The Kähler metric of a blow-up

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Effective 6D theories

Douglas, Moore, 1996

- Place 6D branes at an orbifold singularity of the form $\mathbf{C}^2/\Gamma$, where $\Gamma \subset SU(2)$ – a finite subgroup
- Obtain $\mathcal{N} = (1,0)$ gauge theory on the brane with M hypermultiplets ${}_m\phi^a_A$, $m = 1 \ldots M$, $A = 1,2$ – $SU(2)_R$ index, a – gauge index
- Field content may be determined by Γ-projection. It depends not just on Γ, but also on its representation r

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From 6D theories to geometry

- In algebraic geometry singularities of the above type are called *du Val*
- Do not change the canonical class of X, i.e. preserve Ricci-flatness
- Reminder: $c_1(X) = -c_1(\mathcal{K}_X) = \frac{i}{2\pi} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$
- Example: $\Gamma = \mathbf{Z}_2$ acting as $(z_1, z_2) \to (-z_1, -z_2)$. The invariants $X = z_1^2, Y = z_2^2, Z = z_1 z_2$ satisfy an equation

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ALE spaces

- The A_{m-1}-series: $\Gamma = \mathbf{Z}_m$ leads to
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ALE spaces, 2.

- How is all this related to the 6D brane theory?
- Answer: Look at the potential for the hypermultiplets, that is uniquely determined by the field content:

$$V(\phi) = \frac{1}{2}\Big[\sum_{m=1}^{M} ({}_m\phi^a_A)^* (\sigma_i)_{AB}\, {}_m\phi^a_B + \zeta_i\Big]^2$$

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Curves in complex surfaces

$\boldsymbol{X}$ – a complex surface with canonical class $\mathcal{K}$
$\boldsymbol{C} \subset \boldsymbol{X}$ – a complex curve in $\boldsymbol{X}$, of genus g_C

The main formula:

$$g_{\mathcal{C}} = \frac{\mathcal{C}.(\mathcal{C}+\mathcal{K})}{2} + 1$$

The dot . stands for the intersection pairing (that I will define momentarily).
Here $\mathcal{C}$ and $\mathcal{K}$ should be thought of as 'divisors', i.e. as formal linear combinations of some curves.

The intersection and self-intersection

- If we have two curves $C_1, C_2 \subset X$ intersecting transversely, their intersection number $C_1.C_2$ is the usual geometric intersection number.
- We may also define for C_1, C_2 the integral Poincare dual 2-forms $\omega_1, \omega_2 \in H^2(X, \mathrm{Z})$ and define the intersection of curves as $C_1.C_2 := \int \omega_1 \wedge \omega_2$
- The latter approach allows to introduce *self-intersection*, i.e. $C.C$
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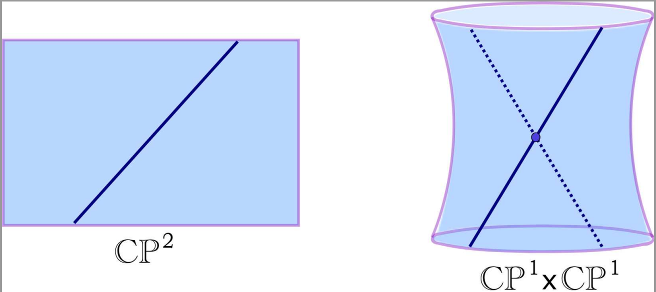
Simple examples

- We will be looking at curves of genus zero: $\mathbf{CP}^1 \subset X$
- Every normal bundle to $\mathbf{CP}^1$ has the form $N_{C/X} = \mathcal{O}(m), \; m \in \mathbf{Z}$



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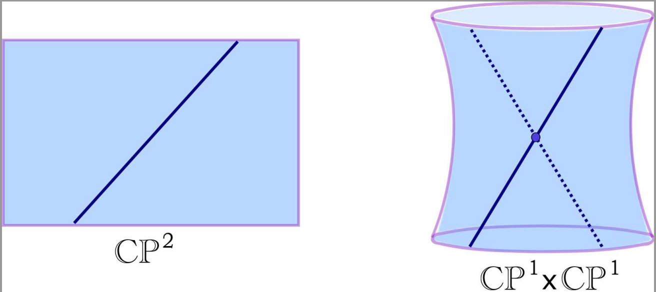

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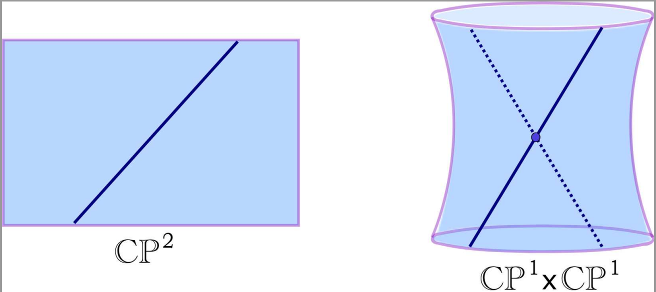

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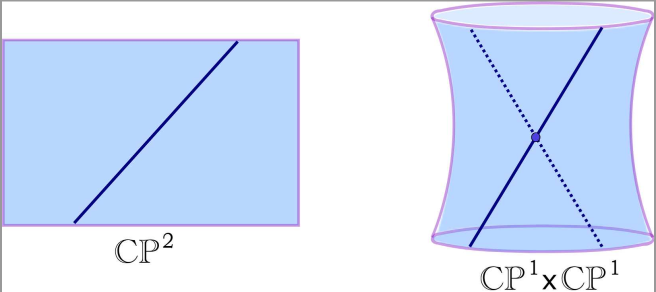

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The Eguchi-Hanson space

- More interesting: negative normal bundles, i.e. $m < 0$ (rigid spheres)
- If X is Calabi-Yau, $\mathcal{K} = 0$, hence the formula $g_{\mathcal{C}} = \frac{\mathcal{C}.(\mathcal{C}+\mathcal{K})}{2} + 1$ implies

$$g_{\mathcal{C}} = 0 \quad \Leftrightarrow \quad \mathcal{C}.\mathcal{C} = -2 \quad \Leftrightarrow \quad N_{\mathcal{C}/X} = \mathcal{O}(-2)$$

- The equation $XY = Z^2$ may be seen as defining the total space of the line bundle $\mathcal{O}(-2)$
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The Eguchi-Hanson space, 2.

- The EH metric may be obtained as a hyper-Kähler quotient $\mathbf{R}^8//U(1)$
- But even easier: $XY = Z^2$ equivalent to $X^2 + Y^2 + Z^2 = 0 \to U(2)$ invariance
- Introduce complex coordinates z_1, z_2 and form a $U(2)$-invariant combination

$$x := |z_1|^2 + |z_2|^2$$

- Look for a Kähler potential $K = K(x)$, solve the Ricci-flatness equation (ODE) $R_{i\bar{j}} = -\partial_i \bar{\partial}_j \log \det(g) = 0$

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- $K = \sqrt{r^2 + 4x^2} + r \log\left(\frac{\sqrt{r^2+4x^2}-r}{2x}\right), \quad r > 0$

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Positive/negative curvature surfaces

- We discussed above that if X is Calabi-Yau, $\mathcal{K} = 0$, then the normal bundle to a $\mathbf{CP}^1$ is *necessarily* $\mathcal{O}(-2)$
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The rank-1 del Pezzo surface

- $\mathbf{CP^2}$ blown-up at one point
- Includes a $\mathbf{CP^1}$ with normal bundle $\mathcal{O}(-1)$
- May be constructed as a GIT quotient $\mathbf{C^4/(C^*)^2}$ with charge vectors $(-2,-2,1,3)$ and $(1,1,1,0)$
- Interpret this as Kähler quotient $\rightarrow$ construct a Kähler potential

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Normal bundle $\mathcal{O}(-1)$

Expand $\boldsymbol{K}$ near the $\mathbf{CP^1}$, i.e. at $\boldsymbol{x \rightarrow 0}$:

$$\boldsymbol{K = a_1 \log(x) + b_1 x + \ldots, \qquad a_1, b_1 > 0}$$

Compare

Expansions at $x \to 0$, i.e. $z_1, z_2 \to 0$

Normal bundle $\mathcal{O}(-1)$:

$$K = a_1 \log(x) + b_1 x + \ldots, \qquad a_1, b_1 > 0$$

Normal bundle $\mathcal{O}(-2)$:

$$K = a_2 \log(x) + b_2 x^2 + \ldots, \qquad a_2, b_2 > 0$$

Conjecture: Normal bundle $\mathcal{O}(-m)$:

$$K = a_m \log(x) + b_m x^m + \ldots, \qquad a_m, b_m > 0$$

Regarding blow-ups

- Consider $\mathbf{C}^2$ with coordinates (z_1, z_2)
- A blow-up of $\mathbf{C}^2$ at the origin:
  Pass from $\mathbf{C}^2$ to the surface
  $\{\ z_1 u_2 = z_2 u_1\ \} \subset \mathbf{C}^2 \times \mathbf{CP}^1$,
  where $(u_1 : u_2)$ live in projective space $\mathbf{CP}^1$
- Blowing-up at a smooth point $\to \mathbf{CP}^1$ with normal bundle $\mathcal{O}(-1)$
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The normal bundle $\mathcal{O}(-m), \;\; m \geqslant 3$

- First Chern class of the total space is negative: $\int_S c_1(Y_m) = 2 - m < 0$
- Hence we can look for a Kähler-Einstein metric with a negative cosmological constant: $R_{i\bar{j}} = -g_{i\bar{j}}$
- Obtain the Kähler potentials

$$Q = xK', \quad x = \prod_{i=1}^{3} (Q - y_i)^{\frac{1}{2+y_i}},$$

where

$$y_i^3 + 3y_i^2 - (m-2)^2(m+1) = 0$$

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Topology of complex surfaces

- What is the underlying manifold for these metrics?
- Characteristic numbers $\int c_2(X)$, $\int c_1(X)^2$
- Equivalently, Euler characteristic and signature
- Signature = signature of intersection form $(.)$ on $H^2(X, \mathrm{R})$
- The underlying manifold is the total space of $\mathcal{O}(-m)$ if $\mathbf{Eu = 2}$ and $\mathbf{Sgn = -1}$

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The Chern-Weil formulas

To check the above we use the Ch.-W.-formulas expressing the topological numbers through the metric:

The prototypical Ch.-W. formulas

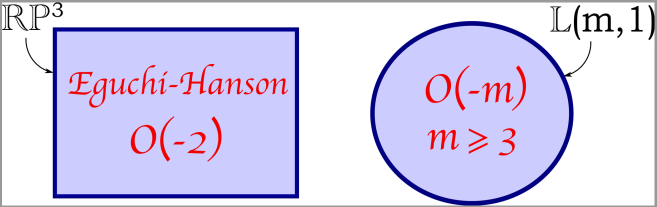
$$\mathbf{Eu}(X, \varnothing) = \frac{1}{32\pi^2} \int d^4x \, \frac{1}{\sqrt{g}} \, \epsilon^{abcd} \, \epsilon^{mnpq} \, R_{abmn} \, R_{cdpq}$$

$$\mathbf{Sgn}(X, \varnothing) = -\frac{1}{24\pi^2} \int d^4x \, \epsilon^{abcd} \, R^m{}_{nab} \, R^n{}_{mcd}$$

For a manifold with boundary ∂X there is also a boundary contribution and a contribution to the signature of the $\eta(\partial X)$-invariant of Atiyah, Patodi, Singer, 1975

The total space of $\mathcal{O}(-m)$, $m \geqslant 3$

Key step: identify points $(z_1, z_2) \sim e^{\frac{2\pi i}{m}}(z_1, z_2)$
As a result, the boundary is
$\partial X = S^3/\mathbf{Z}_m = L(m,1)$ — the lens space.
The η-invariant of the lens space cancels the contribution of the blown-up $\mathbf{CP}^1$ to produce the right answer for the signature $\mathbf{Sgn} = -1$!



Ricci-flat versus negative cosmological constant

Questions / Answers

- We have described the general setup of a sphere holomorphically embedded in a complex surface $\mathbf{CP}^1 \subset X$
- The normal bundle to the sphere is clearly encoded in the asymptotic behavior of the Kähler metric (potential)
- Using these asymptotics, one can build a Kähler-Einstein metric of negative curvature on the total space of $\mathcal{O}(-m), m \geqslant 3$.
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Thank you!