Flag manifold sigma-models

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10.12.2018, Nordita, Stockholm

Sigma-models are theories of maps $X : \mathscr{C}_2 \to \mathscr{M}$ from a worldsheet $\mathscr{C}_2$ to a target space $\mathscr{M}$. The action depends on a metric h_{ij} and a 2-form B_{ij} on $\mathscr{M}$ and has the form

$$\mathcal{S} = \frac{1}{2} \int_{\mathscr{C}} d^2 z \sqrt{\gamma}\, h_{ij}(X)\, \gamma^{\mu\nu} \partial_\mu X^i\, \partial_\nu X^j + \frac{1}{2} \int_{\mathscr{C}} d^2 z\, B_{ij}(X)\, \epsilon_{\mu\nu}\, \partial_\mu X^i\, \partial_\nu X^j \qquad (1)$$

We assume that $\mathscr{M}$ is a homogeneous space:
$\mathscr{M} = G/H, \quad G$ – compact semi-simple Lie group. For the Lie algebra $\mathfrak{g}$ of the group G we use the standard decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad (2)$$

where $\mathfrak{m} \perp \mathfrak{h}$ w.r.t. the Killing metric on $\mathfrak{g}$. The following relations hold:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

Flag manifolds.

We will be interested in the case when $\mathscr{M}$ is a flag manifold (of the group $SU(N)$):

$$\mathscr{F}_{n_1,\ldots,n_m} = \frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_m))}, \qquad \sum_{i=1}^{m} n_i = N \tag{3}$$

Sigma-models with such target spaces naturally arise, for example, as effective continuum theories of spin chains with $SU(N)$-symmetry
[DB '11-'12, Affleck et.al. '17, Tanizaki & Sulejmanpasic '18, Seiberg et.al. '18]

There also exist sigma-models with flag manifold target spaces that are conjecturally integrable
[Young '06, Beisert & Lücker '12, DB '14$^+$, Delduc, Magro, Vicedo, Lacroix '13$^+$]
This talk is dedicated to the analysis of such models

Flag manifolds are complex manifolds, moreover they carry several complex structures. A complex structure $\mathscr{J}$ on $\mathscr{F}$ is defined by an ordering of the factors in the denominator $\frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_m))}$ [Borel & Hirzebruch '58]. Once a complex structure is chosen, $\mathscr{F}$ may be interpreted as the manifold of linear subspaces embedded into each other:

$$0 \in V_1 \subset \ldots \subset V_m = \mathbb{C}^N, \qquad \dim_{\mathbb{C}} V_k = \sum_{i=1}^{k} n_i \,. \tag{4}$$

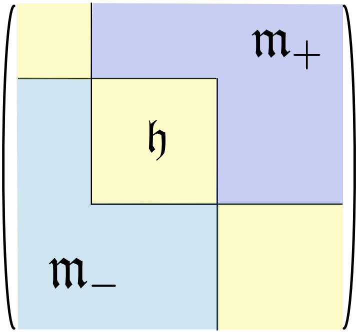
One has a more detailed decomposition of the Lie algebra:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-, \qquad \mathscr{J} \circ \mathfrak{m}_\pm = \pm i\, \mathfrak{m}_\pm \,. \tag{5}$$

Homogeneity and integrability of the complex structure are equivalent to the conditions on the Lie algebra:

$$[\mathfrak{h}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm \,, \qquad [\mathfrak{m}_\pm, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm \,. \tag{6}$$

The complex structure and the Lie algebra.



The decomposition of the Lie algebra.

Definition of the models.

Quite generally, the metric and B-field are constructed as follows. We decompose $\mathfrak{m}_+$ into irreps of the subalgebra $\mathfrak{h}$: $\mathfrak{m}_+ = \oplus_{1\leq i<j\leq m}(\mathfrak{m}_+)_{ij}$ and pick out the corresponding components of the Maurer-Cartan 1-form $J := -g^{-1}dg = \sum_{i,j=1}^{m} J_{ij}$.

Then,

$$ds^2 = h_{ij}dX^i dX^j = \sum_{1\leq i<j\leq m} a_{ij}\,\mathrm{tr}(J_{ij}J_{ji})\,, \qquad a_{ij} > 0 \tag{7}$$

$$B = \sum_{1\leq i<j\leq m} b_{ij}\,\mathrm{tr}(J_{ij}\wedge J_{ji}) \tag{8}$$

As a simplest example, we may set $b_{ij} = a_{ij}$, in which case B is called the fundamental Hermitian form of the metric h w.r.t. one of the complex structures $\mathscr{J}$ on $\mathscr{F}$. One may write $B = h \circ \mathscr{J}$.

Moreover, we will set $a_{ij} = 1$, then the metric is the normal metric on $\mathscr{F}$: $(ds^2 = \mathrm{Tr}(J_{\mathfrak{m}}J_{\mathfrak{m}}))$.

The conjecture of integrability of the models so defined is based on the following evidence:

- The zero-curvature representation

$$A_u = \frac{1+u}{2}\, K_z dz + \frac{1+u^{-1}}{2}\, K_{\bar{z}} d\bar{z}, \qquad u \in \mathbb{C}^*$$

- Involutivity of the integrals of motion
- Explicit solutions of the e.o.m. in certain cases ($\frac{U(3)}{U(1)^3}$)
- Analogy with the case of symmetric spaces (review: [Zarembo '17])
- Explicit form of the quantum anomaly in the non-local charge $\mathcal{Q}_2$, which is similar to the Grassmannian case

Complex symmetric spaces fall in our class, with characteristic property $[\mathfrak{m}_+, \mathfrak{m}_+] = 0$. In fact, this implies $[\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{h}$. Symmetric spaces of the group $SU(N)$ are the Grassmannians

$$\mathbb{G}_{n|N} := \frac{SU(N)}{S(U(n) \times U(N-n))}$$

In this case the canonical one-parametric family of flat connections is

$$\widetilde{A}_\lambda = \frac{1-\lambda}{2}\, \widetilde{K}_z dz + \frac{1-\lambda^{-1}}{2}\, \widetilde{K}_{\bar{z}} d\bar{z},$$

where $\widetilde{K}$ is the canonical Noether current, i.e. the one constructed using the standard action

$$\mathcal{S} = \frac{1}{2} \int\limits_{\mathscr{C}} d^2 z\, h_{ij}(X)\, \partial_\mu X^i\, \partial_\mu X^j \tag{9}$$

Relation to the case of symmetric spaces. 2

The models, which we described above, feature an additional term in their action: $\int_{\mathscr{C}} B$, the integral of the Kähler form. Therefore the Noether current K defined using this action will be different from $\widetilde{K}$, the difference being a 'topological' current:

$$K = \widetilde{K} + *d\mu$$

(In fact, μ is the moment map $G(k,N) \to \mathfrak{su}_N$).

Nevertheless both K and $\widetilde{K}$ are flat. The one-parametric family of connections that we constructed earlier has the form

$$A_u = \frac{1+u}{2}\, K_z dz + \frac{1+u^{-1}}{2}\, K_{\bar{z}} d\bar{z},$$

A natural question arises: How are $\widetilde{A}_\lambda$ and A_u related?

Relation to the case of symmetric spaces. 3

The answer is: $\widetilde{A}_\lambda$ and A_u are related by a gauge transformation Ω:

$$\widetilde{A}_\lambda = \Omega A_u \Omega^{-1} - \Omega d\Omega^{-1}$$

Ω can be written out explicitly ($\tilde{g}$ is the 'dynamical' group element):

$$\Omega = \tilde{g}\Lambda\tilde{g}^{-1}, \quad \text{where} \quad \Lambda = \mathrm{diag}(\underbrace{\lambda^{-1/2},\ldots,\lambda^{-1/2}}_{n},\underbrace{\lambda^{1/2},\ldots,\lambda^{1/2}}_{N-n})$$

Rather important is the nontrivial relation between the spectral parameters:

$$\lambda = u^{1/2}$$

This relation may be confirmed by analyzing the limiting behavior of the holonomies of the connection as $u \to 0$ (such analysis can be borrowed from Hitchin ('90)).

The gauged linear sigma-model (GLSM).

The above B-field is closed in the following case: $dB = 0 \quad \leftrightarrow \quad m = 2$, i.e. $\mathcal{F}$ is a Grassmannian (a symmetric space). For the case of Kähler target spaces the GLSM representation is tantamount to the theory of Kähler quotients. For example, for the Grassmannian one has

$$G(k,N) = \frac{U(N)}{U(k) \times U(N-k)} = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)//U(k) \tag{10}$$

This means that one can write down the Lagrangian

$$\mathscr{L} = \mathrm{Tr}(D_\mu V^\dagger D_\mu V) + \mathrm{Tr}(\lambda(V^\dagger V - r\,\mathbb{1}_k)) \tag{11}$$

Such representations date back to the work of
[Cremmer, Scherk '78, D'Adda, Lüscher, di Vecchia '78]

If one equips the flag manifold with a Kähler metric, the GLSM representation will follow from the theory of quiver representations
[Donagi & Sharpe '08, Ginzburg '12]

In the case $m > 2$ the flag manifold $\mathcal{F}$ is not a symmetric space and the B-field is no longer topological (and the normal metric is not Kähler). Therefore a question arises, how to construct a GLSM representation in this situation.

First recall, that our model depends on the complex structure $\mathscr{J}$. In order to be able construct a $\frac{1}{N}$-expansion, one should consider flags of the form $0 \in V_1 \subset \ldots \subset V_m = \mathbb{C}^N$, where the dimension of the ambient space $N \to \infty$, whereas $M := \dim V_{m-1}$ is fixed. For now we take this as a constraint on the allowed complex structure.

Given this setup, we choose a matrix $V \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)$. Its columns parametrize M vectors that define the flag. They are orthonormal:

$$V^\dagger V = \mathbb{1}_M \,. \tag{12}$$

This is an analog of the moment map constraint.

The gauged linear sigma-model (GLSM).

Now we introduce an analogue of gauge field [DB '17]

$$\mathscr{A}_z := \begin{pmatrix} (A_{11})_z & 0 & 0 & \cdots & 0 \\ (A_{21})_z & (A_{22})_z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (A_{m-1\,1})_z & (A_{m-1\,2})_z & \cdots & \cdots & (A_{m-1\,m-1})_z \end{pmatrix}, \qquad \mathscr{A}_{\bar{z}} = (\mathscr{A}_z)^\dagger \tag{13}$$

and the covariant derivative

$$\mathscr{D}_\mu V := \partial_z V - i\, V\, \mathscr{A}_\mu \,. \tag{14}$$

The Lagrangian has the form:

$$\mathscr{L} = \mathrm{Tr}((\mathscr{D}_\mu V)^\dagger\, \mathscr{D}_\mu V) + \mathrm{Tr}(\lambda(V^\dagger V - r\, \mathbb{1}_M))\,. \tag{15}$$

We have a theory very similar to the Grassmannian $G(M, N)$ sigma-model, but with a 'reduced' gauge field.

The gauged linear sigma-model (GLSM).

Finally, we would like to prove that the representation applies to *any* complex structure $\mathscr{J}$ on the flag manifold. This relies on the fact that for certain complex structures $\mathscr{J}_1$, $\mathscr{J}_2$ the corresponding models are classically equivalent:

$$\mathcal{S}[\mathscr{J}_1] - \mathcal{S}[\mathscr{J}_2] = \int\limits_{\Sigma} \mathscr{O}_{12}, \qquad d\mathscr{O}_{12} = 0\,. \tag{16}$$

To this end we recall that the complex structures on $\mathscr{F}$ are in a one-to-one correspondence with an ordering of the mutually orthogonal spaces $\mathbb{C}^{n_1}, \ldots \mathbb{C}^{n_m}$, composing a flag manifold $\frac{U(N)}{U(n_1)\times \cdots U(n_m)}$.

Proposition. The actions $\mathcal{S}[\mathscr{J}_1]$ and $\mathcal{S}[\mathscr{J}_2]$ differ by a topological term, as in (16), if and only if the corresponding sequences of spaces $\{\mathbb{C}^{n_1}, \ldots \mathbb{C}^{n_m}\}$ differ by a cyclic permutation.

Therefore we can always cyclically permute the subspaces to make sure $\mathbb{C}^{N-M}$ is the largest subspace in the ordering.

The non-local conserved charge.

We consider the Wilson loop $\mathrm{P}\, e^{-\int_\Gamma \mathscr{A}_u}$ of the flat connection $\mathscr{A}_u$ and expand it around the point $u = 1$ to second order. We obtain the following charges:

$$\mathcal{Q}_1 = \int\limits_\Gamma *K \tag{17}$$

$$\mathcal{Q}_2 = \int\limits_\Gamma K - \frac{1}{2} \int\limits_{t<s} dt\, ds\, [(*K)_t, (*K)_s] \tag{18}$$

Here $\Gamma \subset \Sigma$ is an arbitrary closed (or stretching to infinity) contour on the worldsheet.

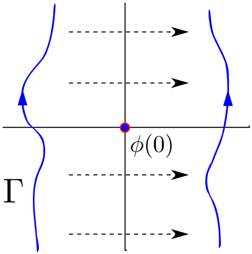
The first one is the conserved charge related to $SU(N)$ symmetry, and the second one is the celebrated non-local conserved charge [Lüscher '78]. These charges generate the Yangian algebra, which underlies the integrable structure of the theory [Bernard '91].

"Conserved" here = depends only on the class of the contour $[\Gamma] \in H_1(\Sigma_{\text{punct}}, \mathbb{Z})$ (recall that $d * K = 0$).

The local conserved charge.

Example. Consider the Lagrangian $\mathcal{L} = \partial_z\phi\,\partial_{\bar{z}}\phi$ with the symmetry $\phi \to \phi + a$. One has the charge $\mathcal{Q} = \int_\Gamma *K = \int_\Gamma i\,(\partial_z\phi\,dz - \partial_{\bar{z}}\phi\,d\bar{z})$. Consider the correlation function

$$\langle \mathcal{Q}\,\phi(0)\rangle = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x\,dy}{x^2+y^2} = -\frac{1}{2}\,\mathrm{sgn}(x). \tag{19}$$



To prove that $\mathcal{Q}_2 = \int\limits_{\Gamma} K - \frac{1}{2} \int\limits_{t<s} dt\, ds\, [(*K)_t, (*K)_s]$ is independent of Γ, we introduce the one-form

$$S(p) := [\left(\int\limits_0^p *K \right), *K]. \tag{20}$$

Then $\mathcal{Q}_2 = \int\limits_{\Gamma} (K - \frac{1}{2} S)$. Since $dS = 2K \wedge K$, we get

$$\mathcal{Q}_2(\Gamma_2) - \mathcal{Q}_2(\Gamma_1) = \int\limits_D (dK - K \wedge K) = 0 \tag{21}$$

$$\partial D = \Gamma_2 - \Gamma_1 \,.$$

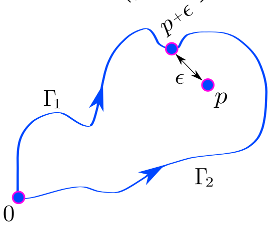
The regularized charge.

In the quantum theory the one-form S is not well-defined. Consider a regularized version (ϵ fixed)

$$S_\epsilon(p) := \left[\left(\int_0^{p+\epsilon} *K \right), *K(p) \right]. \tag{22}$$

The one-form S_ϵ has an ambiguity under $\epsilon \to e^{2\pi i}\epsilon$. Indeed,

$$S_{e^{2\pi i}\epsilon} - S_\epsilon = \left[\left(\oint *K \right), *K(p) \right]. \tag{23}$$



The regularized non-local conserved charge.

We may use the Ward identity

$$[\left(\oint *K\right), *K(p)] = 2 * K(p) \qquad (24)$$

to show that the following operator is ambiguity-free [DB '18]

$$\mathcal{Q}_\epsilon(\Gamma) := \int_\Gamma \left(\left[ia + \frac{1}{2\pi}\log(\epsilon)\right] K_z dz + \left[-ia + \frac{1}{2\pi}\log(\bar{\epsilon})\right] K_{\bar{z}} d\bar{z} - \frac{1}{2} S_\epsilon\right), \qquad (25)$$

This is similar, but not identical to the original definition of Lüscher.

- There exists a limit $\lim_{\epsilon\to 0} \mathcal{Q}_\epsilon$
- The limit depends on the curve Γ through an anomaly 2-form Ω_A, namely

$$\delta_\Gamma\left(\lim_{\epsilon\to 0} \mathcal{Q}_\epsilon\right) = \int_{D_{\delta\Gamma}} \Omega_A, \qquad (26)$$

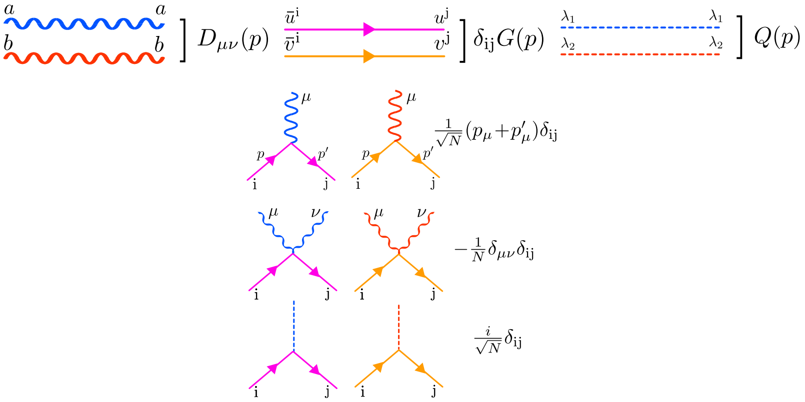
The simplest Lagrangian.

In the case of the $\mathbb{CP}^{N-1}$-model the existence of similar anomalies for local charges was predicted in [Polyakov '77] and [Goldschmidt, Witten '80], and the anomaly in the non-local charge was explicitly computed in [Abdalla, Abdalla, Gomes '81-'84].

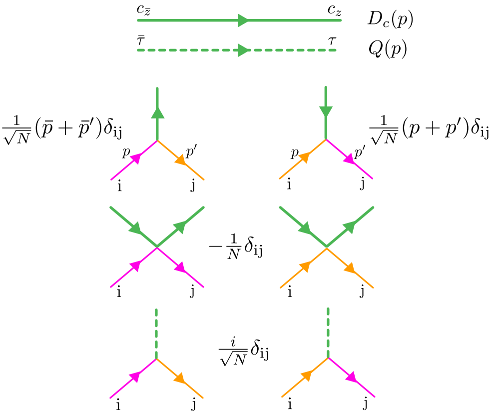
To compute Ω_A one needs to introduce the Feynman rules of the $\frac{1}{N}$-expansion. To this end for the moment we will restrict to the target space $\mathscr{F} = \frac{U(N)}{U(1)\times U(1)\times U(N-2)}$. Then the Lagrangian has the form of two interacting $\mathbb{CP}^{N-1}$ models [DB '18]

$$\begin{aligned}
\mathscr{L} = & |D_\mu^{(a)} u|^2 + |D_\mu^{(b)} v|^2 + \\
& + i\left(c_{\bar z}\,\bar v \circ \partial_z u - c_z\,\partial_{\bar z}\bar u \circ v + c_z\,\bar u \circ \partial_{\bar z} v - c_{\bar z}\partial_z \bar v \circ u\right) + c_z\,c_{\bar z}\left(|u|^2 + |v|^2\right) + \\
& + i\lambda_1\left(\|u\|^2 - N\right) + i\lambda_2\left(\|v\|^2 - N\right) + i\tau\,\bar u \circ v + i\bar\tau\,\bar v \circ u\,.
\end{aligned}$$

The Feynman rules.



The propagators and vertices of two $\mathbb{CP}^{N-1}$ models.



The new vertices and propagators.

The OPE.

The Noether current has the following form:

$$K = 2\left(V\,(\mathscr{D}_z V)^\dagger\, d\bar{z} - (\mathscr{D}_z V)\, V^\dagger\, dz\right). \tag{27}$$

Note that this current is different from the standard one even in the case of symmetric target spaces (Grassmannians).

To prove that there exists a limit $\lim\limits_{\epsilon\to 0} \mathcal{Q}_\epsilon$, one needs the OPE

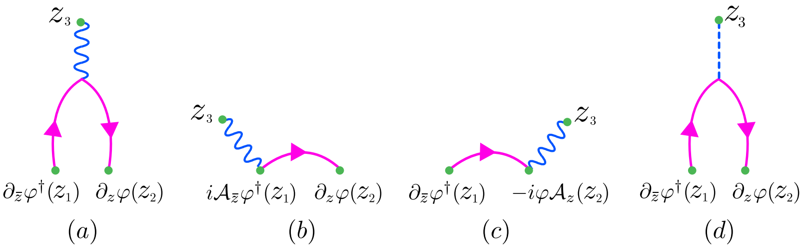
$$[(*K)_z(p+\epsilon), (*K)_z(p)] = \frac{1}{\pi\epsilon}\, K_z(p) + \text{finite terms} \tag{28}$$

(The only commutator singular enough to produce a potential divergence in $\mathcal{Q}_\epsilon$.)

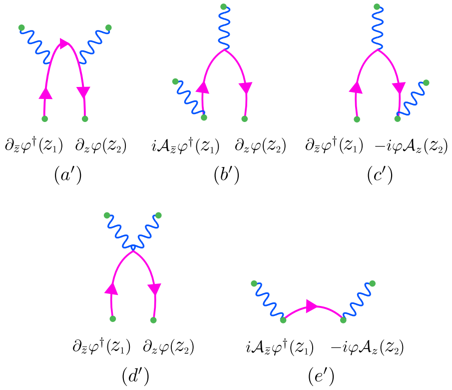
The anomaly 2-form Ω_A is computed from the OPE

$$[*K(p+\epsilon), *K(p)] \sim [K_z(p+\epsilon), K_{\bar{z}}(p)] + [K_z(p), K_{\bar{z}}(p+\epsilon)], \qquad \epsilon \to 0\,. \tag{29}$$

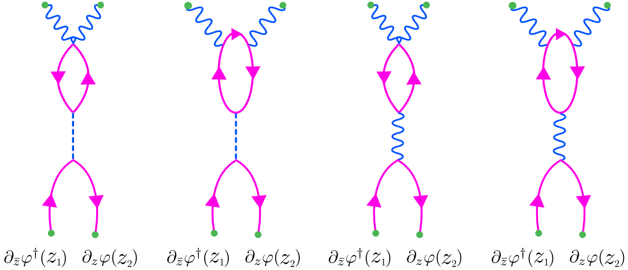
The OPE $[K_z(p+\epsilon), K_{\bar z}(p)]$ is given by the following diagrams ($\varphi = V$)



Order $\frac{1}{\sqrt{N}}$.



Order $\frac{1}{N}$, part 1.



(a'') (b'') (c'') (d'')

Order $\frac{1}{N}$, part 2.

The anomaly 2-form.

The final result for the anomaly 2-form is as follows [DB '18]

$$\Omega_A = \frac{1}{4\pi} V F V^\dagger, \qquad \text{where} \qquad F = d\mathscr{A} - \mathscr{A} \wedge \mathscr{A}\,. \tag{30}$$

Here $V \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)$. Recall that the auxiliary 'gauge field' $\mathscr{A}$ has restricted form, as compared to the gauge field of the would-be Grassmannian $G(M,N)$:

$$\mathscr{A}_z := \begin{pmatrix} (A_{11})_z & 0 & 0 & \cdots & 0 \\ (A_{21})_z & (A_{22})_z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (A_{m-1\,1})_z & (A_{m-1\,2})_z & \cdots & \cdots & (A_{m-1\,m-1})_z \end{pmatrix}, \qquad \mathscr{A}_{\bar{z}} = (\mathscr{A}_z)^\dagger \tag{31}$$

Conclusion and outlook.

- Integrable sigma-models beyond symmetric target spaces [DB '14$^+$]
  "Geometry $\cap$ Integrable models"
- Relation to η-deformed models
  [Fateev '96, Klimcik '09, Delduc, Magro, Vicedo '13$^+$, DB '16]
- GLSM formulation beyond Kähler target spaces [DB '17]
- The anomaly has a form, similar to the case of symmetric spaces
  [Abdalla, Abdalla, Gomes '81-'84]
- Possibly exact to all orders [Abdalla, Abdalla, Gomes '83]
- Potentially possible to cancel it by introducing fermions
- Pohlmeyer reduction