Geometric aspects of branes at toric singularities

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AEI, 23 September 2014

arXiv:1405.2319 + work in progress

Motivation

- We wish to understand the AdS/CFT correspondence away from the maximally supersymmetric case ($AdS_5 \times S^5$), but in situations where dual theories can still be under control ($\mathcal{N} = 1$)

Plan:

- Discuss the relevant supergravity backgrounds
- Elaborate on certain geometric aspects of the problem (new resolution parameters)

Supergravity backgrounds and interpretation

Supergravity solutions. I.

- We start with type IIB theory defined on $\mathbb{R}^{3,1} \times Y$, where Y is a Calabi-Yau threefold
- We place D3-branes with worldvolume $\mathbb{R}^{3,1}$ at an isolated singular point of Y
- We will consider singularities that have the form of complex cone over a surface X
- We will assume that X is toric (hence Y is toric), which means that Y has at least a $U(1)^3$ worth of isometries

Supergravity solutions. II.

- Within supergravity these branes lead to solutions of the form

$$ds^2 = h^{-1/2}(y) \sum_{\mu=0}^{3} dx_\mu dx_\mu + h^{1/2}(y)\,(\overline{ds^2})_Y$$

where $h(y)$ depends only on the coordinates on Y and satisfies Poisson's equation:

$$\triangle h = \sum \delta(\text{sources})$$

Kehagias [1998]

- These are generalizations of the solution of Horowitz, Strominger [1991] (for $Y = \mathbf{C}^3$), which leads in the 'near-horizon' limit to the familiar $AdS_5 \times S^5$ geometry

Supergravity solutions. III.

- The simplest case is when branes are placed exactly at the singular point
- Then $(\overline{ds^2})_Y = dr^2 + r^2 \widetilde{ds^2}_X$, where X is a Sasaki-Einstein 5-manifold (defined below)
- There is also a 5-form flux through X: $\int_X F_5 \sim N$
- The function h is found explicitly: $h \sim N \cdot \frac{1}{r^4}$, and the SUGRA ansatz above leads to a smooth geometry of the form $AdS_5 \times X^5$ Morrison, Plesser [1998]

AdS/CFT with $\mathcal{N} = 1$ SUSY

- For generic $\boldsymbol{X}$ such solutions preserve $\boldsymbol{N = 1}$ SUSY, since $\boldsymbol{X}$ admits one Killing spinor: $\boldsymbol{(\nabla_\mu + F_5 \gamma_\mu)\psi = 0}$

  There are two types of deformations of the above construction:
- One can move the branes off the cone tip
- One can 'resolve' the singularity at the tip
- Both of these are related to symmetry breaking in the $\boldsymbol{N = 1}$ superconformal field theory Klebanov, Witten [1998]

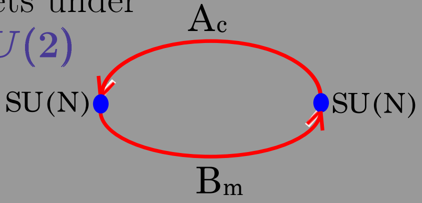
The conifold theory. I.

- The conifold: $XY = UV$ in $\mathbf{C}^4$
- Formal solution:

$$X = a_1 b_1,\ Y = a_2 b_2,\ U = a_1 b_2,\ V = a_2 b_1$$

$\Rightarrow$ Conifold = cone over $\mathbf{CP}^1 \times \mathbf{CP}^1$

- Dual QFT with gauge group $SU(N) \times SU(N)$, two sets of chiral fields: A_b and B_m, doublets under global $SU(2) \times SU(2)$



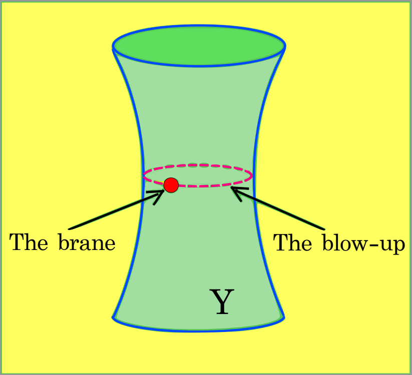
Klebanov, Witten [1998]

The conifold theory. II.

Klebanov, Witten [1998]

- There are vacuum configurations when $\boldsymbol{A_c}$ and $\boldsymbol{B_m}$ are simultaneously diagonalizable
- If the eigenvalues satisfy $\boldsymbol{a_c \neq 0, b_m \neq 0}$, then $\langle \mathbf{tr}(\boldsymbol{A_c B_m}) \rangle$ can be regarded as positions of the branes moved off the tip of the conifold
- If $\boldsymbol{A_c \equiv 0}$, then $\langle \mathbf{det}\, \boldsymbol{B_j} \rangle$ may be thought of as positions on the $\mathbf{CP}^1 \times \mathbf{CP}^1$ glued in at the origin of the resolved cone (hence proportional to the blow-up parameters)

The conifold theory. III.



The conifold theory. IV.

- The metric on the resolved conifold (i.e. on Y) was built in Candelas, de la Ossa [1990] and generalized in Pando Zayas, Tseytlin [2001]
- A background that interpolates between $\mathcal{N} = 1$ conifold theory (UV) and $\mathcal{N} = 4$ theory after symmetry breaking (IR) (i.e. the h-function for the brane solution) was constructed in Klebanov, Murugan [2007]

Geometry of the transverse space Y

Sasaki-Einstein manifolds

- X is Sasaki-Einstein iff the cone over it is Kähler and Ricci-flat:
  $$(\overline{ds^2})_Y = dr^2 + r^2\,(\widetilde{ds^2})_X$$
- $(\overline{ds^2})_Y$ Kähler & Ricci-flat $\Leftrightarrow$ $(\widetilde{ds^2})_X$ Sasaki-Einstein, of positive curvature
- The metric can be written as
  $$(ds^2)_{X^5} = (d\phi - J)^2 + (ds^2)_{\mathcal{M}}$$
  where $(ds^2)_{\mathcal{M}}$ is Kähler-Einstein (but not necessarily smooth), J is the Kähler current
- $r = 0 \rightarrow$ singularity

Resolving the singularity of the cone

- It is possible to resolve the singularity of the conical metric by 'blowing-up' the vertex, i.e. by replacing it with a cycle of non-zero size
- The metric at infinity, i.e. at $r \to \infty$, will still be asymptotic to the cone:

$$(\overline{ds^2})_Y = dr^2 + r^2 \left(\widetilde{ds^2}\right)_X \quad \text{for} \quad r \to \infty$$

- Apart from simplest cases, resolved metrics on the cones are not known $\Rightarrow$ Our study

Some examples

- Eguchi, Hanson, 1978
  Complex dimension 2, singularity of the form
  $$\mathbf{C}^2/\mathbf{Z}_2: \quad (z_1, z_2) \sim (-z_1, -z_2)$$
- Introducing invariant coordinates $X = z_1^2, Y = z_2^2, Z = z_1 z_2$, we get an equation $XY = Z^2 \quad \text{in} \quad \mathbf{C}^3$
- This corresponds to the cone in the embedding of $\mathbf{CP}^1$ by the linear system $|\mathcal{O}(2)|$, i.e. the *anticanonical* embedding

The Eguchi-Hanson metric

- One can look for the Kähler potential of the form $K = K(|z_1|^2 + |z_2|^2)$.
  The metric is, as usual, $ds^2 = \partial_i \bar{\partial}_j K \, dz^i \, d\bar{z}^j$
- For a Kähler metric the Ricci tensor can be expressed as $R_{i\bar{j}} = -\partial_i \bar{\partial}_j \log \det g$
- Set $R_{i\bar{j}} = 0$, solve for the Kähler potential:

The Eguchi-Hanson metric

$$K = \sqrt{r^2 + 4x^2} + r \log\left(\frac{\sqrt{r^2+4x^2} - r}{2x}\right), \quad r > 0$$

More examples

- 2d case: Eguchi-Hanson=anticanonical cone over $\mathbf{CP}^1 \Rightarrow$ SE $X_3 = S^3/\mathbb{Z}_2$
- '3d Eguchi-Hanson'= anticanonical cone over $\mathbf{CP}^2 \Rightarrow$ SE $X_5 = S^5/\mathbb{Z}_3$
- 3d case: Candelas-de la Ossa [1990] = anticanonical cone over $\mathbf{CP}^1 \times \mathbf{CP}^1$ (resolved conifold) $\Rightarrow$ SE $X_5 := T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$

Other cones?

- One can only build Ricci-flat cones over complex manifolds of 'positive curvature' (i.e. with ample anticanonical class)
- For the cone to be of $\mathbf{dim}_{\mathbb{C}} = 3$, we take the underlying base to be of $\mathbf{dim}_{\mathbb{C}} = 2$
- Apart from $\mathbf{CP}^2$ and $\mathbf{CP}^1 \times \mathbf{CP}^1$, there are only 8 other positively curved complex surfaces – the del Pezzo surfaces

$$\mathbf{dP}_1, \ldots, \mathbf{dP}_8$$

Pasquale del Pezzo (1859-1936)

- Duke of Cajanello and Marquis of Campodisola
- Born in Berlin
- Rector of the University of Naples, Mayor of Naples, Senator
- Introduced the rational surfaces that bear his name (Example: cubic in $\mathbf{CP}^3$, i.e. $X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$)

The del Pezzo surface $\mathbf{dP_1}$

- $\mathbf{dP}_n$ can be seen as $\mathbf{CP^2}$, blown-up in n sufficiently generic points
- We will consider the simplest non-homogeneous case, i.e. the cone over $\mathbf{dP_1}$
- Any metric on $\mathbf{dP_1}$ should have at least two parameters – the sizes of $\mathbf{CP^2}$ and of the blown-up $\mathbf{CP^1}$
- Do these parameters persist in the cone over $\mathbf{dP_1}$?

Isometries

- Whereas the automorphism group of $\mathbf{CP^2}$ is $\boldsymbol{PGL(3,\mathbb{C})}$, the automorphism group of the del Pezzo surface is reduced to

$$Aut(\mathrm{dP}_1) = P\begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{pmatrix} \qquad (1)$$

- The isometry group of the metric on the *cone* is the maximal compact subgroup of the parabolic subgroup shown above, i.e.

Isom $= U(1) \times U(2)$

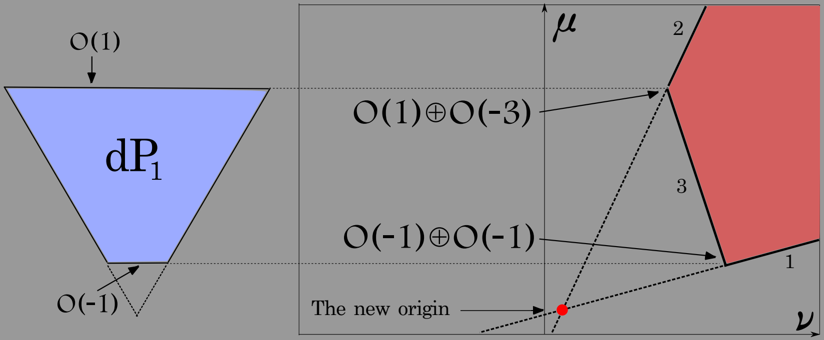
The main equation

- We will look for a Kähler potential of the form $K = K(|u|^2, |z_1|^2 + |z_2|^2) := K(e^t, e^s)$
- Just as in the case of the Eguchi-Hanson metric, we can write out a Ricci-flatness equation
- More convenient to perform a Legendre transform w.r.t. t, s, introducing the dual moment maps $\mu = \frac{\partial K}{\partial t}$, $\nu = \frac{\partial K}{\partial s}$ and a dual potential $G = t\mu + s\nu - K$

The equation

$$e^{G_\mu+G_\nu}\left(G_{\mu\mu}G_{\nu\nu}-G_{\mu\nu}^2\right)=\mu$$

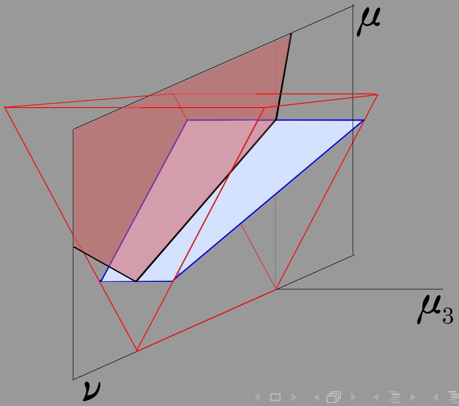
- The domain – the moment polygon



The equation

$$e^{G_\mu+G_\nu}\ \left(G_{\mu\mu}G_{\nu\nu}-G_{\mu\nu}^2\right)=\mu$$

- This is a 2D (μ,ν) section of a 3D polytope



The equation

$$e^{G_\mu + G_\nu} \left(G_{\mu\mu} G_{\nu\nu} - G_{\mu\nu}^2 \right) = \mu$$

- Once we know G, we can recover the metric:

$$ds^2 = \mu\, g_{\mathrm{CP}^1} + \frac{\partial^2 G}{\partial \mu_i \partial \mu_j}\, d\mu_i\, d\mu_j + \left(\frac{\partial^2 G}{\partial \mu^2} \right)^{-1}_{ij} D\phi_i\, D\phi_j$$

where $(\mu_1, \mu_2) = (\mu, \nu)$,
$D\phi_i := d\phi_i - A_i$,
$A_2 = 0$ and $dA_1 = \omega_{\mathrm{CP}^1}$

The expansion at ∞

- We can solve the equation exactly at large μ, ν with fixed 'angle' $\xi = \frac{\mu}{\nu}$, assuming the conical form of the metric
- This gives $G = 3\nu(\log\nu - 1) + \nu\, P_0(\xi)$
- $P_0(\xi)$ satisfies an ODE and can be found exactly. It provides a Sasaki-Einstein metric, which in the dP_1 case is the $Y^{2,1}$ manifold ($Y^{p,q}$ manifolds were constructed in Gauntlett, Martelli, Sparks, Waldram [2004])

M^{th} order and the Heun equation

- We can build a systematic perturbation theory

$$G = 3\nu(\log\nu - 1) + \nu\, P_0(\xi) + \log\nu + \sum_{k=0}^{\infty} \nu^{-k}\, P_{k+1}(\xi)$$

- In order ν^{-M} we obtain the equation

$$\frac{d}{d\xi}\left(Q(\xi)\frac{dP_M}{d\xi}\right) - \left((M-2)^2 - 1\right)\xi\, P_M = \text{r.h.s.},$$

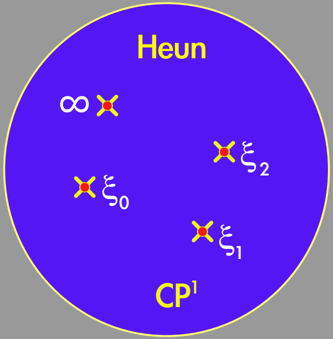
$$\text{where} \quad Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d, \quad d = \frac{16+\sqrt{13}}{64}$$

- This is a Heun equation – an analogue of hypergeometric equation with 4 Fuchsian singularities on $\mathbf{CP}^1$

M^{th} order and the Heun equation

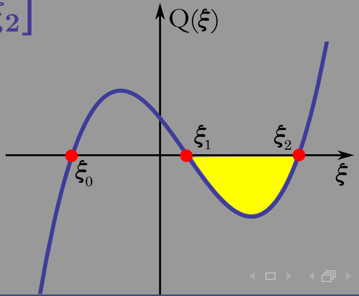
$$\frac{d}{d\xi}\left(Q(\xi)\frac{dP_M}{d\xi}\right) - \left((M-2)^2-1\right)\xi\, P_M = \text{r.h.s.},$$

where $Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d, \quad d = \frac{16+\sqrt{13}}{64}$



Resolution parameters

- All resolution parameters should arise as coefficients in front of the solutions to the homogeneous equation in some order of perturbation theory
- The equation is solved in a 'physical' interval $\xi \in [\xi_1, \xi_2]$



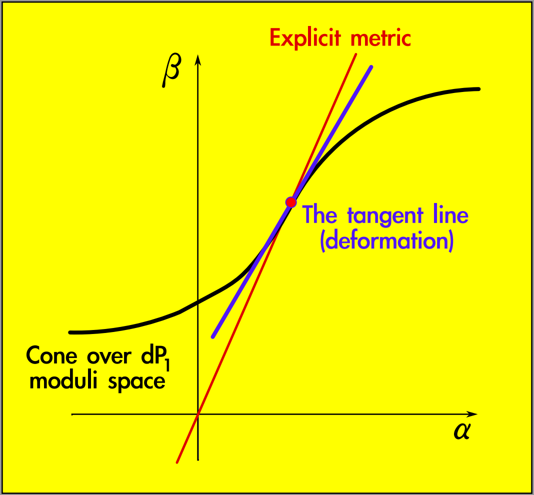
Resolution parameters. 2.

- Regularity of the metric at the boundaries of the moment polytope requires that the solutions should be regular at $\xi = \xi_1, \xi_2$
  $\Rightarrow$ Eigenvalue problem
- Solutions exist for $M = 3, 4$:
  $P_3 = \alpha, \; P_4 = \beta\,(\xi - 1)$
- Conjecture:
  For other M solutions do not exist

Resolution parameters. 3.

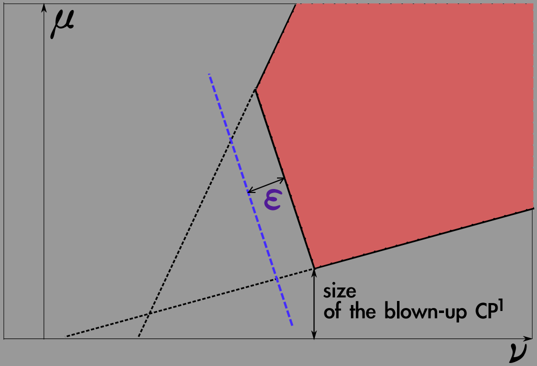
- When $\beta = -\frac{\alpha}{2\xi_0}$, the metric is known explicitly
  Calderbank, Gauduchon [2006], Chen, Lu, Pope [2006]
- Topology imposes one more relation between β and α Martelli, Sparks [2007]
- Hence the explicit metric is just one point in the space of metrics on the cone over $\mathbf{dP_1}$

The explicit metric and its deformation



The deformation

- Linearize the Monge-Ampere equation
- Require sufficient regularity at the edges



- Obtain the restriction: $\varepsilon = \delta\alpha = \delta\beta$

Questions / Answers

- Can one obtain an exact formula with both parameters α, β? As just discussed, there is an exact formula when $\beta = -\frac{\alpha}{2\xi_0}$. Is there a generalization?
- Dual field theories for $AdS_5 \times X^5$ have been conjectured Feng, Hanany, He, 2000
- What is the symmetry breaking pattern corresponding to the new parameter in the metric?

Thank you!