Combinatorial 2-truncated Cubes and Applications

Victor M. Buchstaber and Vadim D. Volodin

Abstract We study a class of simple polytopes, called 2-truncated cubes. These polytopes have remarkable properties and, in particular, satisfy Gal's conjecture. Well-known polytopes (flag nestohedra, graph-associahedra and graph-cubeahedra) are 2-truncated cubes.

1 Introduction

Stasheff polytopes have many geometric realizations which are not affinely equivalent. The starting point of this work is a realization of Stasheff polytopes obtained from cubes by truncations of faces of codimension 2. In the focus of our interest is the family of polytopes that are obtained from cubes by a sequence of truncations of faces of codimension 2. Such polytopes will be called *2-truncated cubes* and a truncation of a face of codimension 2 will be called a *2-truncation*. We will show that 2-truncated cubes have remarkable properties and this family contains classes of polytopes playing important roles in different areas of mathematics. Note that every n-dimensional 2-truncated cube is a simple flag polytope P^n and, moreover, it is an image of the moment map for some smooth toric variety M_P^{2n}. It is well known that the odd Betti numbers of M_P^{2n} are zero and the even Betti numbers are equal to the components of the h-vector of P^n. Truncation of a face of codimension 2 corresponds to a blow-up of the variety M_P^{2n} along a subvariety of complex codimension 2. For the well-known f-, h-, g- and γ-vectors, we explicitly describe their transformation under 2-truncations.

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S. Gal conjectured in [14] that all components of the γ-vector of a flag generalized homological sphere (and, therefore, a γ-vector of a flag simple polytope) are nonnegative. This was a generalization of the Charney-Davis conjecture (see [6]), which was formulated in terms of h-vectors and is equivalent to the nonnegativity of the last component of the γ-vector. In the work of R. Charney and M. Davis and later in the work of S. Gal, the operation dual to 2-truncation was used to support their conjectures. Gal's conjecture became widely known because it is connected not only with the combinatorics of sphere triangulations, but also with problems of differential geometry and the topology of manifolds. This conjecture was proved in special cases.

Using the transformation formula for the γ-vector under a 2-truncation, we obtain a proof of Gal's conjecture for our family. There is a well-known formula in toric geometry, $\sum_{i=0}^{2n}(-1)^i h_i(P^{2n}) = (-1)^n \sigma(M_P^{4n})$, where P^{2n} is the image of the moment map of a smooth toric variety M_P^{4n} and σ is the signature. The left part is equal to the last component $\gamma_n(P^{2n})$ of the γ-vector. Then our result shows that $(-1)^n\sigma(M_P^{4n})$ is nonnegative if the image of the moment map is a 2-truncated cube.

An important class of simple polytopes is that of nestohedra. These polytopes arose in the work of C. De Concini and C. Procesi (see [7]). Nestohedra were constructed as Minkowski sums of certain sets of simplices corresponding to some building set. Attention was drawn to nestohedra due to the work of A. Postnikov, V. Reiner, L. Williams (see [20]), who obtained important results about their combinatorics. In particular, Gal's conjecture was proved for chordal building sets. We show that a nestohedron is a 2-truncated cube if and only if it is a flag polytope. As a corollary, we obtain a proof of Gal's conjecture for the family of all flag nestohedra.

A wide class of flag nestohedra, the graph-associahedra, were introduced by M. Carr and S. Devadoss in [3]. Among them are the Stasheff polytopes (associahedra), Bott-Taubes polytopes (cyclohedra), and the permutohedra. Graph-associahedra can be described as nestohedra, where the building set is constructed in a natural way from a graph. Since every graph-associahedron is a 2-truncated cube, we obtain (see Example 8.9) realizations of Stasheff polytopes that are not equivalent to realizations described in [5], [15] and [20].

S. Fomin and A. Zelevinsky (see [13]) introduced a new class of polytopes corresponding to cluster algebras related to Dynkin diagrams. It was shown by M. Gorsky (see [16]) that the polytopes corresponding to diagrams of the D-series are not nestohedra, but each of them is a 2-truncated cube.

The face lattices of the Stasheff polytopes are the well-known Tamari lattices. Since Stasheff polytopes are 2-truncated cubes, Tamari lattices can be obtained from Boolean lattices by a special operation corresponding to the 2-truncation. As a result, we obtain a family of lattices from Boolean lattices by sequences of these operations.

Stasheff polytopes have extremal properties: their f-, h-, g- and γ- vectors are componentwise minimal in each dimension among the graph-associahedra corresponding to connected graphs. We describe classes of graph-associahedra among which cyclohedra, stellohedra and permutohedra having extremal properties.

We describe geometric operations that transform an n-dimensional graph-associahedron to an $(n+1)$-dimensional one. It allows us to consider series of graph-associahedra and to describe their combinatorics in terms of differential and functional equations for the generating functions of face polynomials. Similar equations were obtained using the ring of simple polytopes (see [1]). For example, the famous Hopf equation describes the generating series of H-polynomials of Stasheff polytopes.

In the work of S.L. Devadoss, T. Heath and W. Vipismakul, it was shown that some moduli spaces of marked bordered surfaces have a polytopal stratification. In [9] a class of simple polytopes, called graph-cubeahedra, were introduced. They generalize polytopes associated with moduli spaces. This class contains some well-known series (for example, associahedra) and a new sequence of polytopes called halohedra.

We introduce a class of simple n-polytopes $NP(P,B)$, called nested polytopes. A member is defined by a pair (P,B), where P is a simple n-polytope with fixed order of facets and B is a building set on $[n]$. We show that the nested polytope $NP(P,B)$ is flag if both the polytope P and the nestohedron P_B are flag. The nested polytope $NP(P,B)$ is a 2-truncated cube if P is a 2-truncated cube and P_B a flag polytope.

We show that graph-cubeahedra are a special case of nested polytopes $NP(P,B)$, where P is the n-cube and B is a graphical building set. As a corollary, we obtain that any graph-cubeahedron is a 2-truncated cube.

2 Simple polytopes

A convex n-dimensional polytope P is called *simple* if each vertex belongs to exactly n facets.

A simple polytope P is called *flag* if every collection of its pairwise intersecting faces has a nonempty intersection.

2.1 Enumerative polynomials

Let f_i be the number of i-dimensional faces of an n-dimensional polytope P. The vector $(f_0,\dots,f_n)$ is called the f-vector of P. The F-polynomial of P is defined by

$$F(P)(\alpha,t) = \alpha^n + f_{n-1}\alpha^{n-1}t + \cdots + f_1\alpha t^{n-1} + f_0 t^n,$$

The h-vector $(h_0,\dots,h_n)$ and the H-polynomial of P are defined by

$$H(P)(\alpha,t) = F(P)(\alpha-t,t) = h_0\alpha^n + h_1\alpha^{n-1}t + \cdots + h_{n-1}\alpha t^{n-1} + h_n t^n .$$

The g-vector of a simple polytope P is the vector $(g_0,g_1,\dots,g_{[\frac{n}{2}]})$, where $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i > 0$.

The following formulas connect the f-, h- and g-vectors:

$$f_i(P) = \sum_{j=i}^{n} \binom{j}{i} h_{n-j}(P); \qquad h_i(P) = \sum_{j=0}^{i} g_j(P)\,. \tag{1}$$

According to the Dehn-Sommerville equations (see [22]), $H(P)$ is symmetric for any simple polytope (see [1]). It can therefore be expressed as a polynomial of $\alpha+t$ and αt:

$$H(P) = \sum_{i=0}^{[\frac{n}{2}]} \gamma_i (\alpha t)^i (\alpha+t)^{n-2i}\,. \tag{2}$$

The γ-vector of P is the vector $(\gamma_0, \gamma_1, \dots, \gamma_{[\frac{n}{2}]})$. The γ-polynomial of P is defined by

$$\gamma(P)(\tau) = \gamma_0 + \gamma_1 \tau + \dots + \gamma_{[\frac{n}{2}]} \tau^{[\frac{n}{2}]}\,.$$

The next formula (see [1]) connects the g- and the γ-vectors.

$$g_i(P) = (n-2i+1) \sum_{j=0}^{i} \frac{1}{n-i-j+1} \binom{n-2j}{i-j} \gamma_j(P)\,. \tag{3}$$

Proposition 2.1. *Let $\gamma_i(P_1) \le \gamma_i(P_2), i = 0, \dots, [\frac{n}{2}]$, where P_1 and P_2 are simple n-polytopes. Then*

1) $g_i(P_1) \le g_i(P_2)$,
2) $h_i(P_1) \le h_i(P_2)$,
3) $f_i(P_1) \le f_i(P_2)$.

Proof. This follows from the nonnegativity of the coefficients in (1) and (3) □

Gal's conjecture (see [14]) in the case of convex polytopes can be formulated as follows.

Conjecture 2.2. *Any flag simple n-polytope P satisfies $\gamma_i(P^n) \ge 0, i = 0, \dots, [\frac{n}{2}]$.*

3 Class of 2-truncated cubes

Truncation of faces plays a key role in this work. We will use it as a combinatorial operation (dual to stellar subdivision) and as a geometric operation. We consider the combinatorial construction first, because it is stricter.

The stellar subdivision of a complex K over a simplex $\sigma \in K$ is denoted by $(v_0, \sigma)K$, where v_0 is a new vertex. The link of the simplex $\sigma \in K$ is denoted by $\mathrm{lk}(\sigma, K)$. The join of complexes K and L is denoted by $K \star L$. We will use $\approx$ to indicate combinatorial equivalence.

Recall that, for every polytope P, there exists a *dual polytope* P^*. Its partially ordered set of faces is inverse to the partially ordered set of faces of P. If P is simple, then P^* is simplicial (all its faces are simplices). If P is simplicial, then P^* is simple.

Definition 3.1. We say that a simple polytope Q is obtained from a simple polytope P by truncation of the face $G \subset P$, if the simplicial complex ∂Q^* is obtained from the simplicial complex ∂P^* by stellar subdivision over the simplex σ_G corresponding to the face G, i.e., if $\partial Q^* = (v_0, \sigma_G)\partial P^*$. The polytope Q has a facet corresponding to the vertex $v_0 \in \partial Q^*$ (added to ∂P^*).

Geometric realization of the truncation. Let $P \subset \mathbb{R}^n$ be a simple polytope containing 0 in its relative interior, and let G be a face of P. Let $l_G \in \mathbb{R}^{n*}$ be a linear function such that $l_G(P) \leq 1$ and $\{x \in P : l_G x = 1\} = G$. The polytope Q, obtained from P by truncation of the face G, can be realized as $Q = \{x \in P : l_G x \leq 1 - \varepsilon\}$. Here $\varepsilon > 0$ is so small that all the vertices of P, except the vertices of G, satisfy $l_G x < 1 - \varepsilon$. Informally, the polytope Q is obtained from P by shifting the support hyperplane of G inside the polytope P. The new facet F_0 of the polytope Q, corresponding to a new vertex $v_0 \in \partial Q^*$, lies in the hyperplane of the section. We will call it the *section facet* F_0.

Remark 3.2. Let P be a simple polytope and the face $G \subset P$ correspond to the simplex $\sigma_G \subset \partial P^*$. Then the complexes ∂G^* and $\mathrm{lk}(\sigma_G, \partial P^*)$ are isomorphic.

Proposition 3.3. *Let the polytope Q be obtained from the simple n-polytope P by truncation of the face G of dimension k. Then the section facet F_0 is combinatorially equivalent to $G \times \Delta^{n-k-1}$.*

Proof. Truncation of the face G corresponds to a stellar subdivision, so we have $\partial Q^* = (v_0, \sigma_G)\partial P^*$. From the properties of stellar subdivisions, we obtain

$$\mathrm{lk}(v_0, \partial Q^*) \simeq \mathrm{lk}(\sigma_G, \partial P^*) \star \partial \Delta^{n-k-1} .$$

The proof is completed by

$$\partial F_0^* \simeq \mathrm{lk}(v_0, \partial Q^*) \simeq \partial G^* \star \partial \Delta^{n-k-1} \simeq \partial (G \times \Delta^{n-k-1})^* . \qquad \square$$

Proposition 3.4. *Let the polytope Q be obtained from the simple n-polytope P by truncation of a face G of dimension k. Then*

1) $H(Q) = H(P) + \alpha t H(G) H(\Delta^{n-k-2})$,
2) $\gamma(Q) = \gamma(P) + \tau \gamma(G) \gamma(\Delta^{n-k-2})$.

Proof. The truncation removes the face G and creates the face $G \times \Delta^{n-k-1}$, so that

$$F(Q) = F(P) + tF(G)F(\Delta^{n-k-1}) - t^{n-k}F(G) .$$

Thus

$$\begin{aligned}
H(Q) &= H(P) + tH(G)H(\Delta^{n-k-1}) - t^{n-k}H(G) \\
&= H(P) + tH(G)\Big(\sum_{i=0}^{n-k-1} \alpha^i t^{n-i} - t^{n-k-1}\Big) \\
&= H(P) + \alpha t H(G)\left(\sum_{i=0}^{n-k-2} \alpha^i t^{n-i}\right) = H(P) + \alpha t H(G) H(\Delta^{n-k-2}),
\end{aligned}$$

and, moreoever,

$$H(Q) = \sum_{i=0}^{[\frac{n}{2}]} \gamma_i(P)(\alpha t)^i(\alpha+t)^{n-2i}$$

$$+ \alpha t \left(\sum_{i=0}^{[\frac{k}{2}]} \gamma_i(G)(\alpha t)^i(\alpha+t)^{k-2i}\right)\left(\sum_{j=0}^{[\frac{n-k-2}{2}]} \gamma_j(\Delta^{n-k-2})(\alpha t)^j(\alpha+t)^{n-k-2-2j}\right)$$

$$= \sum_{i=0}^{[\frac{n}{2}]} \gamma_i(P)(\alpha t)^i(\alpha+t)^{n-2i} + \sum_{i=0}^{[\frac{k}{2}]} \sum_{j=0}^{[\frac{n-k-2}{2}]} \gamma_i(G)\gamma_j(\Delta^{n-k-2})(\alpha t)^{i+j+1}(\alpha+t)^{n-2(i+j+1)}.$$

Then, by definition of the γ-vector, $\gamma_i(Q) = \gamma_i(P) + \sum_{p+q=i-1} \gamma_p(G)\gamma_q(\Delta^{n-k-2})$. The proof is now complete. □

Definition 3.5. A truncation of a face of codimension 2 will be called a 2-*truncation*. A combinatorial polytope, obtained from a cube by 2-truncations, will be called a 2-*truncated cube*.

The following is the dual of a result that can be found in [14].

Corollary 3.6. *Let the polytope Q be obtained from a simple polytope P by 2-truncation of the face G. Then*

1) $H(Q) = H(P) + \alpha t H(G)$,
2) $\gamma(Q) = \gamma(P) + \tau\gamma(G)$.

Remark 3.7. Let K be a simplicial complex and $K' = (v_0, A)K$ its stellar subdivision over $A = \{v_1, \dots, v_k\} \in K$. An arbitrary set V of vertices of K' forms a simplex iff one of the following conditions holds:

a) $v_0 \notin V$ and $\{v_1, \dots, v_k\} \not\subseteq V$ and $V \in K$,
b) $v_0 \in V$ and $\{v_1, \dots, v_k\} \not\subseteq V$ and $\{v_1, \dots, v_k\} \cup (V \setminus \{v_0\}) \in K$.

Lemma 3.8. *Any* 2-*truncation keeps flagness.*

Proof. Let Q be obtained from P by truncation of the face $G = F_1 \cap F_2$. Then $\partial Q^* = (v_0, \sigma_G)\partial P^*$, where $\sigma_G = \{v_1, v_2\}$, and v_1, v_2 are the vertices corresponding to facets F_1, F_2. Let the vertices $V \subset \partial Q^*$ be pairwise adjacent. Note that one of the vertices v_1, v_2 is not contained in V. The vertices $V \setminus \{v_0\}$ are pairwise adjacent in the complex ∂P^*, and then $V \setminus \{v_0\} \in \partial P^*$. If $v_0 \notin V$, then $V \in \partial Q^*$ according to a) of Remark 3.7. If $v_0 \in V$, then $V \setminus \{v_0\} \in \mathrm{lk}(v_0, \partial Q^*) = \mathrm{lk}(\{v_1, v_2\}, \partial P^*)$, hence $V \in \partial Q^*$ according to b) of Remark 3.7. □

Proposition 3.9. *Every face of a 2-truncated cube is a 2-truncated cube.*

Proof. We show that if P is a 2-truncated cube, then all the facets of P are 2-truncated cubes. The proof is by induction on the number of truncated faces. Let the polytope Q be obtained from a 2-truncated cube P by 2-truncation of a face G of codimension 2. Then the section facet has the form $F_0 \approx G \times I$, and is thus a 2-truncated cube by the induction assumption. Every other facet F' of the polytope Q is either some facet of P, or obtained from some facet F'' of P by 2-truncation of a face $G' \subseteq F''$. $\square$

Proposition 3.10. *Every 2-truncated cube P satisfies $\gamma_i(P) \geq 0$, i.e., Gal's conjecture holds for 2-truncated cubes.*

Proof. According to Proposition 3.9, each face of a 2-truncated cube is a 2-truncated cube. We can therefore complete the proof by induction on the dimension of P, using the formula $\gamma(Q) = \gamma(P) + \tau\gamma(G)$. $\square$

Definition 3.11. The convex polytope $P \subset \mathbb{R}^n$ is called a *Delzant polytope* if at each vertex the normal vectors of the facets through the vertex can be chosen to form a $\mathbb{Z}$-basis for the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Note that the required normal vectors can be chosen as outer integer prime normals to the corresponding facets, i.e., integer vectors directed outside of P having coprime components.

Proposition 3.12. *Let $P \subset \mathbb{R}^n$ be a Delzant polytope and $G = F_1 \cap \cdots \cap F_k$ a face. Let the polytope Q be obtained from P by truncation of G in such a way that the outer normal to the section facet F_0 is $v_0 = v_1 + \cdots + v_k$, where v_i are the outer integer prime normals to the facets F_i. Then the polytope Q is a Delzant polytope.*

Proof. It is sufficient to prove that, for each vertex $q \in F_0$ of the polytope Q, outer integer prime normals containing q form a basis of the integer lattice. By Remark 3.7, $q = F_0 \cap F_1 \cap \cdots \cap \widehat{F_i} \cap \cdots \cap F_k \cap G'$, where F_i is omitted, and the intersection $G \cap G'$ is a vertex in P (i.e., F_i is replaced by F_0 in the expression $F_1 \cap \cdots \cap F_k \cap G'$). Without loss of generality, we can assume $i = 1$.

Let $v_1, \ldots, v_k$ be outer integer prime normals to facets $F_1, \ldots, F_k$ and $v_{k+1}, \ldots, v_n$ outer integer prime normals to facets containing G'. Then $\det(v_0, v_2, \ldots, v_n) = \det(v_1 + \cdots + v_k, v_2, \ldots, v_n) = \det(v_1, v_2, \ldots, v_n) = \pm 1$. $\square$

Every Delzant n-polytope P^n with m facets has a canonical characteristic function $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^n$ (see [2]), which is a linear map given by the $n \times m$ matrix A which has as its j-th column the vector of components of the outer integer prime normal to the j-th facet F_j. Using Proposition 3.12, we can explicitly construct the matrix A for any 2-truncated cube P^n with m facets. The n-cube is realized by $-1 \leq x_i \leq 1, i = 1, \ldots, n$. Therefore the first n columns form the identity matrix E, the next n columns form the matrix $-E$, and every other column has the form $A_j = A_{j_1} + A_{j_2}$, where $j_1, j_2 < j$ and F_j is the section facet appearing in the $(j-2n)$-th step after truncation of the face $F_{j_1} \cap F_{j_2}$.

4 Smooth toric varieties over 2-truncated cubes

Every 2-truncated cube is a Delzant polytope, so it is the image of the moment map for some smooth toric variety. There is a well-known formula in toric geometry.

Theorem 4.1 ([17, Theorem 3.12]). *If the n-polytope P^n is the image of the moment map for a smooth toric variety M_P^{2n}, then*

$$\sigma(M_P^{2n}) = \sum_{k=0}^{n}(-1)^k h_k(P^n)\,.$$

The signature $\sigma(M_P^{2n})$ is zero for odd-dimensional polytopes. Consider a polytope P^{2n} and the corresponding toric variety M_P^{4n}. Setting $\alpha = 1$ and $t = -1$ in formula (2), we obtain

$$\sigma(M_P^{4n}) = \sum_{k=0}^{2n}(-1)^k h_k(P^{2n}) = (-1)^n \gamma_n(P^{2n})\,. \tag{4}$$

Corollary 4.2. $(-1)^n\sigma(M_P^{4n}) \geq 0$ *for every 2-truncated cube P^{2n}.*

Corollary 4.3. $\sigma(M_P^{4n}) = 0$ *for every 2-truncated cube P^{2n} with less than $5n$ facets.*

Proof. The assumption is equivalent to the number of 2-truncations being less than n. By Corollary 3.6, we have $\gamma(Q) = \gamma(P) + \tau\gamma(G)$, where Q is obtained from P by a 2-truncation of the face G. Then, using induction on the dimension of the polytope and the number of facets, we obtain the stated result. □

5 Small covers of 2-truncated cubes

In [8], the notion of a *small cover* M^n of a simple n-polytope P^n with mod 2 characteristic function on it was introduced. This is a smooth manifold with $\mathbb{Z}_2^n$ action (locally isomorphic to the standard representation of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$) such that $M^n/\mathbb{Z}_2^n \simeq P^n$. The small cover M^n of any 2-truncated cube P^n is an Eilenberg-Mac Lane space $K(G,1)$ (see [18], Proposition 4.10), since 2-truncated cubes are flag. The group G can be determined (see [8], Corollary 4.5) in terms of a right-angled Coxeter group and the characteristic function of P^n, which is explicitly constructed in §3. It was proved (see [8], Theorem 3.1) that mod 2 Betti numbers of M^n are equal to the components of the h-vector of P^n. For P^{2n-1}, the Euler characteristic of the small cover M^{2n-1} is zero. Setting $\alpha = 1$ and $t = -1$ in (2), we obtain the following formula for the Euler characteristic of the small cover M^{2n} of P^{2n}.

$$\chi(M^{2n}) = \sum_{k=0}^{2n}(-1)^k h_k(P^{2n}) = (-1)^n \gamma_n(P^{2n})\,. \tag{5}$$

We note that the Euler characteristic of the small cover of P is equal to the signature of the smooth toric variety associated with P.

Corollary 5.1. $(-1)^n\chi(M^{2n}) \geq 0$ *for every* 2*-truncated cube* P^{2n}.

Using Corollary 4.3, we obtain the following result.

Corollary 5.2. $\chi(M^{2n}) = 0$ *for every* 2*-truncated cube* P^{2n} *with less than* $5n$ *facets.*

6 Nestohedra and graph-associahedra

In this section we recall some well-known facts about nestohedra (see [12], [19], [21]).

Notation 6.1. By $[n]$ and $[i,j]$ we denote the sets $\{1,\dots,n\}$ and $\{i,\dots,j\}$, respectively.

Definition 6.2. A collection B of nonempty subsets of $[n+1]$ is called a *building set* on $[n+1]$ if the following conditions hold:

1) If $S_1, S_2 \in B$ and $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \in B$;
2) $\{i\} \in B$ for every $i \in [n+1]$.

The building set B is *connected* if $[n+1] \in B$.

Remark 6.3. It is often convenient to consider an arbitrary set A, $|A| = n+1$, instead of the set $[n+1]$. Building sets are considered up to the following equivalence: a building set B_1 on A_1 is equivalent to a building set B_2 on A_2 if there exists a bijection $\sigma\colon A_1 \to A_2$ which induces a bijection between B_1 and B_2. We will assume that $A = [n+1]$, if not specified else.

The *restriction* of a building set B to $S \in B$ is the following building set on S:

$$B|_S = \{S' \in B\colon S' \subseteq S\}\,.$$

The *contraction* of a building set B with respect to $S \in B$ is the following building set on $[n+1] \setminus S$:

$$B/S = \{S' \subseteq [n+1] \setminus S\colon S' \in B \text{ or } S' \cup S \in B\} = \{S' \setminus S, S' \in B\}\,.$$

The *product* $B_1 \cdot B_2 = B_1 \sqcup B_2$ of building sets B_1 and B_2 on A and B, where $A \cap B = \emptyset$, is the building set on $A \sqcup B$ consisting of all elements of both building sets.

Remark 6.4. Any building set B on $[n+1]$ is the product of connected building sets: $B = B_1 \sqcup \dots \sqcup B_k$, where B_i is a connected building set on A_i. The sets A_i are maximal in B (i.e., there is no $A_i' \in B$ containing A_i), and $B_i = B|_{A_i}$. We will denote the collection $\{A_i\}$ by $B_{\max}$. If B is connected, then $B_{\max} = \{[n+1]\}$.

Recall that a graph is called *simple* if it has no loops or multiple edges.

Definition 6.5. Let Γ be a simple graph on the node set $[n+1]$. The *graphical building set* $B(\Gamma)$ is the collection of nonempty subsets $S \subseteq [n+1]$ such that the induced subgraph $\Gamma|_S$ on the node set S is connected.

Remark 6.6. A building set $B(\Gamma)$ is connected iff Γ is connected.

Remark 6.7. Let Γ be a connected graph on $[n+1]$ and $S \in B(\Gamma)$. Then $B|_S$ and B/S are both graphical building sets corresponding to connected graphs $\Gamma|_S$ and Γ/S. The node set of Γ/S is the set $[n+1] \setminus S$. Vertices v and w are adjacent in Γ/S if they are either adjacent in Γ, or if they are both adjacent to some vertices from S in Γ.

Let M_1 and M_2 be subsets of $\mathbb{R}^n$. The *Minkowski sum* of M_1 and M_2 is the following subset of $\mathbb{R}^n$:

$$M_1 + M_2 = \{x \in \mathbb{R}^n : x = x_1 + x_2,\ \ x_1 \in M_1,\ x_2 \in M_2\}\,.$$

If M_1 and M_2 are convex polytopes, so is $M_1 + M_2$.

Definition 6.8. Let e_i be the endpoints of the basis vectors of $\mathbb{R}^{n+1}$. We define the *nestohedron* P_B corresponding to the building set B as follows,

$$P_B = \sum_{S \in B} \Delta^S, \text{ where } \Delta^S = \operatorname{conv}\{e_i, i \in S\}\,.$$

If $B(\Gamma)$ is a graphical building set, then $P_\Gamma := P_{B(\Gamma)}$ is called a *graph-associahedron.*

Definition 6.9. A building set B is called *flag* if the corresponding nestohedron P_B is a flag polytope.

Example 6.10. We are especially interested in the following series of graph-associahedra:

- Let L_{n+1} be the path graph on $[n+1]$. Then the polytope $P_{L_{n+1}}$ is called *associahedron* (Stasheff polytope) and denoted by As^n.
- Let C_{n+1} be the cyclic graph on $[n+1]$. Then the polytope $P_{C_{n+1}}$ is called *cyclohedron* (Bott-Taubes polytope) and denoted by Cy^n.
- Let K_{n+1} be the complete graph on $[n+1]$. Then the polytope $P_{K_{n+1}}$ is called *permutohedron* and denoted by Pe^n.
- Let $K_{1,n}$ be the complete bipartite graph on $[n+1]$. Then the polytope $P_{K_{1,n}}$ is called *stellohedron* and denoted by St^n.

Proposition 6.11 (cf. [12, §2]). *Let B be a connected building set on $[n+1]$. Then*

1) *the nestohedron P_B is a simple n-polytope given by the intersection of the hyperplane $H = \{\sum_{i=1}^{n+1} x_i = |B|\}$ with half-spaces $H_S = \{\sum_{i \in S} x_i \geq |B|_S|\}$, where $S \in B \setminus [n+1]$;*
2) *every facet of P_B has the form $F_S = P_B \cap \partial H_S$, where $S \in B \setminus [n+1]$, and it is combinatorially equivalent to the nestohedron $P_{B|_S} \times P_{B/S}$.*

Proposition 6.12 ([20, Corollary 7.2]). *Every graph-associahedron is a flag polytope, i.e., every graphical building set is flag.*

Definition 6.13. Let B be a building set on $[n+1]$. The collection $\mathscr{S} = \{S_1, \dots, S_k\} \subseteq B$ is called a *nested set* if the following conditions hold:

1) $\forall S_i, S_j$: either $S_i \subset S_j$, or $S_i \supset S_j$, or $S_i \cap S_j = \emptyset$,
2) $\forall S_{i_1}, \dots, S_{i_p}$ such that $S_{i_j} \cap S_{i_l} = \emptyset$: $S_{i_1} \sqcup \dots \sqcup S_{i_p} \notin B$.

Definition 6.14. Let B be a building set on $[n+1]$. The *nested set complex* $\mathscr{N}(B)$ is the simplicial complex on the node set $B \setminus B_{\max}$ consisting of all the nested sets $\mathscr{S} \subseteq B \setminus B_{\max}$.

The next proposition allows us to describe the combinatorics of P_B in terms of elements of a building set B.

Proposition 6.15 ([19, Theorem 7.4], [12, Theorem 3.14]). *Let B be a building set on $[n]$. The nestohedron P_B is a simple polytope of dimension $n - |B_{\max}|$ and the simplicial complexes ∂P_B^* and $\mathscr{N}(B)$ are isomorphic. The facets $F_{S_1}, \dots, F_{S_k}$ of the polytope P_B have a nonempty intersection iff $\{S_1, \dots, S_k\} \in \mathscr{N}(B)$.*

Remark 6.16. If $B = B_1 \sqcup \dots \sqcup B_k$, then $\mathscr{N}(B) \simeq \mathscr{N}(B_1) \star \dots \star \mathscr{N}(B_k)$ and $P_B \approx P_{B_1} \times \dots \times P_{B_k}$.

Notation 6.17. Occasionally, we will write "facet S" instead of "facet F_S".

7 Building set as a structure

Let us show (see Corollary 7.6) that every nestohedron is combinatorially equivalent to some nestohedron corresponding to a connected building set. Since we consider building sets up to equivalence, $[k]$ and $[l]$ will eventually denote disjoint sets consisting of k, respectively l elements.

Construction 7.1 ([10]). Let $B_1, \dots, B_{n+1}$ be connected building sets on $[k_1]$, …, $[k_{n+1}]$. Then, for every connected building set B on $[n+1]$, there is a connected building set $B(B_1, \dots, B_{n+1})$ on $[k_1] \sqcup \dots \sqcup [k_{n+1}] = [k_1 + \dots + k_{n+1}]$, consisting of elements $S^i \in B_i$ and $\bigsqcup_{i \in S} [k_i]$, where $S \in B$.

Notation 7.2. When $B_1, \dots, B_n$ are singletons, $\{1\}, \dots, \{n\}$, we will simply write $B(1, 2, \dots, n, B_{n+1})$ instead of $B(\{1\}, \{2\}, \dots, \{n\}, B_{n+1})$.

Lemma 7.3 ([10]). *Let $B, B_1, \dots, B_{n+1}$ be connected building sets on $[n+1]$, $[k_1]$, …, $[k_{n+1}]$, and $B' = B(B_1, \dots, B_{n+1})$. Then $P_{B'} \approx P_B \times P_{B_1} \times \dots \times P_{B_{n+1}}$.*

Proof. Consider $B'' = B \sqcup B_1 \sqcup \dots \sqcup B_{n+1}$ and the map $\varphi : B'' \to B'$ defined as follows,

$$\varphi(S) = \begin{cases} S & \text{if } S \in B_i \\ \bigsqcup_{i \in S} [k_i] & \text{if } S \in B\,. \end{cases}$$

Obviously, φ generates a bijection between $B'' \setminus B''_{\max}$ and $B' \setminus [n+1]$. Let $\mathscr{S} \subset B \setminus [n+1]$ and $\mathscr{S}_i \subset B_i \setminus [k_i]$. Notice that $\bigcup_{i=1}^{n+1} \varphi(\mathscr{S}_i) \cup \varphi(\mathscr{S}) \in \mathscr{N}(B')$ iff $\mathscr{S} \in \mathscr{N}(B)$

and $\mathscr{S}_i \in \mathscr{N}(B_i)$. Consequently, $\mathscr{N}(B') \simeq \mathscr{N}(B'') \simeq N(B) \star N(B_1) \star \cdots \star N(B_{n+1})$, and therefore $P_{B'} \approx P_{B''} \approx P_B \times P_{B_1} \times \cdots \times P_{B_{n+1}}$. $\square$

Example 7.4. Let B, B_1, B_2 be building sets equivalent to $\{\{1\},\{2\},\{1,2\}\}$ corresponding to a segment I. Let us describe the building set $B(B_1, B_2)$. In the building set $\{\{a\},\{b\},\{a,b\}\}$, we substitute a by $B_1 = \{\{1\},\{2\},\{1,2\}\}$ and b by $B_2 = \{\{3\},\{4\},\{3,4\}\}$. As a result, we obtain the building set B' on $[4]$, consisting of $\{i\},\{1,2\},\{3,4\},[4]$.

The facet correspondence between the polytopes $P_B \times P_{B_1} \times P_{B_2}$ and $P_{B'}$ is defined as follows,

$$\begin{aligned} &\{1\} \in B_1 \mapsto \{1\} \in B', &&\{2\} \in B_1 \mapsto \{2\} \in B',\\ &\{3\} \in B_2 \mapsto \{3\} \in B', &&\{4\} \in B_2 \mapsto \{4\} \in B',\\ &\{a\} \in B_2 \mapsto \{1,2\} \in B', &&\{b\} \in B_2 \mapsto \{3,4\} \in B'. \end{aligned}$$

Example 7.5. Let $B = \{\{i\},[n+1]\}$ be a building set corresponding to the simplex Δ^n and $B_1, \ldots, B_{n+1}$ be arbitrary connected building sets on $[k_1], \ldots, [k_{n+1}]$. Then $B' = B(B_1, \ldots, B_{n+1}) = (B_1 \sqcup \cdots \sqcup B_{n+1}) \cup [k_1 + \cdots + k_{n+1}]$, and $P_{B'} \approx \Delta^n \times P_{B_1} \times \cdots \times P_{B_{n+1}}$.

Corollary 7.6. *Every nestohedron corresponds to some connected building set, i.e., for each nestohedron P there exists a connected building set B such that $P_B \approx P$.*

Proof. Indeed, any building set B' can be represented as $B_1 \sqcup \cdots \sqcup B_k$, where B_i are connected building sets on $[k_i + 1]$. Define a building set $B'' = B_1(1, \ldots, k_1, B_2) \sqcup B_3 \sqcup \cdots \sqcup B_k$, giving the same polytope. The building set B' is a product (disjoint union) of k connected building sets and B'' is a product of $(k-1)$ connected building sets. Then we apply again a substitution to B'', and so on. At some step we obtain a connected building set B. $\square$

From now on, we can assume without loss of generality that every nestohedron corresponds to a connected building set.

Proposition 7.7. *Let B be a connected building set on $[n+1]$. Then the polytope P_B is flag iff for every element $S \in B$ with $|S| > 1$ there exist elements $S_1, S_2 \in B$ such that $S_1 \sqcup S_2 = S$.*

Proof. Suppose P_B is a flag polytope. Consider an element $S \in B$. Choose $S_1, \ldots, S_k \in B \setminus \{S\}$, such that $S_1 \sqcup \cdots \sqcup S_k = S$, and k minimal among such disjoint representations of S. Note that $\forall J \subset [k], 1 < |J| < k : \bigsqcup_{j \in J} S_j \notin B$, since otherwise k can be decreased. Therefore, $k = 2$. Indeed, if $k > 2$, then the facets $F_{S_1}, \ldots, F_{S_k}$ intersect pairwise, but have empty intersection.

Suppose for each element $S \in B, |S| > 1$, there exist elements $S_1, S_2 \in B$, such that $S_1 \sqcup S_2 = S$. Let $F_{S_1}, \ldots, F_{S_k}$ be the minimal collection of facets that intersect pairwise but have empty intersection. By Proposition 6.15, $S_1 \sqcup \cdots \sqcup S_k = S \in B$. We will find a nontrivial subcollection of facets having empty intersection in the collection $F_{S_1}, \ldots, F_{S_k}$. For that let us find a set $\widetilde{S} \in B|_S$ intersecting more than one S_i, but not intersecting every S_i. Then the collection F_{S_i}, satisfying $S_i \cap \widetilde{S} \neq \emptyset$, will be the desired subcollection, since by definition of a building set

$$\bigsqcup_{S_i : S_i \cap \widetilde{S} \neq \emptyset} S_i \in B\,.$$

By our assumption, $S = S' \sqcup S''$, where $S', S'' \in B$. Choose as S^1 one of the sets S' and S'', intersecting more elements S_i than the other. If S^1 intersects all the sets S_i, then $S^1 = S'^1 \sqcup S''^1$, where $S'^1, S''^1 \in B$. Choose as S^2 one of the sets S'^1 and S''^1, intersecting more elements S_i than the other. If S^2 intersects all the sets S_i, choose S^3 in the same way, and so on. The resulting sequence must be finite. Therefore, in some step we will find the desired set $\widetilde{S} \in B$ as one of the sets S'^i, S''^i. □

The necessity of the condition in Proposition 7.7 has also been proved in [20].

Definition 7.8. A *Hasse diagram* of a partially ordered set $(X, \prec)$ is the oriented graph on the vertex set X, where vertices $x \in X$ and $y \in X$ are connected by an edge from x to y iff $x \prec y$ and no element $z \in X$ with $x \prec z \prec y$ exists.

Every building set B is partially ordered by inclusion. Its Hasse diagram will be called *B-Hasse diagram.*

Proposition 7.9. *Let B be a connected building set on $[n+1]$. The polytope P_B is combinatorially equivalent to a product of simplices iff the B-Hasse diagram is a tree. In this case $P_B \approx \Delta^{k_1-1} \times \cdots \times \Delta^{k_l-1}$, where $k_1, \ldots, k_l$ are numbers of input edges for all internal nodes of the B-Hasse diagram.*

Proof. Let the B-Hasse diagram be a tree, and $k_1, \ldots, k_l$ numbers of input edges for all its internal nodes. We prove that $P_B \approx \Delta^{k_1-1} \times \cdots \times \Delta^{k_l-1}$. For $n = 1$ there is nothing to prove. Let us assume the statement holds for $m < n$. We prove it for $m = n$. Without loss of generality, we can assume that the number of input edges for the maximal vertex $[n+1]$ is k_l and that this vertex is adjacent to the vertices $S_1, \ldots, S_{k_l}$. Therefore $S_1 \sqcup \cdots \sqcup S_{k_l} = [n+1]$. Indeed, for every vertex $\{q\}$ there exists a path to the vertex $[n+1]$, this path necessarily contains some S_i. Then $\{q\} \subset S_i$. If $\{q\} \subset S_i \cap S_j \neq \emptyset$, then there exist pathes from the vertex $\{q\}$ to the vertices S_i and S_j, but S_i and S_j are adjacent to $[n+1]$, so there is a cycle, which is a contradiction (since we consider a tree). Therefore $B = (B|_{S_1} \sqcup \cdots \sqcup B|_{S_{k_l}}) \cup [n+1]$ and the Hasse diagrams of $B|_{S_i}$ are trees, and $k_1, \ldots, k_{l-1}$ are the numbers of all their input edges. Using Example 7.5, and the induction assumption, we obtain $P_B \approx \Delta^{k_l-1} \times P_{B|_{S_1}} \times \cdots \times P_{B|_{S_{k_l}}} \approx \Delta^{k_1-1} \times \cdots \times \Delta^{k_l-1}$.

Let $P_B \approx \Delta^{k_1-1} \times \cdots \times \Delta^{k_l-1}$. We prove that each vertex S has no more than one output edge. This implies that the B-Hasse diagram is a tree. Suppose a vertex S has output edges to vertices $S', S'' \in B$. Then $S' \setminus S'' \neq \emptyset$, $S'' \setminus S' \neq \emptyset$, $S' \cap S'' \supset S$, and thus $F_{S'} \cap F_{S''} = \emptyset$. As a consequence, the facets $F_{S'}$ and $F_{S''}$ correspond to the opposite points of $\Delta^1 = I$ in the product of simplices, i.e., they are the bases of the cylinder P_B. Let us consider the collection $S_1, \ldots, S_k \in B$, such that $S_1 \sqcup \cdots \sqcup S_k = S' \setminus S''$, and k is the minimal among such disjoint representations of the set $S' \setminus S''$. Then $\bigsqcup_{j \in J} S_j \notin B$

$\forall J \subset [k], 1 < |J| < k$, since otherwise k could be decreased. Therefore the side face $\bigcap_{i=1}^{k} F_{S_i}$ does not intersect the base $F_{S''}$, which is a contradiction. □

The last proposition implies the following result.

Corollary 7.10. *Let B be a connected building set on $[n+1]$. The polytope P_B is combinatorially equivalent to a cube iff the B-Hasse diagram is a binary tree.*

The sufficiency of the statement was proved in [20].

8 Flag nestohedra as 2-truncated cubes

We are going to construct a sequence of building sets $B_0 \subset \cdots \subset B_N = B$, where B_0 corresponds to a cube and B_i, $i > 0$, is obtained from B_{i-1} by adding one element S_i. From [11] (Theorem 4.2), it follows that, for connected building sets $B' \subset B''$, the polytope $P_{B''}$ is obtained from $P_{B'}$ by a sequence of truncations. Our goal is to show that if P_B is flag, then B_i can be chosen such that all truncated faces have codimension 2.

Proposition 8.1 ([20, Proposition 7.1]). *If B is a flag building set on $[n+1]$, then there exists a building set $B_0 \subseteq B$ such that P_{B_0} is a combinatorial cube with* $\dim P_{B_0} = \dim P_B$.

Proof. By Remarks 6.4 and 6.16, we need to consider only connected building sets. For $n = 1$, the proposition is true. Assuming that the assertion holds for $m < n$, we shall prove it for $m = n$. By Proposition 7.7, we have $[n+1] = S_1 \sqcup S_2$, where $S_1, S_2 \in B$. By the induction assumption, the building sets $B|_{S_1}$ and $B|_{S_2}$ have subsets B_1 and B_2 corresponding to cubes. The building set $B_0 = (B_1 \sqcup B_2) \cup [n+1]$ is the desired one (see Example 7.5). □

Now we determine which faces have to be truncated in order to obtain $P_{B''}$ from $P_{B'}$, where $B' \subset B''$ are connected building sets.

Construction 8.2 (Decomposition of $S \in B_1$ by elements of B_0). Let B_0 and B_1 be building sets on $[n+1]$, $B_0 \subset B_1$, and $S \in B_1$. We define a decomposition of S by elements of B_0 as $S = S_1 \sqcup \cdots \sqcup S_k$, $S_j \in B_0$, where k is minimal among such disjoint representations of S. Denote the collection $S_1, \ldots, S_k$ by $B_0(S)$. One can see that this decomposition exists and is unique.

Lemma 8.3. *If two simplicial complexes $K \subseteq L$ are the boundaries of simplicial n-polytopes, then $K = L$.*

Proof. The lemma holds for $n = 1$. Assuming it holds for $m < n$, we shall prove it for $m = n$. Let us assume that $L \setminus K \neq \emptyset$ and choose a simplex $A \in L \setminus K$, containing a vertex $v \in K$ (which exists since L is connected). The complexes $\mathrm{lk}(v, K)$ and $\mathrm{lk}(v, L)$ are the boundaries of simplicial $(n-1)$-polytopes and $\mathrm{lk}(v, K) \subsetneq \mathrm{lk}(v, L)$, since $A \setminus \{v\} \in \mathrm{lk}(v, L) \setminus \mathrm{lk}(v, K)$. This contradicts the induction assumption. □

The next lemma can be extracted from [11], Theorem 4.2. For convenience, we provide a simpler proof in the present more restricted context.

Lemma 8.4. *Let B_0 and B_1 be connected building sets on $[n+1]$ and $B_0 \subset B_1$. Then $\mathcal{N}(B_1)$ is obtained from $\mathcal{N}(B_0)$ by stellar subdivisions over simplices $\sigma_i = \{S_1^i, \dots, S_{k_i}^i\}$, corresponding to decompositions $S^i = S_1^i \sqcup \dots \sqcup S_{k_i}^i \in B_1 \setminus B_0$ of elements S^i, numbered in any order that is reverse to inclusion (i.e., $S^i \supseteq S^{i'} \Rightarrow i \leq i'$).*

Proof. We use induction on the number $N = |B_1| - |B_0|$. For $N = 1$, we have $B_1 = B_0 \cup \{S^1\}$. We show that $K \simeq \mathcal{N}(B_1)$, where $K = (S^1, B_0(S^1)).\mathcal{N}(B_0)$. By Lemma 8.3, it is sufficient to prove that $\mathcal{N}(B_1) \subseteq K$. Let $\mathcal{S} \in \mathcal{N}(B_1)$. Note that $B_0(S^1) = \{S_1^1, \dots, S_{k_1}^1\} \nsubseteq \mathcal{S}$. If $S^1 \notin \mathcal{S}$, then $\mathcal{S} \in \mathcal{N}(B_0)$, therefore $\mathcal{S} \in K$ according to a) of Remark 3.7. If $S^1 \in \mathcal{S}$, then for each element $S \in \mathcal{S}$, intersecting with S^1, either $S^1 \subset S$, or $S^1 \supset S$ (otherwise $S \cup S^1 \in B_1$). In the last case, $\exists j : S \subset S_j^1$ (otherwise, merging S_j^1, intersecting S, we can decrease k_1, which contradicts the definition of the decomposition of S^1). Consequently, $B_0(S^1) \cup (\mathcal{S} \setminus \{S^1\}) \in \mathcal{N}(B_0)$, thus $\mathcal{S} \in K$ according to b) of Remark 3.7.

Assuming the lemma holds for $M < N$, we prove it for $M = N$. The collection of sets $B_0' = B_0 \cup \{S^1\}$ is a building set. By the induction assumption, $\mathcal{N}(B_0')$ is obtained from $\mathcal{N}(B_0)$ by stellar subdivision over the simplex corresponding to the decomposition of S^1, and $\mathcal{N}(B_1)$ is obtained from $\mathcal{N}(B_0')$ by a sequence of stellar subdivisions over simplices corresponding to decompositions of S^i, $i = 2, \dots, N$. This completes the proof. □

Corollary 8.5. *Let B_0 and B_1 be connected building sets on $[n+1]$, and $B_0 \subset B_1$. Then P_{B_1} is obtained from P_{B_0} by truncations of faces $G^i = \bigcap_{j=1}^{k_i} F_{S_j^i}$, corresponding to decompositions of $S^i = S_1^i \sqcup \dots \sqcup S_{k_i}^i \in B_1 \setminus B_0$, numbered in any order that is reverse to inclusion (i.e., $S^i \supseteq S^{i'} \Rightarrow i \leq i'$).*

Theorem 8.6. *Let B_1 and B_2 be flag connected building sets on $[n+1]$ and $B_1 \subset B_2$. Then $\mathcal{N}(B_2)$ is obtained from $\mathcal{N}(B_1)$ by a sequence of subdivisions of edges (or, equivalently, P_{B_2} is obtained from P_{B_1} by a sequence of 2-truncations).*

Proof. We will prove the following: if B_1 and B_3 are connected flag building sets on $[n+1]$ and $B_1 \subsetneq B_3$, then there exists a building set B_2 such that $B_1 \subsetneq B_2 \subseteq B_3$ and $\mathcal{N}(B_2)$ is obtained from $\mathcal{N}(B_1)$ by a sequence of subdivisions of edges. This then implies the statement in the theorem, since stellar subdivisions over edges are dual to 2-truncations (which keep flagness).

Set $B_2 = \widehat{B_1 \cup \{S\}}$, i.e., the minimal (by inclusion) building set containing $B_1 \cup \{S\}$, where S is the minimal (by inclusion) element of $B_3 \setminus B_1$. By Proposition 7.7, there exist $I, J \in B_3$ such that $I \sqcup J = S$. From the choice of S, it follows that $I, J \in B_1$. It is easy to show that the collection of sets $B_1 \cup \{S' = S_1 \sqcup S_2 \mid S_i \in B_1, I \subseteq S_1, J \subseteq S_2\}$ is a minimal building set containing $B_1 \cup \{S\}$. Then the decomposition of any element $B_2 \setminus B_1$ consists of two elements. Therefore, $\mathcal{N}(B_2)$ is obtained from $\mathcal{N}(B_1)$ by a sequence of subdivisions of edges. □

We conclude that for each flag building set B there exists a sequence of building sets $B_0 \subset B_1 \subset \cdots \subset B_N = B$, where P_{B_0} is a combinatorial cube, $B_i = B_{i-1} \cup \{S_i\}$, and the polytope P_{B_i} is obtained from the polytope $P_{B_{i-1}}$ by a 2-truncation of the face $F_{S_{j_1}} \cap F_{S_{j_2}} \subset P_{B_{i-1}}$, where $S_i = S_{j_1} \sqcup S_{j_2}$, and $S_{j_1}, S_{j_2} \in B_{i-1}$.

Using results of the last two sections, we obtain the following.

Theorem 8.7. *Every flag nestohedron is a 2-truncated cube.*

Theorem 8.8.

1) $\gamma_i(P_B) \geq 0$ *for every flag nestohedron* P_B*, i.e., Gal's conjecture holds for flag nestohedra.*
2) $\gamma_i(P_{B_1}) \leq \gamma_i(P_{B_2})$ *for connected flag building sets* B_1, B_2 *with* $B_1 \subseteq B_2$*. Moreover, equality holds for all* i *iff* $B_1 = B_2$.

Example 8.9. Let us construct a geometric realization of the 3-dimensional Stasheff polytope as a 2-truncated cube. The building set corresponding to As^3 is

$$B = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}\,.$$

In order to obtain As^3 from I^3 by 2-truncations, we have to find a building set $B_0 \subset B$, such that $P_{B_0} \approx I^3$, and to order the elements of $B \setminus B_0$ such that adding a new element to the building set corresponds to a 2-truncation. On the left-hand side of Fig. 1, B_0 consists of $\{i\},\{1,2\},\{3,4\},[4]$. The associahedron P_B is obtained from $P_{B_0} \approx I^3$ by stepwise truncation of the faces $F_{\{1,2\}} \cap F_{\{3\}}, F_{\{2\}} \cap F_{\{3,4\}}, F_{\{2\}} \cap F_{\{3\}}$ in this order. In the drawing, the facets $F_{\{1,2\}}$ and $F_{\{3,4\}}$ are the top and the bottom section facets, respectively, while $F_{\{2\}}$ and $F_{\{3\}}$ are the right and front facets, respectively.

On the right-hand side of Fig. 1, B_0 consists of $\{i\},\{1,2\},\{1,2,3\},[4]$. First we truncate the face $F_{\{2\}} \cap F_{\{3\}}$ of P_{B_0} and derive the new facet $F_{\{2,3\}}$. Then we truncate the faces $F_{\{2,3\}} \cap F_{\{4\}}$ and $F_{\{3\}} \cap F_{\{4\}}$. In the drawing, $F_{\{2\}}, F_{\{3\}}$ and $F_{\{4\}}$ are the front, right and top facets.

In [4] it was shown that the geometric realizations of As^n that appeared in [5], [15] and [20] are not affinely equivalent. None of our realizations above is equivalent to one of those.

9 Recursion formulas

Let J be the building set $\{\{1\},\{2\},\{1,2\}\}$, which corresponds to the interval $P_J \approx I$. Recall that a subset V of vertices of a graph Γ is called a *clique* if the induced subgraph $\Gamma|_V$ is complete.

Construction 9.1. Let Γ_{n+1} be a connected graph on $[n+1]$ and the set $V \subseteq [n]$ of vertices adjacent to the vertex $\{n+1\}$ be a clique. Set $\Gamma_n = \Gamma_{n+1} \setminus \{n+1\}$, i.e., Γ_n is the graph obtained from Γ_{n+1} by deletion of the vertex $\{n+1\}$.

According to Lemma 7.3, the building set $B_1 = J(B(\Gamma_n), n+1) = B(\Gamma_n) \cup \{n+1\} \cup [n+1]$ corresponds to $P_{B_1} \approx P_{\Gamma_n} \times I$: the bottom and top bases of the cylinder

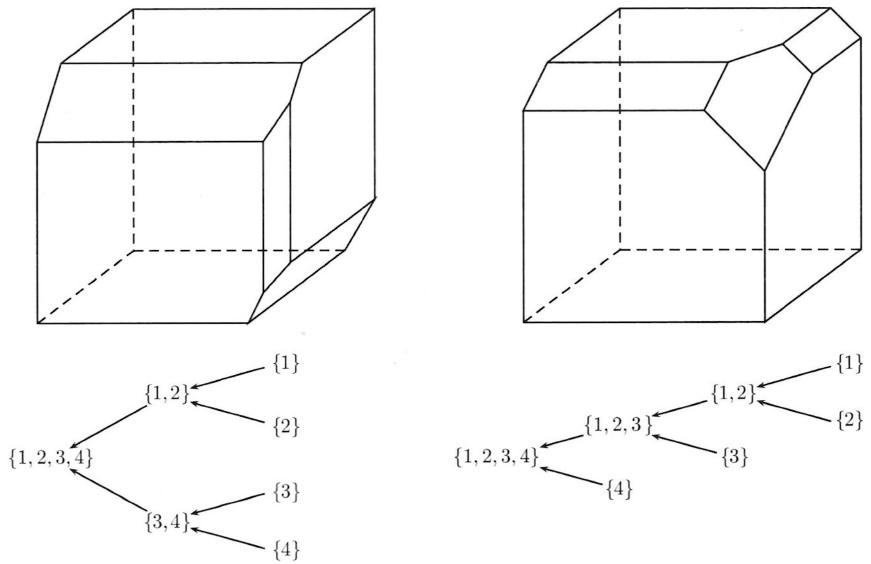

Fig. 1 Realizations of the 3-dimensional Stasheff polytope via two different choices of the building subset B_0 of B, represented by their Hasse diagrams (below the respective polytope).

$P_{\Gamma_n} \times I$ correspond to $[n], \{n+1\} \in B_1$, the side facets correspond to $S \in B(\Gamma_n) \setminus [n] \subset B_1$. Thus, the side facets $F_S, S \in B_1$ of the cylinder $P_{\Gamma_n} \times I$ are naturally identified with the facets $F_S, S \in B(\Gamma_n)$ of the base P_{Γ_n}.

We have $B(\Gamma_{n+1}) \setminus B_1 = \{S \sqcup \{n+1\}, S \in \mathscr{S}\}$, where $\mathscr{S} = \{S \in B(\Gamma_n) \setminus [n] \colon S \cap V \neq \emptyset\}$. By Corollary 8.5, $P_{\Gamma_{n+1}}$ is obtained from $P_{\Gamma_n} \times I$ by truncations of intersections of the top base $F_{\{n+1\}}$ with side facets F_S for $S \in \mathscr{S}$. Since the top base does not change after truncation of any of its facets, the truncated faces have type $P_{\Gamma_n|S} \times P_{\Gamma_n/S}, S \in \mathscr{S}$. By Proposition 3.6, we have

$$\gamma(P_{\Gamma_{n+1}}) = \gamma(P_{\Gamma_n}) + \tau \sum_{S \in \mathscr{S}} \gamma(P_{\Gamma_n|S})\gamma(P_{\Gamma_n/S}), \tag{6}$$

$$H(P_{\Gamma_{n+1}}) = (\alpha + t)H(P_{\Gamma_n}) + \alpha t \sum_{S \in \mathscr{S}} H(P_{\Gamma_n|S})H(P_{\Gamma_n/S}). \tag{7}$$

We required that V is a clique of Γ_{n+1}, because in this case every element of $B(\Gamma_{n+1}) \setminus B_1$ has a decomposition consisting of two elements ($S \sqcup \{n+1\}$, where $\{n+1\}, S \in B_1$), and we know the combinatorial type of the truncated faces of codimension 2.

9.1 Associahedra

Let us apply Construction 9.1 to the path graph L_{n+1}. After deletion of the vertex $\{n+1\}$, we obtain the graph L_n. Here $V = \{n\}$ and $\mathscr{S} = \{[i,n], i = 2,\dots n\}$. Therefore, the truncated faces have type $As^{i-1} \times As^{n-i-1}$, $i = 1,\dots,n-1$, and we obtain the recursion formulas

$$\gamma(As^n) = \gamma(As^{n-1}) + \tau \sum_{i=1}^{n-1} \gamma(As^{i-1})\gamma(As^{n-i-1}), \tag{8}$$

$$H(As^n) = (\alpha+t)H(As^{n-1}) + \alpha t \sum_{i=1}^{n-1} H(As^{i-1})H(As^{n-i-1}). \tag{9}$$

The recursion formulas for associahedra are equivalent to the equations

$$\gamma_{As}(x) = 1 + x\gamma_{As}(x) + \tau x^2 \gamma_{As}^2(x) \qquad \text{where} \qquad \gamma_{As}(x) = \sum_{n=0}^{\infty} \gamma(As^n)x^n,$$

$$H_{As}(x) = (1+\alpha x H_{As}(x))(1+txH_{As}(x)) \quad \text{where} \qquad H_{As}(x) = \sum_{n=0}^{\infty} H(As^n)x^n.$$

The last equation is equivalent to:

$$\frac{xH_{As}(x)}{(1+\alpha x H_{As}(x))(1+txH_{As}(x))} = x.$$

Set $U(x) = xH_{As}(x)$. Then $U(0) = 0, U'(0) = 1$, and

$$\frac{U}{(1+\alpha U)(1+tU)} = x.$$

Applying the classical Lagrange Inversion Formula, we obtain

$$\begin{aligned} U(x) &= -\frac{1}{2\pi i}\oint_{|z|=\varepsilon} \ln\left[1 - \frac{x}{z}(1+\alpha z)(1+tz)\right] dz \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint_{|z|=\varepsilon}\left[\frac{(1+\alpha z)^n(1+tz)^n}{z^n}\right] dz\right)\frac{x^n}{n} \\ &= \sum_{n=1}^{\infty}\left(\sum_{i+j=n-1}\binom{n}{i}\binom{n}{j}\alpha^i t^j\right)\frac{x^n}{n}. \end{aligned}$$

Therefore,

$$H(As^n) = \frac{1}{n+1}\sum_{i+j=n}\binom{n+1}{i}\binom{n+1}{j}\alpha^i t^j = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1}{i+1}\alpha^{n-i}t^i.$$

Lemma 9.2. *For every connected graph Γ_{n+1} on $[n+1]$, we have $\gamma_i(P_{\Gamma_{n+1}}) \geq \gamma_i(As^n)$. Moreover, equality holds for all i iff Γ_{n+1} is a linear graph L_{n+1}.*

Proof. It is sufficient to prove the lemma for trees, since for every connected graph Γ, there exists a tree $T \subseteq \Gamma$ on the same nodes. Then $B(T) \subseteq B(\Gamma)$ and we can apply Theorem 8.8.

For $n = 1$ there is nothing to prove. Assume that the lemma holds for $m \leq n$. Let Γ_{n+1} be a tree on $[n+1]$. Without loss of generality, assume that $\{n+1\}$ is adjacent only to $\{n\}$. Then we can use Construction 9.1, setting $\Gamma_n = \Gamma_{n+1} \setminus \{n+1\}$ and $V = \{n\}$. For every $i \in [1, n-1]$, there exists a connected subgraph of Γ_n on i vertices containing $\{n\}$, i.e., there exists $S \in \mathscr{S}$: $|S| = i$. Therefore, comparing (6) and (8), and using the induction assumption and Remark 6.7, we obtain the proof of the stated inequalities.

Note that if Γ_{n+1} is not a linear graph, then either Γ_n is not a linear graph or, for some $i \in [1, n-1]$, there exist more than one element $S \in \mathscr{S}$ with $|S| = i$. In both cases, for some i the inequality in the lemma is strict. □

9.2 Cyclohedra

Let C_{n+1} be a cyclic graph on $[n+1]$. We apply a construction different from Construction 9.1.

Construction 9.3. According to Lemma 7.3, the building set $B_1 = B(C_n)(1, \ldots, n-1, J(n, n+1))$ corresponds to $P_{B_1} \approx Cy^{n-1} \times I$: the bottom and top bases of $Cy^{n-1} \times I$ correspond to $\{n\}, \{n+1\} \in B_1$, and the side facets correspond to elements $S \in B_1 \setminus [n+1]$ that either contain $\{n, n+1\}$, or do not intersect $\{n, n+1\}$. Thus, by contraction of the set $\{n, n+1\}$ to the point $\{n\}$, we identify side facets F_S, $S \in B_1$, of the cylinder $Cy^{n-1} \times I$ with facets $F_{S \setminus \{n+1\}}, S \setminus \{n+1\} \in B(C_n)$ of the base Cy^{n-1}.

We have

$$B(C_{n+1}) \setminus B_1 = \{S \sqcup \{n\}, S \in \mathscr{S}_n\} \cup \{S \sqcup \{n+1\}, S \in \mathscr{S}_{n+1}\},$$

where $\mathscr{S}_n = \{[i, n-1], i = 2, \ldots, n-1\}$ and $\mathscr{S}_{n+1} = \{[1, i], i = 1, \ldots, n-2\}$. By Corollary 8.5, Cy^n is obtained from $Cy^{n-1} \times I$ by truncations of intersections of the bottom base $F_{\{n\}}$ with the side facets F_S for $S \in \mathscr{S}_n$, and by truncations of intersections of the top base $F_{\{n+1\}}$ with the side facets F_S for $S \in \mathscr{S}_{n+1}$. Since the bases do not change after truncations of the facets, truncated faces of each base have the form $As^{i-1} \times Cy^{n-i-1}, i = 1, \ldots, n-1$ ($P_{C_n|S} \times P_{C_n/S}, S \in \mathscr{S}_n$ for bottom base and $P_{C_n|S} \times P_{C_n/S}, S \in \mathscr{S}_{n+1}$ for top base). By Proposition 3.6, we obtain the recursion formulas

$$\gamma(Cy^n) = \gamma(Cy^{n-1}) + 2\tau \sum_{i=1}^{n-1} \gamma(As^{i-1})\gamma(Cy^{n-i-1}), \tag{10}$$

$$H(Cy^n) = (\alpha+t)H(Cy^{n-1}) + 2\alpha t \sum_{i=1}^{n-1} H(As^{i-1})H(Cy^{n-i-1}). \tag{11}$$

The recursion formulas for cyclohedra are equivalent to the equations

$$\gamma_{Cy}(x) = 1 + x\gamma_{Cy}(x) + 2\tau x^2 \gamma_{As}(x)\gamma_{Cy}(x),$$
$$H_{Cy}(x) = 1 + (\alpha+t)xH_{Cy}(x) + 2\alpha t x^2 H_{As}(x)H_{Cy}(x),$$

where

$$\gamma_{Cy}(x) = \sum_{n=0}^{\infty} \gamma(Cy^n)x^n, \qquad H_{Cy}(x) = \sum_{n=0}^{\infty} H(Cy^n)x^n.$$

Setting $V(x) = xH_{Cy}(x)$, we obtain

$$\frac{V}{1+(\alpha+t)V+2\alpha tUV} = x,$$

and thus

$$\frac{U}{(1+\alpha U)(1+tU)} = \frac{V}{1+(\alpha+t)V+2\alpha tUV},$$

which implies

$$V = \frac{U}{1-\alpha tU^2}.$$

Recall that a graph Γ is called *Hamiltonian* if it contains a Hamiltonian cycle, i.e., a closed loop that visits each vertex of Γ exactly once.

Lemma 9.4. *$\gamma_i(P_{\Gamma_{n+1}}) \geq \gamma_i(Cy^n)$ for any Hamiltonian graph Γ_{n+1} on $[n+1]$. Moreover, equality for all i is achieved iff Γ_{n+1} is a cyclic graph C_{n+1}.*

Proof. Since Γ_{n+1} is Hamiltonian, there exists a cyclic subgraph $C_{n+1} \subseteq \Gamma_{n+1}$. Therefore, $B(C_{n+1}) \subseteq B(\Gamma_{n+1})$ and Theorem 8.8 allows to complete the proof. □

9.3 Permutohedra

Let us apply Construction 9.1 to the complete graph K_{n+1}. After deletion of the vertex $\{n+1\}$, we obtain the graph K_n. Here $V = [n]$ and $\mathscr{S} = 2^{[n]} \setminus \{\emptyset, [n]\}$. Therefore, we truncate $\binom{n}{i}$ faces of the form $Pe^{i-1} \times Pe^{n-i-1}$, $i = 1, \ldots, n-1$, and obtain the recursion formulas for permutohedra

$$\gamma(Pe^n) = \gamma(Pe^{n-1}) + \tau \sum_{i=1}^{n-1} \binom{n}{i} \gamma(Pe^{i-1})\gamma(Pe^{n-i-1}), \tag{12}$$

$$H(Pe^n) = (\alpha + t)H(Pe^{n-1}) + \alpha t \sum_{i=1}^{n-1} \binom{n}{i} H(Pe^{i-1})H(Pe^{n-i-1}). \tag{13}$$

They are equivalent to the differential equations

$$\frac{d\gamma_{Pe}(x)}{dx} = 1 + \gamma_{Pe}(x) + \tau\gamma_{Pe}^2(x) \qquad \text{where} \qquad \gamma_{Pe}(x) = \sum_{n=0}^{\infty} \gamma(Pe^n)\frac{x^{n+1}}{(n+1)!},$$

$$\frac{dH_{Pe}(x)}{dx} = (1 + \alpha H_{Pe}(x))(1 + tH_{Pe}(x)) \quad \text{where} \qquad H_{Pe}(x) = \sum_{n=0}^{\infty} H(Pe^n)\frac{x^{n+1}}{(n+1)!}.$$

One can explicitly solve the last equation to obtain

$$H_{Pe}(x) = \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} - te^{\alpha x}}.$$

Let $A(n,k) = |\{\sigma \in \mathrm{Sym}(n) \colon \mathrm{des}(\sigma) = k\}|$. Then, by a well-known formula,

$$\frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} - te^{\alpha x}} = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} A(n+1,k)\alpha^k t^{n-k} \Big) \frac{x^{n+1}}{(n+1)!}.$$

9.4 Stellohedra

Let us apply Construction 9.1 to the complete bipartite graph $K_{1,n}$, or n-star with apex $\{1\}$. After deletion of the vertex $\{n+1\}$, we obtain the graph $K_{1,n-1}$. Here $V = \{1\}$ and $\mathscr{S} = \{\{1\} \cup S, S \subsetneq [2,n]\}$. Therefore, we truncate $\binom{n-1}{i-1}$ faces of the form $St^{i-1} \times Pe^{n-i-1}, i = 1,\dots,n-1$ and obtain

$$\gamma(St^n) = \gamma(St^{n-1}) + \tau \sum_{i=1}^{n-1} \binom{n-1}{i-1} \gamma(St^{i-1})\gamma(Pe^{n-i-1}), \tag{14}$$

$$H(St^n) = (\alpha + t)H(St^{n-1}) + \alpha t \sum_{i=1}^{n-1} \binom{n-1}{i-1} H(St^{i-1})H(Pe^{n-i-1}). \tag{15}$$

These recursion formulas for stellohedra are equivalent to the differential equations

$$\frac{d\gamma_{St}(x)}{dx} = \gamma_{St}(x)(1 + \tau\gamma_{Pe}(x)) \qquad \text{where} \quad \gamma_{St}(x) = \sum_{n=0}^{\infty} \gamma(St^n)\frac{x^n}{n!},$$

$$\frac{dH_{St}(x)}{dx} = H_{St}(x)(\alpha + t + \alpha t H_{Pe}(x)) \quad \text{where} \quad H_{St}(x) = \sum_{n=0}^{\infty} H(St^n)\frac{x^n}{n!}.$$

The last equation can be solved:

$$H_{St}(x) = \frac{(\alpha - t)\, e^{(\alpha+t)x}}{\alpha e^{tx} - t e^{\alpha x}}.$$

Lemma 9.5. *For every tree Γ_{n+1} on $[n+1]$, we have $\gamma_i(P_{\Gamma_{n+1}}) \le \gamma_i(St^n)$. Moreover, equality for all i is achieved iff Γ_{n+1} is a star graph $K_{1,n}$.*

Proof. For $n = 1$ there is nothing to prove. Assume that the Lemma holds for $m \le n$. Let Γ_{n+1} be a tree on $[n+1]$. Without loss of generality, assume that $\{n+1\}$ is adjacent only to $\{n\}$. Then we can use Construction 9.1, setting $\Gamma_n = \Gamma_{n+1} \setminus \{n+1\}$ and $V = \{n\}$. For every $i \in [1, n-1]$, there are no more than $\binom{n-1}{i-1}$ elements $S \in \mathscr{S}$ with $|S| = i$, and for each such S we have $\gamma(P_{\Gamma_n|_S})\gamma(P_{\Gamma_n/S}) \le \gamma(St^{i-1})\gamma(Pe^{n-i-1})$. Therefore, comparing (6) and (14), and using the induction assumption, we obtain the statement in the lemma.

We note that if Γ_{n+1} is not a star graph, then either Γ_n is not a star graph or, for some $i \in [1, n-1]$, the number of elements $S \in \mathscr{S}$ with $|S| = i$ is less than $\binom{n-1}{i-1}$. In both cases, for some i the inequality in the lemma is strict. □

10 Bounds of face polynomials for flag nestohedra and graph-associahedra

Summarizing Lemmas 9.2, 9.4, 9.5 and Theorem 8.8, we obtain the following results.

Theorem 10.1. *For any flag n-dimmensional nestohedron P_B, we have*

1) $\gamma_i(I^n) \le \gamma_i(P_B) \le \gamma_i(Pe^n)$,
2) $g_i(I^n) \le g_i(P_B) \le g_i(Pe^n)$,
3) $h_i(I^n) \le h_i(P_B) \le h_i(Pe^n)$,
4) $f_i(I^n) \le f_i(P_B) \le f_i(Pe^n)$.

Moreover, the lower bound is achieved iff $P_B \approx I^n$, the upper bound is achieved iff $P_B \approx Pe^n$.

Theorem 10.2. *For any connected graph Γ_{n+1} on $[n+1]$, we have*

1) $\gamma_i(As^n) \le \gamma_i(P_{\Gamma_{n+1}}) \le \gamma_i(Pe^n)$,
2) $g_i(As^n) \le g_i(P_{\Gamma_{n+1}}) \le g_i(Pe^n)$,
3) $h_i(As^n) \le h_i(P_{\Gamma_{n+1}}) \le h_i(Pe^n)$,
4) $f_i(As^n) \le f_i(P_{\Gamma_{n+1}}) \le f_i(Pe^n)$.

Moreover, the lower bound is achieved iff Γ_{n+1} is a linear graph L_{n+1}, the upper bound is achieved iff Γ_{n+1} is a complete graph K_{n+1}.

Theorem 10.3. *For any Hamiltonian graph Γ_{n+1} on $[n+1]$, we have*

1) $\gamma_i(Cy^n) \le \gamma_i(P_{\Gamma_{n+1}}) \le \gamma_i(Pe^n)$,
2) $g_i(Cy^n) \le g_i(P_{\Gamma_{n+1}}) \le g_i(Pe^n)$,

3) $h_i(Cy^n) \leq h_i(P_{\Gamma_{n+1}}) \leq h_i(Pe^n)$,
4) $f_i(Cy^n) \leq f_i(P_{\Gamma_{n+1}}) \leq f_i(Pe^n)$.

Moreover, the lower bound is achieved iff Γ_{n+1} is a cyclyc graph C_{n+1}, the upper bound is achieved iff Γ_{n+1} is a complete graph K_{n+1}.

Theorem 10.4. *For any tree Γ_{n+1} on $[n+1]$, we have*

1) $\gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(St^n)$,
2) $g_i(As^n) \leq g_i(P_{\Gamma_{n+1}}) \leq g_i(St^n)$,
3) $h_i(As^n) \leq h_i(P_{\Gamma_{n+1}}) \leq h_i(St^n)$,
4) $f_i(As^n) \leq f_i(P_{\Gamma_{n+1}}) \leq f_i(St^n)$.

Moreover, the lower bound is achieved iff Γ_{n+1} is a linear graph L_{n+1}, the upper bound is achieved iff Γ_{n+1} is a star graph $K_{1,n}$.

The bounds can be written explicitly, using results about the f-, h-, g- and γ-vectors of the respective series (cf. [20] and [1]):

$$h_i(I^n) = \binom{n}{i}, \quad h_i(As^n) = \frac{1}{n+1}\binom{n+1}{i}\binom{n+1}{i+1}, \quad h_i(Cy^n) = \binom{n}{i}^2,$$

$$h_i(Pe^n) = A(n+1,i), \quad h_i(St^n) = \sum_{k=i}^{n}\binom{n}{k}A(k,i-1),\ i>0,$$

$$\gamma_i(I^n) = 0,\ i>0, \quad \gamma_i(As^n) = \frac{1}{i+1}\binom{2i}{i}\binom{n}{2i}, \quad \gamma_i(Cy^n) = \binom{n}{i,i,n-2i}.$$

The derived bounds for the f-, h-, g- and γ-vectors determine bounds for the corresponding polynomials, for which generating functions were obtained in [1].

11 Nested polytopes and graph-cubeahedra as 2-truncated cubes

Let Γ be a connected graph, $B(\Gamma)$ its building set. Using the notation of [9], elements of $B(\Gamma)$ are called *tubes* and $\{S_1,\ldots,S_k\} \in \mathcal{N}(B(\Gamma))$ is called a *tubing*. Two tubes forming a tubing are called *compatible*.

In [9], a class of simple polytopes called graph-cubeahedra was introduced. To formulate a definition, we have to introduce the notion of 'design tubing'.

Definition 11.1. Let Γ be a connected graph. A *round tube* is a set of nodes S of Γ whose induced subgraph $\Gamma|_S$ is connected. A *square tube* is a single node $S=\{v\}$ of Γ. Such tubes are called *design tubes* of Γ. Two design tubes are *compatible* if one of the following conditions holds,

1) they are both round and compatible,
2) one of the tubes is square and does not intersect the other.

A *design tubing* U of Γ is a collection of design tubes of Γ such that every pair of tubes in U is compatible.

For a graph Γ with n nodes, we define $\square_\Gamma$ to be the n-cube where each pair of opposite facets corresponds to a particular node of Γ. Specifically, one facet in the pair represents that node as a round tube and the other represents it as a square tube. Each subset of nodes of Γ, chosen to be either round or square, corresponds to a unique face of $\square_\Gamma$, defined by the intersection of the faces associated with those nodes. The empty set corresponds to the face that is the entire polytope $\square_\Gamma$.

Definition 11.2. For a graph G, truncating faces of G that correspond to round tubes in increasing order of dimension, results in a convex polytope CG called *graph-cubeahedron.*

Theorem 11.3 ([9, Theorem 12]). *For a graph G with n nodes, the graph-cubeahedron CG is the simple convex polytope of dimension n whose face poset is isomorphic to the set of design tubings of G, ordered such that $U < U'$ if U is obtained from U' by adding tubes.*

Definition 11.4. Let B be a building set (not necessary connected) on $[n+1]$. Define the complex $\mathscr{K}(B)$ as the simplicial complex on the node set B consisting of all the nested sets $\mathscr{S} \subseteq B$.

Proposition 11.5. *For any building set B, we have $\mathscr{K}(B) \simeq K\mathscr{N}(B_1) \star \cdots \star K\mathscr{N}(B_k)$, where $B = B_1 \sqcup \cdots \sqcup B_k$ and B_i are connected. In particular, if B is connected, then $\mathscr{K}(B)$ is a cone over $\mathscr{N}(B)$.*

Proof. Let B be a connected building set on $[n+1]$, then the node set of the complex $\mathscr{K}(B)$ consists of the node set of the complex $\mathscr{N}(B)$ and the vertex $[n+1]$. By definition, $\{S_1,\dots,S_k\} \in \mathscr{N}(B)$ iff $\{S_1,\dots,S_k\} \in \mathscr{K}(B)$ iff $\{[n+1],S_1,\dots,S_k\} \in \mathscr{K}(B)$. Therefore $\mathscr{K}(B) \simeq K\mathscr{N}(B)$. The proof is complete, since $\mathscr{K}(B_1 \sqcup B_2) \simeq \mathscr{K}(B_1) \star \mathscr{K}(B_2)$. $\square$

Lemma 11.6. *Let B_0 and B_1 be building sets on $[n+1]$, and $B_0 \subset B_1$. Then $\mathscr{K}(B_1)$ is obtained from $\mathscr{K}(B_0)$ by stellar subdivisions over $\sigma_i = \{S^i_1,\dots,S^i_{k_i}\}$, corresponding to decompositions $S^i = S^i_1 \sqcup \cdots \sqcup S^i_{k_i} \in B_1 \setminus B_0$ of elements S^i, numbered in any order that is reverse to inclusion (i.e., $S^i \supseteq S^{i'} \Rightarrow i \leq i'$).*

Proof. We prove the lemma by induction on the number $N = |B_1| - |B_0|$.

Let $N = 1$, then $B_1 = B_0 \cup S^1$. We show that $\mathscr{K}(B_1) \simeq (S^1, B_0(S^1))\mathscr{K}(B_0)$. Let $B_0 = B^1_0 \sqcup \cdots \sqcup B^l_0$, where B^i_0 are connected building sets on $[k_{i-1}+1, k_i]$. There are only two possibilities: $S^1 \subset [k_{i-1}+1, k_i]$ for some $i \in [l]$, or $S^1 = \bigsqcup_{i\in\sigma}[k_{i-1}+1, k_i]$ for some $\sigma \subseteq [l]$.

Consider the first case and assume $i = 1$. We have $B_1 = B^1_1 \sqcup \cdots \sqcup B^l_1$, where $B^1_1 = B^1_0 \cup \{S^1\}$ and $B^i_1 = B^i_0$ for $i > 1$. By Lemma 8.4, $\mathscr{N}(B^1_1) \simeq (S^1, B^1_0(S^1))\mathscr{N}(B^1_0)$. Using Proposition 11.5, we obtain $\mathscr{K}(B_1) \simeq (S^1, B_0(S^1))\mathscr{K}(B_0)$.

Consider the second case. Without loss of generality, we can assume that $\sigma = [l]$. Using Example 7.5, $\mathscr{N}(B_1) = \partial\Delta^{l-1} \star \mathscr{N}(B^1_0) \star \cdots \star \mathscr{N}(B^l_0)$, and then $\mathscr{K}(B_1) = K(\partial\Delta^{l-1} \star \mathscr{N}(B^1_0) \star \cdots \star \mathscr{N}(B^l_0)) = K(\partial\Delta^{l-1}) \star \mathscr{N}(B^1_0) \star \cdots \star \mathscr{N}(B^l_0)$. This complex is the stellar subdivision of $\mathscr{K}(B_0) \simeq K\mathscr{N}(B^1_0) \star \cdots \star K\mathscr{N}(B^l_0)$ over the simplex $\{[1,k_1],\ldots,[k_{l-1}+1,n+1]\} \in \mathscr{K}(B_0)$, which corresponds to the join of apexes of the cones $\mathscr{K}(B_0^i)$.

Let us assume the lemma holds for $M < N$ and prove it for $M = N$. The collection of sets $B_0' = B_0 \cup \{S^1\}$ is a building set. By the induction assumption, $\mathscr{K}(B_0')$ is obtained from $\mathscr{K}(B_0)$ by stellar subdivision over the simplex corresponding to the decomposition of S^1, and $\mathscr{K}(B_1)$ is obtained from $\mathscr{K}(B_0')$ by a sequence of stellar subdivisions over simplices corresponding to decompositions $S^i, i = 2,\ldots,N$. This completes the proof. □

Theorem 11.7. *Let B_1 and B_2 be flag building sets on $[n+1]$ and $B_1 \subset B_2$. Then $\mathscr{K}(B_2)$ is obtained from $\mathscr{K}(B_1)$ by a subdivision of edges.*

Proof. The proof repeats the proof of Theorem 8.6. □

Let us denote by $B_{\Delta,n}$ the building set consisting of singletons $\{i\}$, $i \in [n]$.

Definition 11.8. Let P be a simple n-polytope with fixed order of facets, $F_1, F_2, \ldots, F_m$, such that $F_1 \cap \cdots \cap F_n \neq \emptyset$, and let B be a building set on n. By Lemma 11.6, $\mathscr{K}(B)$ is obtained from $\mathscr{K}(B_{\Delta,n}) \simeq \Delta^{n-1}$ by a sequence of stellar subdivisions. Pick the facet Δ_v of ∂P^* associated with the vertex $v = F_1 \cap \cdots \cap F_n$, identify vertices $\{i\} \in \mathscr{K}(B_{\Delta,n})$ with vertices Δ_{F_i} of Δ_v for $i = 1,\ldots,n$, and apply the sequence of stellar subdivisions to Δ_v inside the complex ∂P^*, to obtain the boundary of some simplicial polytope Q. By definition, the *nested polytope* $NP(P,B)$ is the dual of the polytope Q.

Proposition 11.9. *If $P = I^n$ and $B = B(\Gamma)$ is a connected graphical building set, then $NP(I^n,B)$ is combinatorially equivalent to the graph-cubeahedron associated with Γ.*

Proof. This follows from Lemma 11.6. □

Theorem 11.10.

1) *If P is a flag polytope and B a flag building set, then $NP(P,B)$ is a flag polytope.*
2) *If P is a 2-truncated cube and B a flag building set, then $NP(P,B)$ is a 2-truncated cube.*

Proof. By Theorem 11.7, the complex $\mathscr{K}(B)$ can be obtained from Δ_{n-1} by stellar subdivisions over edges. Therefore $NP(P,B)$ can be obtained from P by 2-truncations. □

Corollary 11.11. *Any graph-cubeahedron is a 2-truncated cube.*

Acknowledgement We are grateful to A. Gaifullin, S. Forcey, F. Müller-Hoissen, T. Panov and J. Stasheff for valuable discussions and comments. Our work has been supported by the Russian government project 11.G34.31.0053.

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