplicative transformations $U^*(\cdot) \to h^*(\cdot)$ can be identified with any of the following sets:

– power series $\varphi(u_h) \in \Omega_h[[u_h]]$ of the form $\varphi(u_h) = u_h + (u_h^2)$;
– $\mathbb{C}$-orientations of the bundle $\eta \to \mathbb{C}P^\infty$ in the theory $(h^*(\cdot), u_h)$.

Let $(h^*(\cdot), u_h)$ be a $\mathbb{C}$-oriented cohomology theory, and let $\mu_U^h\colon U(\cdot) \to h^*(\cdot)$ be the corresponding multiplicative transformation. Then

$$\mu_U^h(cf_k(\zeta)) = cf_k^h(\zeta).$$

For the canonical orientations described above we obtain the transformations:

$$\mu_U^H\colon U^*(\cdot) \to H^*(\,\cdot\,;\mathbb{Z}), \quad \mu_{SO}^H\colon SO^*(\cdot) \to H^*(\,\cdot\,;\mathbb{Z}), \quad \mu_U^K\colon U^*(\cdot) \to K^*(\cdot),$$

where μ_U^H and μ_{SO}^H are the canonical transformations corresponding to the map from the Thom spectrum $MU = \{MU(n)\}$ and $MSO = \{MSO(2n)\}$ to the Eilenberg–MacLane spectrum $K(\mathbb{Z}) = \{K(\mathbb{Z}, n)\}$, and μ_U^K is the so-called Riemann–Roch–Conner–Floyd transformation (see [23]).

It is clear that $\mu_U^H = \mu_U^{SO} \cdot \mu_{SO}^H$. The transformations μ_U^H and μ_U^K map the Chern–Conner–Floyd characteristic classes in complex cobordism to the classical Chern classes in cohomology and to the Chern–Grothendieck classes in K-theory, respectively.

On the level of rings of scalars, the above transformations induce:

(a) the homomorphism $\mu_U^H\colon \Omega_U \to \mathbb{Z}$, where $\mu_U^H(1) = 1$ and $\mu_U^H(a) = 0$ for $\dim a < 0$, in other words, $\mu_U^H = \varepsilon$ is the augmentation;
(b) the homomorphism $\mu_U^K\colon \Omega_U \to \mathbb{Z}[\beta^{-1}, \beta]$ such that $\mu_U^K([M^{2n}]) = \mathrm{Td}(M^{2n})\beta^n$, where $\mathrm{Td}(M^{2n}) \in \mathbb{Z}$ is the classical Todd genus of a stably complex manifold;
(c) the homomorphism $\mu_U^{SO}\colon \Omega_U \to \Omega_{SO}$ is described in Theorem 25 (e).

Recall that the classical Todd genus of the complex projective space $\mathbb{C}P^n$ is equal to 1 for $n \geqslant 0$; the orientation $u_K = (1 - [\bar\eta])\beta^{-1}$ above was chosen so that the Todd genus considered here coincides with the classical one.

7.6. The algebra $\mathcal{A}_U$ of cohomology operations.

The algebra $\mathcal{A}_U$ of cohomology operations in complex cobordism was described by S. P. Novikov in [65]. This algebra contains a subalgebra S arising from the *Chern–Conner–Floyd characteristeic classes* $cf_k(\xi) \in U^{2k}(X)$, $k = 1, 2, \dots,$ of complex vector bundles $\xi \to X$. It has been introduced independently by S. P. Novikov [65] and Landweber [51] and has since become known as the Landweber–Novikov algebra.

The properties of the algebra S are conveniently described using the universal multiplicative cohomological operation $\mathcal{S}\colon U^*(\cdot) \to U^*(\cdot) \otimes A$, where $A = \mathbb{Z}[a_2, \dots, a_{2n}, \dots]$ is a graded algebra on an infinite set of variables $a_2, \dots, a_{2n}, \dots,$ $\deg a_{2n} = -2n$. The operation $\mathcal{S}$ is uniquely determined by its value $\mathcal{S}u = u + \sum\limits_{n \geqslant 1} a_{2n} u^{n+1}$ on the canonical element $u = Th(\eta) \in U^2(\mathbb{C}P^\infty)$.

For the Thom class $u_N = Th(\eta_N) \in U^2(MU(N))$, we have

$$\mathcal{S}u_N = u_N \left(1 + \sum_{n>0} \sum_{|\omega|=n} a_\omega c_\omega(\eta_N)\right) = u_N + \sum_{n>0} \sum_{|\omega|=n} a_\omega s_\omega u_N,$$

where $c_\omega(\eta_N)$ is the Chern–Conner–Floyd class in the theory $U^*(\cdot)$ (corresponding to the symmetrization of the monomial $u_1^{i_1} \cdots u_k^{i_k}$), $\omega = (i_1, \dots, i_k)$, $i_1 \geqslant \ldots \geqslant i_k > 0$, $|\omega| = i_1 + \ldots + i_k$, $a_\omega = a_{2i_1} \cdots a_{2i_k}$, and $s_\omega u_N \in U^{2n+2|\omega|}(MU(N))$.

The unit 1 of the algebra S is a generator of the group $S_0 = \mathbb{Z}$. The group S_{2n}, $n > 0$, has basis consisting of operations s_ω corresponding to all partitions $\omega = (i_1, \dots, i_k)$ of the integer n.

For any elements of $x, y \in U^*(X)$, the formula

$$s_\omega xy = \sum_{(\omega', \omega'')=\omega} (s_{\omega'} x)(s_{\omega''} y)$$

immediately follows from the multiplicativity of Thom spaces and the definition of the multiplication in the theory $U^*(\cdot)$. It follows from the definition of the cobordism group $\Omega_U^{-2k} = \lim_{n\to\infty} [S^{2n}, MU(n-k)]$ that $s_\omega[M^{2k}] = \bigl(c_\omega(\nu), \langle M^{2k}\rangle\bigr)$, where ν is the normal bundle of the embedding $M^{2k} \subset \mathbb{C}^n$ and $\langle M^{2k}\rangle \in U_{2k}(M^{2k})$ is the fundamental class defined by the identity map $M^{2k} \to M^{2k}$.

7.7. Chern–Dold character in complex cobordism.

A *Chern–Dold character* in a theory $h^*(\cdot)$ with $\Omega_h = h^*(\mathrm{pt})$ is a transformation of cohomology theories

$$\mathrm{ch}_h \colon h^*(\cdot) \to H^*(\,\cdot\,; \Omega_h \otimes \mathbb{Q}) = \sum_{n\geqslant 0} \sum_{i=0}^{n} H^i(\,\cdot\,; \Omega_h^{n-i} \otimes \mathbb{Q}),$$

which reduces to the canonical homomorphism $\Omega_h \to \Omega_h \otimes \mathbb{Q}$ in the case $X = \mathrm{pt}$. Here $H^*(\cdot)$ denotes the classical cohomology.

The existence and uniqueness of a Chern–Dold character was first proved by Dold (see [25]). In the case when $h^*(\cdot)$ is the complex K-theory, the Chern–Dold character ch_h coincides with the classical Chern character ch (see [25]). This explains the terminology and the notation ch_h (see [12]).

For any generalised homology theory $h_*(\cdot) = \sum h_n(\cdot)$, there exists a homology Chern–Dold character

$$\mathrm{ch}^h \colon h_*(\cdot) \to H_*(\,\cdot\,; \Omega_h \otimes \mathbb{Q}),$$

which is defined similarly.

For the stable homotopy theory π_*^s, the ring of scalars $\Omega_{\pi_*^s}$ is the ring of stable homotopy groups of spheres. Therefore, $\Omega_{\pi_*^s} \otimes \mathbb{Q} = \pi_0^s \otimes \mathbb{Q} = \mathbb{Q}$, and the transformation $\mathrm{ch}_{\pi_*^s}$ is decomposed into a composition with the classical Hurewicz homomorphism $H \colon \pi_*^s(\cdot) \to H_*(\,\cdot\,; \mathbb{Z})$. The latter corresponds to the map of spectra $\mu_s^H \colon S \to K(\mathbb{Z})$ such that $\mu_s^H(1) = 1 \in \widetilde{H}_0(S^0)$ for $1 \in \pi_0^s(S^0)$.

The Chern–Dold characters ch_h and ch^h are defined for *any* cohomology theory $h^*(\cdot)$ and homology theory $h_*(\cdot)$. The theory $h^*(\cdot)$ does not need to be multiplicative or $\mathbb{C}$-orientable.

Here are the main properties of the Chern–Dold character ch_h:

(a) For any finite CW-complex X, the homomorphism $\mathrm{ch}_h \otimes \mathbb{Q} \colon h^*(X) \otimes \mathbb{Q} \to H^*(X; \Omega_h \otimes \mathbb{Q})$ is an isomorphism.

(b) If the cohomology theory $h^*(\cdot)$ is multiplicative, then the Chern–Dold character ch_h is a ring homomorphism.

Let $\mathcal{A}_h$ be the ring of stable cohomology operations in the theory $h^*(\cdot)$. We turn $H^*(X; \Omega_h \otimes \mathbb{Q})$ into an $\mathcal{A}_h$-module using the action of the ring $\mathcal{A}_h$ on Ω_h.

(c) The Chern–Dold character ch_h is a homomorphism of $\mathcal{A}_h$-modules.

(d) Let $\varphi \colon h_1(\cdot) \to h_2(\cdot)$ be a transformation of cohomology theories. Then $\mathrm{ch}_{h_2}\varphi = \varphi\mathrm{ch}_{h_1}$.

These properties follow easily from the construction of Chern–Dold character given in [25].

We define the ring $\Omega_U(\mathbb{Z}) = \sum_{n\geqslant 0} \Omega_U^{-2n}(\mathbb{Z})$ by

$$\Omega_U^{-2n}(\mathbb{Z}) = \{\sigma \in \Omega_U^{-2n} \otimes \mathbb{Q} \colon s\sigma \in \Omega_U^0 = \mathbb{Z} \text{ for } s \in S_{2n}\},$$

where $S = \sum_{n\geqslant 0} S_{2n}$ is the Landweber–Novikov algebra. As it is shown in [12],

$$\Omega_U(\mathbb{Z}) \simeq \mathbb{Z}\left[\frac{[\mathbb{C}P^n]}{n+1},\ n = 1, 2, \dots\right]. \tag{2}$$

Let $g(u)=u+\sum[\mathbb{C}P^n]\frac{u^{n+1}}{n+1}$ be the logarithm of the formal group of geometric cobordism (the so-called Mishchenko series). It follows from (2) that $g(u)\in\Omega_U(\mathbb{Z})[[u]]$ and $g^{-1}(t)\in\Omega_U(\mathbb{Z})[[t]]$. The subring $\Omega_U(\mathbb{Z})\subset\Omega_U\otimes\mathbb{Q}$ is obviously invariant with respect to the action of the algebra $\mathcal{A}_U$. There exists an isomorphism of S-modules $\Omega_U(\mathbb{Z})\cong S^*$, where the S-module structure on S^* is defined by the canonical left action l of the Landweber–Novikov Hopf algebra S on the dual Hopf algebra S^*.

The *Chern–Dold character in cobordism* (see [12]) is decomposed into the composition

$$\mathrm{ch}_U\colon U^*(X)\overset{\widehat{\mathrm{ch}}_U}{\to} H^*(X;\Omega_U(\mathbb{Z}))\overset{j}{\to} H^*(X;\Omega_U\otimes\mathbb{Q}),$$

where j is the transformation induced by the injection $j\colon\Omega_U(\mathbb{Z})\to\Omega_U\otimes\mathbb{Q}$. The transformation $\widehat{\mathrm{ch}}_U\colon U^*(\cdot)\to H^*(\,\cdot\,;\Omega_U(\mathbb{Z}))$ is called the *reduced Chern–Dold character*. It is a multiplicative transformation uniquely determined by the formula $\mathrm{ch}_U cf_1(\xi)=g^{-1}(c_1(\xi))$ for a complex line bundle $\xi\to X$.

There exists the corresponding transformation of homology theories

$$\widehat{\mathrm{ch}}^U\colon U_*(\cdot)\to H_*(\,\cdot\,;\Omega_U(\mathbb{Z})).$$

Using the canonical isomorphisms $\Omega_U(\mathbb{Z})\simeq S_*\simeq H_*(MU;\mathbb{Z})$, the transformations $\widehat{\mathrm{ch}}_U$ and $\widehat{\mathrm{ch}}^U$ can be identified with the transformations defined by the map of spectra $S\wedge MU\to K(\mathbb{Z})\wedge MU$. The transformation $\widehat{\mathrm{ch}}^U$ is therefore sometimes called a Hurewicz homomorphism, by analogy with the classical Hurewicz homomorphism $H\colon\pi_*^s(X)\to H_*(X;\mathbb{Z})$ defined by the inclusion $S\to K(\mathbb{Z})$.

There exist transformations $\mu_H\colon U^*(\cdot)\to H^*(\cdot;\mathbb{Z})$ and $\mu^H\colon U_*(\cdot)\to H_*(\cdot;\mathbb{Z})$ defined by the map of spectra $MU\to K(\mathbb{Z})$. It is clear that the augmentation $\varepsilon\colon\Omega_U(\mathbb{Z})\to\mathbb{Z}\colon\varepsilon(1)=1$, $\varepsilon(\sigma)=0$, $\deg\sigma<0$, takes $\widehat{\mathrm{ch}}_U$ to μ_H and takes $\widehat{\mathrm{ch}}^U$ to μ^H. The image of the homomorphism $\mu^H\colon U_*(X)\to H_*(X,\mathbb{Z})$ consists of cycles realisable by the images of fundamental cycles of stably complex manifolds; the calculation of this image is known as the Steenrod problem.

According to the Milnor–Novikov theorem, the ring Ω_U is torsion-free (see Theorem 25 (c)). Therefore, in the case when the group $H_*(X;\mathbb{Z})$ is torsion-free, the homomorphism μ_U^H is an epimorphism.

Theorem 28 (V. M. Buchstaber, [12])**.**
There exists a functorial multiplicative transformation of $\mathcal{A}_U$-modules

$$\widehat{\mathrm{ch}}_U\colon U^*(X)\otimes_{\Omega_U}\Omega_U(\mathbb{Z})\to H^*(X;\Omega_U(\mathbb{Z})),$$

which is an isomorphisms for spaces X with no torsion in homology.

For $X=\mathbb{C}P^\infty$ we obtain an isomorphism of $\mathcal{A}_U$-modules

$$\widehat{\mathrm{ch}}_U\colon\Omega_U(\mathbb{Z})[[u]]=U^*(\mathbb{C}P^\infty)\otimes_{\Omega_U}\Omega_U(\mathbb{Z})\to\Omega_U(\mathbb{Z})[[t]]$$

such that $\widehat{\mathrm{ch}}_U(g(u))=t$ and $\widehat{\mathrm{ch}}_U u=g^{-1}(t)$, where $u\in U^2(\mathbb{C}P^\infty)$, $t\in H^2(\mathbb{C}P^\infty;\mathbb{Z})$, and

$$g(u)=u+\sum[\mathbb{C}P^n]\,\frac{u^{n+1}}{n+1},\quad g^{-1}(t)=t+\sum[\mathcal{M}^{2n}]\,\frac{t^{n+1}}{(n+1)!}\,.$$

For any operation $s\in S$, $s\neq 1$, we have $st=0$ by construction. Hence $0=st=s\widehat{\mathrm{ch}}_U g(u)=\widehat{\mathrm{ch}}_U sg(u)$. So $sg(u)=0$.

For $u\in U^2(\mathbb{C}P^\infty)$, we have $su=0$ if $s\neq s_n$ and $s_nu=u^{n+1}$. Hence $sg^{-1}(t)=0$ if $s\neq s_n$, and $s_ng^{-1}(t)=(g^{-1}(t))^{n+1}$.

Using these facts, it is easy to obtain formulae for the action of operations $s\in S$ on the coefficients $[\mathbb{C}P^n]$ and $[\mathcal{M}^{2n}]$, $n=1,2,\dots$, of the series $g(u)$ and $g^{-1}(t)$. Thus, we obtain a canonical isomorphism between

$$\Omega_U(\mathbb{Z})\simeq\mathbb{Z}\left[\frac{[\mathbb{C}P^1]}{2},\dots,\frac{[\mathbb{C}P^n]}{n+1},\dots\right]\simeq\mathbb{Z}\left[\frac{[\mathcal{M}^2]}{2!},\dots,\frac{[\mathcal{M}^{2n}]}{(n+1)!},\dots\right]$$

and the dual Hopf algebra S_* of the Landweber–Novikov Hopf algebra S.

From the formula

$$\mathrm{ch}_U u = t + \sum_{n\geqslant 1} [\mathcal{M}^{2n}] \, \frac{t^{n+1}}{(n+1)!} \,,$$

where $[\mathcal{M}^{2n}] \in \Omega_U$, we obtain the following result:

Corollary 29. *Let X be a finite CW-complex of dimension n and let $p(X)$ be equal to the maximal prime p in the prime factorization of the order of the group* $\operatorname{Tors} H^*(X;\mathbb{Z})$*. If $p(X) > n$ then there is an isomorphism*

$$\widehat{\mathrm{ch}}_U \colon U^*(X) \otimes_{\Omega_U} \Omega_U(\mathbb{Z}) \to H^*(X;\Omega_U(\mathbb{Z})).$$

Theorem 30.
The reduced Chern–Dold character $\widehat{\mathrm{ch}}_U$ for $X = S^{2n}$ is decomposed into a composition

$$\widehat{\mathrm{ch}}_U \colon U^*(S^{2n}) \to H^*(S^{2n};\Omega_U) \to H^*(S^{2n};\Omega_U(\mathbb{Z})).$$

The corresponding result for the classical Chern character $\mathrm{ch} = \mathrm{ch}_K$ follows from the Bott periodicity theorem in complex K-theory. The integrality theorems for the Chern character ch based on the Bott periodicity theorem where obtained by F. Adams (see [1], [3]). In particular, the Adams invariant $e\colon \pi^s_{2n-1} \to \mathbb{Q}/\mathbb{Z}$ was constructed. It gives a strong lower bound for the order of the group $J_{2n-1} \subset \pi^s_{2n-1}$, where $J_n = J(S^n)$ is the image of the Whitehead stable J-homomorphism $J\colon \pi_n(SO) \to \pi^s_n$ with $SO = \lim SO(n)$ (see section 5 and [2]).

A complex cobordism analogue e_U of the Adams invariant e can be found in [18], Section 8.

7.8. **The universal Todd genus in complex cobordism.**

Following the theory developed in [12], we introduce a characteristic class $Td_U(\xi)$ of complex bundles ξ such that $u_H(\xi) Td_U(\xi) = \mathrm{ch}_U u(\xi)$, where $u(\xi)$ and $u_H(\xi) = \mu^H_U u(\xi)$ are the universal Thom classes in complex cobordism and cohomology. It follows directly from the definition that $Td_U(n) = 1$, where n is the trivial complex n-dimensional bundle, and for any complex bundles ξ_1 and ξ_2 we have the formula $Td_U(\xi_1 \oplus \xi_2) = Td_U(\xi_1) Td_U(\xi_2)$. Thus, we obtain a multiplicative homomorphism $Td_U\colon K(X) \to H^*(X;\Omega_U(\mathbb{Z})$. It is given by its value $Td_U(\eta) = 1 + \sum\limits_{n\geqslant 1} a_{2n} t^n$ on the universal one-dimensional complex bundle $\eta \to \mathbb{C}P^\infty$, where $a_{2n} = \frac{[\mathcal{M}^{2n}]}{(n+1)!} \in \Omega_U(\mathbb{Z})$ are the coefficients of the series $\mathrm{ch}_U u$. Consequently,

$$Td_U(\xi) = 1 + \sum_{n\geqslant 1} \sum_{|\omega|=n} a_\omega c_\omega(\xi),$$

where $\omega = (i_1, \dots, i_k)$, $|\omega| = n$ is a partition of the number n, $a_\omega = a_2^{i_1}, \dots, a_{2k}^{i_k}$, and $c_\omega(\xi)$ are the characteristic Chern classes in cohomologies corresponding to monom $t_1^{i_1}, \dots, t_k^{i_k}$.

It is clear that for any stably complex manifold M^{2n} the formula

$$\big(Td_U(\nu(M^{2n})), \langle M^{2n}\rangle\big) = [M^{2n}] = \sum_{|\omega|=n} a_\omega \left(c_\omega(\nu), \langle M^{2n}\rangle\right), \ a_\omega \in \Omega_U(\mathbb{Z}),$$

holds, where $[M^{2n}] \in \Omega_U^{-2n}$ is the complex cobordism class of M^{2n} and $\langle M^{2n}\rangle$ is its fundamental cycle in homology.

A Hirzebruch genus L_U of stably complex manifolds is defined similarly to a Hirzebruch genus L_{SO} of oriented manifolds, see Theorem 10. Such a genus is defined by a series $\beta(t) = t + \sum\limits_{n\geqslant 1} \beta_n t^{n+1}$ and, accordingly, is denoted by $L_U(\beta)$. There is a formula $L_U^\nu(\beta)[\mathcal{M}^{2n}] = (n+1)!\beta_n$.

The well-known two-parameter complex Todd genus is defined by the series

$$Q(t) = \frac{t}{\beta(t;a,b)}, \quad \text{where} \quad \beta(t;a,b) = \frac{e^{at} - e^{bt}}{ae^{at} - be^{bt}}.$$

This series is symmetric as a function of the parameters a and b. Important special cases are

1. $\beta = \beta(t;1,0) = 1 - e^{-t}$. Then $L_U^\nu(\beta)[M^{2n}] = Td(M^{2n})$ is the classical Todd genus of M^{2n}.
2. $\beta = \beta(t;1,-1) = \tanh(t)$. Then $L_U^\nu(\beta)[M^{2n}] = \tau(M^{2n})$ is the signature of M^{2n}.

3. $\beta = \beta(t;1,1) = \frac{t}{1+t}$. Then $L_U(\beta)[M^{2n}] = c_n(M^{2n})$ is the minimal Chern number of M^{2n}.

Theorem 31. *The classical Hirzebruch genera take the following values on the coefficients $[\mathcal{M}^{2n}]$ of the series* ch_U*:*

$$Td(\mathcal{M}^{2n}) = (-1)^n, \quad \tau(\mathcal{M}^{4n}) = \frac{4^{n+1}(4^{n+1}-1)}{2n+2}B_{2n+2}, \quad c_n(\mathcal{M}^{2n}) = (-1)^n(n+1)!,$$

where B_k is the k-th Bernoulli number, $c_n = c_{(1,\dots,1)}$.

Examples:

$$\tau(\mathcal{M}^4) = -2; \quad \tau(\mathcal{M}^8) = 16; \quad \tau(\mathcal{M}^{12}) = -16\cdot 17.$$

We introduce the group

$$\Omega_U^{-2n}(1) = \{[M^{2n}] \in \Omega_U^{-2n} \ : \ c_\omega(M^{2n}) = 0 \ \text{ if } |\omega| = n \ \text{ and length } \omega > 1\}.$$

Corollary 32. *The coefficient $[\mathcal{M}^{2n}]$ of the series* ch_U *is a generator of the group $\Omega_U^{-2n}(1)$.*

8. The loop space of S^3 and coefficients of Chen–Dold character.

The loop spaces ΩG of compact Lie groups G are classical objects of algebraic topology. The Morse theory has found fundamental applications in problems on cohomology rings $H^*(\Omega G;\mathbb{Z})$ and Pontryagin algebras $H_*(\Omega G;\mathbb{Z})$ (see [10]). Analytical and differential-geometrical properties of the loop spaces ΩG attracted the attention of specialists in the theory of integrable systems, string theory, and the theory of infinite-dimensional Kähler's manifolds (see [7], [76]).

The loop space of a 3-dimensional sphere S^3 is a remarkable object. On the one hand, the manifold S^3, as a Lie group, is $SU(2) = Spin(3) = SP(1)$, and on the other hand, as a CW-complex, it is ΣS^2.

In this section, we use algebraic-topological and geometric constructions to describe the cobordism ring $U^*(\Omega S^3)$ with the action of the algebra $\mathcal{A}_U$ and the bordism group $U_*(\Omega S^3)$ as the ring with the Pontryagin multiplication. We describe the canonical map $\varphi\colon \Omega S^3 \to \mathbb{C}P^\infty$ and use it to construct cobordism classes of U-, SU-, $Spin$- and Sp-manifolds. Thus, we obtain results on the characteristic numbers of manifolds, generalizing the results that were discussed in the first part of our article.

8.1. ch_U-integral spaces.

We say that an element $a \in U^k(X)$ is ch_U-integral if $\mathrm{ch}_U a \in H^*(X;\Omega_U) \subset H^*(X;\Omega_U(\mathbb{Z}))$. An element $a \in U^k(X)$ is called cospherical one if there exists a map $f\colon \Sigma^{2N-k}X \to S^{2N}$ such that $f^*b_N = S^{2N-k}a$, where b_N is a generator of $\widetilde{\mathcal{U}}^{2N}(S^{2N}) \simeq \widetilde{\mathcal{U}}^0(S^0) = \mathbb{Z}$

Theorem 30 implies that a cospherical element is ch_U-integral. A space X is said to be ch_U-integral if its Chern–Dold character decomposes into a composition

$$\mathrm{ch}_U\colon U^*(X) \to H^*(X;\Omega_U) \to H^*(X;\Omega_U(\mathbb{Z})).$$

As a corollary we obtain that if the group $H^*(X;\mathbb{Z})$ is torsion-free and X is ch_U-integral, then the Chern–Dold character defines an isomorphism of $\mathcal{A}_U$-modules $U^*(X) \to H^*(X;\Omega_U)$.

8.2. James spaces.

Recall the necessary facts about the spaces of James and Dold–Thom, see details in [38].

Let X be a CW-complex with a base point $*$. The James space $J(X)$ is a *free noncommutative* monoid generated by the space X, in which the base point $*$ plays the role of unit. This space is a CW-complex $J(X) = \lim_{n\to\infty} J_n(X)$. Here $J_n(X)$ is the image of the canonical mapping $j_n\colon X^n \to J(X) \ : \ j_n\colon (x_1,\dots,x_n) \to (x_1 \circ \cdots \circ x_n)$, where $\circ$ denotes the product in the monoid $J(X)$. The direct limit is taken with respect to the embeddings $i_n\colon J_n(X) \subset J_{n+1}(X)$ that are induced by the embeddings $X^n \subset X^{n+1}\colon (x_1,\dots,x_n) \to (x_1,\dots,x_n,*)$.

Theorem 33 (I. M. James, [43])**.**

1. *There is a homotopy equivalence $j_X\colon J(X) \to \Omega\Sigma X$.*

2. *There is a homotopy equivalence $\Sigma j_X\colon \Sigma J(X) \to \Sigma \bigvee_{n\geqslant 1} X^{(n)}$, where ΩX is the loop space of X, and $X^{(n)} = X \wedge \cdots \wedge X$ is the n-th smashed power of a space X.*

Corollary 34. *Let X be a ch_U-integral space. Then $J(X) \simeq \Omega\Sigma X$ is a ch_U-integral space.*

8.3. **Dold–Thom spaces.**

The Dold–Thom space $DT(X)$ is a *free commutative* monoid generated by the space X, in which the base point $*$ plays the role of unit. This space is a CW-complex $DT(X) = \lim\limits_{n\to\infty} DT_n(X)$. Here $DT_n(X)$ is the quotient space of the direct product X^n by the action of the permutation group S_n. The direct limit is taken with respect to the embeddings $i_n\colon DT_n(X) \subset DT_{n+1}(X)$ that are induced by the embeddings $X^n \subset X^{n+1}\colon (x_1, \dots, x_n) \to (x_1, \dots, x_n, *)$ that are equivariant with respect to the corresponding embeddings of the groups $S_n \to S_{n+1}$. The Dold–Thom space $DT(X)$ is also called the infinite symmetric product and is denoted by $SP(X)$.

Both constructions $J(X)$ and $DT(X)$ are functorial, and there is a canonical functorial continuous map of CW-complexes $J(X) \to DT(X)$, the so-called "abelinization" homomorphism of monoids.

Set $A_n(X) = H_n(X;\mathbb{Z})$ and $A(X) = \sum\limits_{n\geqslant 0} A_n(X) = H_*(X;\mathbb{Z})$. We introduce the Eilenberg–MacLane space $K(A(X)) = \lim\limits_{n\to\infty} \prod\limits_{m=1}^{n} K(A_m(X), m)$. In the case when X is an n-dimensional sphere, we have $K(A(S^n)) = K(\mathbb{Z}; n)$.

Theorem 35 (A. Dold, R. Thom, [24])**.** *There is a functorial homotopy equivalence $k(X)\colon DT(X) \to K(A(X))$.*

It is well known that the n-symmetric degree $\mathrm{Sym}^n(\mathbb{C}P^1)$ of the complex projective line $\mathbb{C}P^1$ can be identified with the complex projective space $\mathbb{C}P^n$. Therefore, $\lim\limits_{n\to\infty} \mathrm{Sym}^n(\mathbb{C}P^1)$ can be identified with the space $\mathbb{C}P^\infty = \lim\limits_{n\to\infty} \mathbb{C}P^n$, which is a model of the Eilenberg–MacLane space $K(\mathbb{Z}; 2)$. Thus, the Dold–Thom Theorem extends this fundamental result to arbitrary CW-complexes.

8.4. **Complex bordism and cobordism of the loop space of S^3.**

Let X be some CW-monoid with multiplication $\mu_X\colon X\times X \to X$. The map μ_X induces a homomorphism $\mu_{X,*}\colon U_*(X\times X) \to U_*(X)$, making $U_*(X)$ into an Ω^U-algebra with multiplication $x\cdot y = \mu_{X,*}(x \otimes y)$, which was introduced by Pontryagin in the case of classical homology.

Whenever we talk about the bordism algebra of a monoid X, we always refer to this Ω^U-algebra.

It is clear that a homomorphism of monoids $\lambda\colon X \to Y$ induces a homomorphism of Ω_U-algebras $\lambda_*\colon U_*(X) \to U_*(Y)$. We identify the space $X = \Omega S^3$ with the free monoid $J(S^2)$ and identify the space $\mathbb{C}P^\infty$ with the commutative free monoid $DT(S^2)$, see Subsections 8.2 and 8.3 above.

Let $\varphi\colon \Omega S^3 \to \mathbb{C}P^\infty$ denote the map defined by the abelinization homomorphism $J(S^2) \to DT(S^2)$. Consider the one-dimensional quaternionic space $\mathbb{H}^1$ and set $S^3 = \{q \in \mathbb{H}^1 : |q| = 1\}$. Let S^1 be a maximal commutative subgroup of $S^3 = Sp(1)$. Denote by $\mathbb{C}P^1_*$ the homogeneous space $SU(2)/S^1$. The tangent bundle $\mathcal{T}\mathbb{C}P^1_*$ is stably equivalent to the trivial bundle $\eta \oplus \overline{\eta}$, where $\eta \to \mathbb{C}P^1_*$ is the tautological line bundle. Therefore, the bordism class $[\mathbb{C}P^1_*] \in \Omega^U_2$ is zero.

The canonical embedding $\mathbb{C}P^1_* \subset \Omega S^3$ defines a generator w of the group $\widetilde{U}_2(\Omega S^3) \simeq \mathbb{Z}$, which maps by $\varphi_*\colon U_*(\Omega S^3) \to U_*(\mathbb{C}P^\infty)$ to a generator of the group $\widetilde{U}_2(\mathbb{C}P^\infty) \simeq \mathbb{Z}$.

Theorem 36. *There is an isomorphism of Ω^U-algebras $U_*(\Omega S^3) \simeq \Omega_U[w]$.*

The proof of this Theorem directly follows from the construction of the monoid $J(S^2) = \lim\limits_{n\to\infty} J_n(S^2)$, whose homology group $H_{2n}(J(S^2))$ is generated by the image of the fundamental cycle of the manifold $(\mathbb{C}P^1_*)^n$ under the canonical projection $j_n\colon (\mathbb{C}P^1_*)^n \to J_n(S^2)$.

Thus, $U_*(\Omega S^3)$ is a free Ω_U-module with generators $w_{2n} \in U_{2n}(\Omega S^3)$, $n = 0, 1, \ldots$, where $w_{2n} = w^n$. The standard embedding $S^{2n} \subset MU(n)$ defines a generator $u(n)$ of the group $\widetilde{U}^{2n}(S^{2n}) \simeq \mathbb{Z}$. Using the homotopy equivalence $\Sigma j_{S^2}\colon \Sigma J(S^2) \to \Sigma\Bigl(\bigvee_{n\geqslant 1} S^{2n}\Bigr)$, we define the classes $b_{2n} \in U^{2n}(\Omega S^3)$ by the formula

$$(\Sigma j_{S^2})^* \sigma u(n+1) = \sigma b_{2n+2},$$

where σ is the suspension isomorphism. The classes b_{2n} generate the free Ω_U-module $U^*(\Omega S^3)$. There are similarly defined classes $b_{2n} \in h^{2n}(\Omega S^3)$ which generate the free Ω_h-module $h^*(\Omega S^3)$ in the theories $h = SU$, $Spin$ and Sp.

Theorem 37.

1. *There is an Ω^U-algebra isomorphism*
$$U^*(\Omega S^3) \simeq \Omega_U[[b_2, \ldots, b_{2n}, \ldots]]/J,$$
where $J = \{b_2^n - n! b_{2n}\}$, $n = 2, 3, \ldots$; and therefore $b_{2i}b_{2j} = \binom{i+j}{i} b_{2(i+j)}$.
2. *There is the identity $\operatorname{ch}_U \varphi_U^* u = \varphi_H^* \operatorname{ch}_U u$, where φ_U^* and φ_H^* are the ring homomorphisms induced by the map $\varphi\colon \Omega S^3 \to \mathbb{C}P^\infty$,*
$$\varphi_U^* u = b_2 + \sum_{n\geqslant 1} [\mathcal{M}^{2n}] b_{2n+2}, \quad \operatorname{ch}_U u = t + \sum_{n\geqslant 1} [\mathcal{M}^{2n}] \frac{t^{n+1}}{(n+1)!},$$
and $\operatorname{ch} b_{2n} = \mu_U^H b_{2n} = \widehat{b}_{2n}$.
3. *The Ω_U-isomorphism $\mu_U^H\colon \Omega_U[[b_2, \ldots, b_{2n}, \ldots]] \to \Omega_U[[\widehat{b}_2, \ldots, \widehat{b}_{2n}, \ldots]]$, where $\widehat{b}_{2n} = \mu_U^H b_{2n}$, defines an isomorphism of $\mathcal{A}_U$-modules $U^*(\Omega S^3) \to H^*(\Omega S^3; \Omega_U)$.*

The proof of this Theorem follows from the following results obtained above:

(1) $s_\omega b_{2n} = 0$ for any operation $s_\omega \subset S \subset \mathcal{A}_U$ with $|\omega| > 0$;

(2) $s_\omega[\mathcal{M}^{2n}] = 0$ if $|\omega| = n$, $|\omega| \neq (n)$, and $s_{(n)}[\mathcal{M}^{2n}] = (n+1)!$

8.5. **Representatives of the coefficients of ch_U.**

Consider the Ω_U-bilinear scalar product

$$(\cdot,\cdot)\colon U_n(\Omega S^3) \otimes U^m(\Omega S^3) \to \Omega^U_{n-m}$$

Let $\{w_{2n}\}$ and $\{b_{2n}\}$ be the bases of the free Ω_U-modules $U_*(\Omega S^3)$ and $U^*(\Omega S^3)$ described above. By construction, we have $(w_{2n}, b_{2m}) = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker symbol. Using statement 2 of Theorem 37, we obtain:

Corollary 38. *There is the identity $(w_{2n+2}, \varphi^* u) = [\mathcal{M}^{2n}]$, $n \geqslant 1$.*

Theorem 39. *The bordism class $[\mathcal{M}^{2n}] \in \Omega^U_{2n}$ is represented by the submanifold $\mathcal{M}^{2n} \subset (\mathbb{C}P^1_*)^{n+1}$ dual to the complex line bundle $\eta_1 \otimes \cdots \otimes \eta_{n+1} \to (\mathbb{C}P^1_*)^{n+1}$.*

Proof. For any n, there is an isomorphism $U_{2n+2}(\mathbb{C}P^\infty) \simeq U_{2n+2}(\mathbb{C}P^{2n+2})$. From the definition of the multiplication in the bordism groups and the construction of the map $\varphi\colon J_{n+1}(S^2) \to \mathbb{C}P^{n+1}$, we obtain that the bordism class $\varphi_* w_{2n+2}$ is realized by the map $f_{n+1}\colon (\mathbb{C}P^1_*)^{n+1} \to \mathbb{C}P^{n+1}$ such that $f_{n+1}^* \eta = \eta_1 \otimes \cdots \otimes \eta_{n+1}$.

According to Corollary 38, we have $[\mathcal{M}^{2n}] = (\varphi_* w_{2n+2}, u)$. Now using the properties of the $\frown$-product

$$\frown\colon U_{2n+2}(\mathbb{C}P^{2n+2}) \otimes U^2(\mathbb{C}P^{2n+2}) \to U_{2n}(\mathbb{C}P^{2n+2}),$$

we complete the proof of the theorem. □

9. The signature of partially framed manifolds.

In this section, we introduce the concept of partially framed U- and Sp-manifolds and obtain results on the divisibility of their signature. More about partially framed U-manifolds see [21].

9.1. **The signature of U-manifolds.**

We say that a U-manifold M^n has a stable structure of a partially (with defect 1) framed U-manifold if there exists a one-dimensional complex bundle $\xi \to M^n$ such that the bundle $\mathcal{T}M^n \oplus \xi$ is stably trivial. For brevity, we refer to such a U-manifold as a $(fr(1), U)$-manifold.

Clearly, the bundle ξ defining a $(fr(1), U)$-structure on a U-manifold M^n is isomorphic to the complex conjugate of the determinant bundle $\det \mathcal{T}M^n$. In particular, a $(fr(1), U)$-manifold M^n is a framed manifold when it is an SU-manifold.

We introduce the bordism group $\Omega_n^{(fr(1),U)}$ of manifolds with a $(fr(1), U)$-structure, and denote by $\widehat{\Omega}_n^{(fr(1),U)}$ its canonical image in the complex bordism group Ω_n^U.

Theorem 40.

1. *There are isomorphisms* $\widehat{\Omega}_{2n}^{(fr(1),U)} \simeq \mathbb{Z}$ *and* $\widehat{\Omega}_{2n+1}^{(fr(1),U)} \simeq 0$.
2. *The group* $\widehat{\Omega}_{2n}^{(fr(1),U)}$ *is generated by the bordism class of the manifold* $\mathcal{M}^{2n} \subset (\mathbb{C}P^1_*)^{n+1}$.

The proof of the theorem follows from the fact that the tangent bundle of $(\mathbb{C}P^1_*)^{2n+2}$ is stably trivial, and that the bordism class of $\mathcal{M}^{2n} \subset (\mathbb{C}P^1_*)^{n+1}$ dual to the Euler class of a one-dimensional complex bundle (see Theorem 39).

Corollary 41. *If* $[M^{2n}] \in \widehat{\Omega}_{2n}^{(fr(1),U)}$, *then the signature* $\tau(M^{2n})$ *is divisible by* $\tau(\mathcal{M}^{2n})$.

9.2. **Characteristic classes of quaternionic bundles and the Chen–Dold character** ch_{Sp}**.**

In the theory $Sp^*(\cdot)$, the characteristic classes $p_k^{Sp}(\zeta) \in Sp^{4k}(X)$ of symplectic bundles $\zeta \to X$ are defined. These classes were introduced by Conner and Floyd, see [23]. The classes $p_k^{Sp}(\zeta)$ map to the Borel classes in cohomology under the canonical transformation $\mu_{Sp}^H\colon Sp^*(\cdot) \to H^*(\cdot; \mathbb{Z})$.

As noted above, the Chern–Conner–Floyd classes $c_k^{SO}(\eta) \in SO^{2k}(X)$ of complex vector bundles $\eta \to X$ are defined in the theory $SO^*(\cdot)$. This allows us to introduce the Pontryagin classes $p_k^{SO}(\xi) \in SO^{4k}(X)$ of real vector bundles $\xi \to X$ by the classical formula $p_k^{SO}(\xi) = (-1)^k c_{2k}^{SO}(c\xi)$. In the case when $\zeta \to X$ is a quaternionic bundle, we obtain the formula $\mu_{Sp}^{SO} p_k^{Sp}(\zeta) = p_k^{SO}(r\xi)$.

The universal bundle $\eta(1) \to \mathbb{C}P^\infty$ *is not orientable* in the theory $Sp^*(\cdot)$; therefore, it is not possible to define the Chern characteristic classes in the theory $Sp^*(\cdot)$. For any complex bundle $\eta \to X$, the bundle $\zeta = \eta \oplus \overline{\eta}$ has the structure of a quaternionic bundle; and there is the identity $\mu_{Sp}^U p_k^{Sp}(\zeta) = \sum\limits_{i+j=k} c_i^U(\eta) c_j^U(\overline{\eta})$.

Let $\zeta \to X$ be a quaternionic bundle and $\dim_{\mathbb{H}} \zeta = n$. Then $\mu_{Sp}^U p_n^{Sp}(\zeta) = c_{2n}^U(\xi)$; but for $k < n$ the identity $\mu_{Sp}^U p_k^{Sp}(\zeta) = c_{2k}^U(\xi)$ does not hold.

We use the general notation Ω_{Sp} for the isomorphic rings $\Omega^*_{Sp} \simeq \Omega_*^{Sp}$. According to the paper of S. P. Novikov [61], there is an isomorphism $\Omega_{Sp} \otimes \mathbb{Z}[1/2] \simeq \mathbb{Z}[1/2][a_4, \ldots, a_{4n}, \ldots]$, $\deg a_{4n} = 4n$. The calculation of the ring $\operatorname{Tors} \Omega_{Sp}$ is a famous unresolved problem in algebraic topology.

We introduce a ring $\Omega_{Sp}(\mathbb{Z}) = \sum\limits_{n \geqslant 0} \Omega_{Sp}^{-4n}(\mathbb{Z})$, where $\Omega_{Sp}^{-4n}(\mathbb{Z})$ is the subgroup of $\Omega_{Sp}^{-4n} \otimes \mathbb{Q}$ consisting of all classes for which all the classical Pontryagin numbers are integral. There is the reduced Chen–Dold character $\widehat{\mathrm{ch}}_{Sp}\colon Sp^*(X) \to H^*(X; \Omega_{Sp}(\mathbb{Z}))$, which is a module homomorphism over the algebra of cohomological operations $\mathcal{A}_{Sp}$ in the theory $Sp^*(\cdot)$. The transformation $\widehat{\mathrm{ch}}_{Sp}$ is completely determined by the series $\widehat{\mathrm{ch}}_{Sp} x \in H^*(\mathbb{H}P^\infty; \Omega_{Sp}(\mathbb{Z})) = \Omega_{Sp}(\mathbb{Z})[[z]]$, where $x = Th(\zeta(1)) = p_1^{Sp}(\zeta(1)) \in Sp^4(\mathbb{H}P^\infty)$, $z = \mu_{Sp}^H x \in H^4(\mathbb{H}P^\infty; \mathbb{Z})$, and $\zeta(1) \to \mathbb{H}P^\infty$ is the universal one-dimensional quaternionic bundle.

We set $\widehat{\Omega}_{Sp} = \Omega_{Sp}/\mathrm{Tors}$ and identify the ring $\widehat{\Omega}_{Sp}$ with its image in Ω_U.

Theorem 42 (V. M. Buchstaber, S. P. Novikov, [13])**.**

There is a formula $\mathrm{ch}_{Sp}\, x = z + \sum\limits_{n\geqslant 1} [\mathcal{N}^{4n}] \frac{z^{n+1}}{(2n+2)!}$, *where* $[\mathcal{N}^{4n}] \in \widehat{\Omega}^{-4n}_{Sp}$; *moreover, for all* $k \geqslant 0$, *there are classes* $[\widetilde{\mathcal{N}}^{8k+4}] \in \Omega_U^{-8k-4}$ *such that* $[\mathcal{N}^{8k+4}] = 2[\widetilde{\mathcal{N}}^{8k+4}]$.

The embedding $S^1 \subset Sp(1)$ defines a bundle $\pi\colon \mathbb{C}P^\infty \to \mathbb{H}P^\infty$ with a fiber $S^3/S^1 \simeq \mathbb{C}P^1_*$. We have $\pi^*\zeta(1) = \eta \oplus \overline{\eta}$. Let us denote by $\psi\colon \Omega S^3 \to \mathbb{H}P^\infty$ the composition $\pi\varphi$. Statement 2 of Theorem 33 implies that the ring $Sp^*(\Omega S^3)$ is a free Ω_{Sp}-module.

In the notation of Theorem 37, we have:

Theorem 43.
1. *The image of the homomorphism* $\mu^U_{Sp}\colon Sp^*(\Omega S^3) \to U^*(\Omega S^3)$ *is the ring* $\widehat{\Omega}_{Sp}[[b_2, \dots, b_{2n}, \dots]]/J$.
2. *There is the formula* $\mu^U_{Sp}\, \psi^*_{Sp}\, x = \psi^*_H\, \widehat{\mathrm{ch}}_{Sp}\, x$, *where* $\mu^U_{Sp}\, \psi^*_{Sp}\, x = 2b_4 + \sum\limits_{n\geqslant 1} [\mathcal{N}^{4n}] b_{4(n+1)}$.
3. *The* $\widehat{\Omega}_{Sp}$*-isomorphism* $\mu^H_{Sp}\colon \widehat{\Omega}_{Sp}[[b_2, \dots, b_{2n}, \dots]] \to \widehat{\Omega}_{Sp}[[\widehat{b}_2, \dots, \widehat{b}_{2n}, \dots]]$ *defines an isomorphism of* $\mathcal{A}_{Sp}$*-modules* $Sp^*(\Omega S^3)/\mathrm{Tors} \to H^*(\Omega S^3; \widehat{\Omega}_{Sp})$.

The proof of this theorem is similar to the proof of Theorem 37.

9.3. Representatives of the coefficients of ch_{Sp}.

Consider the Ω_{Sp}-bilinear scalar product

$$(\cdot,\cdot)\colon Sp_n(\Omega S^3) \otimes Sp^m(\Omega S^3) \to \widehat{\Omega}^{Sp}_{n-m}.$$

Let $\{w_{2n}\}$ and $\{b_{2n}\}$ be as before.

Corollary 44. *There is the formula* $(w_{4n+4}, \mu^U_{Sp}\, \psi^*_{Sp}\, x) = [\mathcal{N}^{4n}]$, $n \geqslant 1$.

Theorem 45. *The bordism class* $[\mathcal{N}^{4n}]$ *is represented by the submanifold* $\mathcal{N}^{4n} \subset (\mathbb{C}P^1_*)^{2n+2}$ *dual to the quaternionic bundle* $\eta \oplus \overline{\eta} \to (\mathbb{C}P^1_*)^{2n+2}$, *where* $\eta = \eta_1 \otimes \dots \otimes \eta_{2n+2}$.

9.4. The signature of Sp-manifolds.

We say that an Sp-manifold M^n has a stable structure of a partially (with defect 1) framed Sp-manifold if there exists an one-dimensional quaternionic bundle $\zeta \to M^n$ such that the bundle $\mathcal{T}M^n \oplus \zeta$ is stably trivial. For brevity, we refer to such an Sp-manifold as a $(fr(1), Sp)$-manifold.

We introduce the bordism group $\Omega^{(fr(1),Sp)}_n$ of manifolds with a $(fr(1), Sp)$-structure, and denote by $\widehat{\Omega}^{(fr(1),Sp)}_n$ its canonical image in the group Ω^U_n.

Theorem 46.
1. *There are isomorphisms* $\widehat{\Omega}^{(fr(1),Sp)}_{4n} \simeq \mathbb{Z}$ *and* $\widehat{\Omega}^{(fr(1),Sp)}_k \simeq 0$ *if* $k \not\equiv 0 \mod 4$.
2. *The group* $\widehat{\Omega}^{(fr(1),Sp)}_{4n}$ *is generated by the bordism class of the manifold* $\mathcal{N}^{4n} \subset (\mathbb{C}P^1_*)^{2n+2}$.
3. $Td(\mathcal{N}^{4n}) = (-1)^n a_n$, *where* $a_{2k-1} = 2$ *and* $a_{2k} = 1$, $k = 1, 2, \dots$
4. $\tau(\mathcal{N}^{4n}) = (-1)^{n+1} \frac{4^{n+2}(4^{n+2}-1)}{2(n+2)} B_{2(n+2)}$, *where* B_k *is the* k*-th Bernoulli number.*

Examples:

(1) $\tau(\mathcal{N}^4) = 16$, and therefore the group $\Omega^{-4}_{Sp} \simeq \mathbb{Z}$ is generated by the bordism class of $\mathcal{N}^4$.
(2) $\tau(\mathcal{N}^8) = 16 \cdot 17$; $\tau(\mathcal{N}^{12}) = 16^2 \cdot 31$.

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