Flag manifold sigma-models

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– $X : \Sigma \to \mathscr{M}$. The action:

$$\mathcal{S} = \frac{1}{2} \int\limits_{\Sigma} d^2 z \sqrt{\gamma}\, h_{ij}(X)\, \gamma^{\mu\nu} \partial_\mu X^i\, \partial_\nu X^j + \frac{1}{2} \int\limits_{\Sigma} d^2 z\, B_{ij}(X)\, \epsilon_{\mu\nu}\, \partial_\mu X^i\, \partial_\nu X^j$$

– $\mathscr{M}$ is a homogeneous space:
$\mathscr{M} = G/H, \quad G$ compact semi-simple.

The Lie algebra $\mathfrak{g}$ admits the standard decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

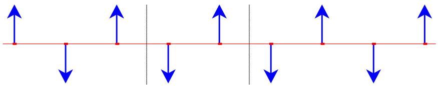
where $\mathfrak{m} \perp \mathfrak{h}$ w.r.t. the Killing metric on $\mathfrak{g}$.

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

We will be interested in the case when $\mathscr{M}$ is a flag manifold (of the group $SU(N)$):

$$\mathscr{F}_{n_1,\ldots,n_m} = \frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_m))}, \qquad \sum_{i=1}^{m} n_i = N$$

There are several encounters of such sigma-models in mathematical physics.
First, they arise as effective continuum theories of spin chains with $SU(N)$-symmetry. ($SU(2)$-case: [Haldane '83])
The idea is that the flag manifold is the space of Néel vacua of the classical chain:



– Geometric theory: [DB '11-'12]
– Analysis of spin chains: [Affleck et.al. '17 ($SU(3)$), '19 ($SU(N)$)]
– Discrete 't Hooft anomalies [Tanizaki & Sulejmanpasic '18, Seiberg et.al. '18]

Flag manifolds.

– Flag manifold sigma-models arise in the [Cho '80]-[Faddeev-Niemi '97-'99] approach to $SU(N)$ gauge theories, and in the case of complete flags these admit Hopfion solutions, since $\pi_3\left(\frac{U(N)}{U(1)^N}\right) \simeq \mathbb{Z}$ [Amari, Sawado '18]

– Describe the worldsheet theory of non-Abelian vortices in certain 4D (SUSY) gauge theories [Bolokhov, Shifman, Yung, '09] [Ireson '19]

– In this talk I will discuss mostly the sigma-models that are (conjecturally) integrable. These were considered in:
[Young '06, Beisert & Lücker '12] – $\mathbb{Z}_m$-graded spaces,
[DB '14-19] – General complex homogeneous spaces,
[Costello & Yamazaki '19] – Chern-Simons theory,
[Bytsko '94, Brodbeck & Zagermann '00, Delduc et. al. '19] – Ultralocality of Poisson brackets: $\{\mathcal{L}_\lambda(x), \mathcal{L}_\mu(y)\} \sim [\mathbf{r}(\lambda-\mu), \mathcal{L}_\lambda \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_\mu]\, \delta(x-y)$

Complex structures.

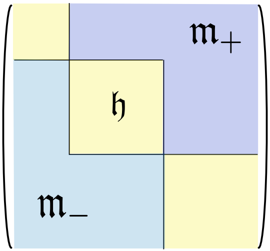
A complex structure $\mathscr{J}$ on $\mathscr{F}$ is defined by an ordering of the factors in the denominator $\frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_m))}$ [Borel & Hirzebruch '58].
$\mathscr{F}$ may then be interpreted as the manifold of embedded linear subspaces:

$$0 \in V_1 \subset \ldots \subset V_m = \mathbb{C}^N, \qquad \dim_{\mathbb{C}} V_k := d_k = \sum_{i=1}^{k} n_i \,.$$

One has a more detailed decomposition of the Lie algebra:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-, \qquad \mathscr{J} \circ \mathfrak{m}_{\pm} = \pm i\, \mathfrak{m}_{\pm} \,.$$

$$[\mathfrak{h}, \mathfrak{m}_{\pm}] \subset \mathfrak{m}_{\pm}\,, \qquad \text{(Homogeneity of } \mathscr{J}\text{)}$$
$$[\mathfrak{m}_{\pm}, \mathfrak{m}_{\pm}] \subset \mathfrak{m}_{\pm} \qquad \text{(Integrability of } \mathscr{J}\text{)}.$$



The decomposition of the Lie algebra.

Decompose $\mathfrak{m}_+$ into irreps of the subalgebra $\mathfrak{h}$: $\mathfrak{m}_+ = \oplus_{1\leq i<j\leq m} (\mathfrak{m}_+)_{ij}$ and pick the corresponding components of the Maurer-Cartan 1-form $J := -g^{-1}dg = \sum_{i,j=1}^{m} J_{ij}$.

$$ds^2 = h_{ij} dX^i dX^j = \sum_{1\leq i<j\leq m} a_{ij}\, \mathrm{tr}(J_{ij} J_{ji})\,, \qquad a_{ij} > 0$$

$$B = \sum_{1\leq i<j\leq m} b_{ij}\, \mathrm{tr}(J_{ij} \wedge J_{ji})$$

If $b_{ij} = a_{ij}$, B is called the fundamental Hermitian form of the metric h w.r.t. one of the complex structures $\mathscr{J}$ on $\mathscr{F}$: $B = h \circ \mathscr{J}$.

- Kähler metric: $a_{ij} = z_i - z_j$ (Kirillov-Kostant-Souriau form).
- Normal metric: $a_{ij} = 1$, $ds^2 = \mathrm{Tr}(J_{\mathfrak{m}} J_{\mathfrak{m}})$. Geodesics are homogeneous [Alekseevsky & Arvanitoyeorgos '07].

Integrability.

The conjecture of integrability of the models is based on the following evidence:

- The zero-curvature representation

$$A_u = \frac{1+u}{2}\, K_z dz + \frac{1+u^{-1}}{2}\, K_{\bar{z}} d\bar{z}, \qquad u \in \mathbb{C}^*\,,$$

  where K is the Noether current that is *flat*. This property holds for a more general class of hom. spaces: the complex hom. spaces [DB '16]. These were classified by [Wang '54] and are toric bundles over flag manifolds.
- Involutivity of the integrals of motion [Delduc et. al. '19]
- Explicit solutions of the e.o.m. in certain cases $\left(\frac{U(3)}{U(1)^3}\right)$ [DB '16]
- Analogy with the case of symmetric spaces (review: [Zarembo '17])
- Explicit form of the quantum anomaly in the non-local charge $\mathcal{Q}_2$, which is similar to the Grassmannian case [DB '19]

Relation to the case of symmetric spaces

Complex symmetric spaces fall in our class, with characteristic property $[\mathfrak{m}_+, \mathfrak{m}_+] = 0$. This implies $[\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{h}$. Symmetric spaces of $SU(N)$ are the Grassmannians

$$G(n, N) := \frac{SU(N)}{S(U(n) \times U(N-n))}$$

In this case the canonical one-parametric family of flat connections is

$$\widetilde{A}_\lambda = \frac{1-\lambda}{2}\, \widetilde{K}_z dz + \frac{1-\lambda^{-1}}{2}\, \widetilde{K}_{\bar{z}} d\bar{z},$$

where $\widetilde{K}$ is the canonical Noether current, constructed using the standard action

$$\mathcal{S} = \frac{1}{2} \int\limits_{\Sigma} d^2 z\, h_{ij}(X)\, \partial_\mu X^i\, \partial_\mu X^j$$

Relation to the case of symmetric spaces. 2

The models, which we described above, feature an additional term in their action: $\int\limits_{\Sigma} B$, the integral of the Kähler form. It modifies the Noether current:

$$K = \widetilde{K} + *d\mu, \qquad \mu = \text{ (moment map } G(k,N) \to \mathfrak{su}_N)$$

Both K and $\widetilde{K}$ are flat (in the case of $\mathbb{CP}^1$ first observed by [Bytsko '94]). Define

$$A_u = \frac{1-u}{2}\,K_z dz + \frac{1-u^{-1}}{2}\,K_{\bar{z}} d\bar{z},$$

$\widetilde{A}_\lambda$ and A_u are related by an explicit gauge transformation $\Omega(\lambda)$: $\widetilde{A}_\lambda = \Omega A_u \Omega^{-1} - \Omega d\Omega^{-1}$, where $\lambda = u^{1/2}$. This relation is confirmed by analyzing the limiting behavior of the holonomies as $u \to 0$ [Hitchin ('90)].

The gauged linear sigma-model (GLSM).

$dB = 0 \quad \leftrightarrow \quad m = 2$, i.e. $\mathcal{F}$ is a Grassmannian (a symmetric space). In the Kähler case the GLSM representation is tantamount to the theory of Kähler quotients. For the Grassmannian one has

$$G(k,N) = \frac{U(N)}{U(k)\times U(N-k)} = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)//U(k)$$

This means that one can write down the Lagrangian

$$\mathscr{L} = \mathrm{Tr}((D_{\bar{z}}U)^\dagger (D_{\bar{z}}U)), \qquad U^\dagger U = \mathbb{1}_k\,.$$

Such representations date back to
[Cremmer, Scherk '78, D'Adda, Lüscher, di Vecchia '78]

For a flag manifold with a Kähler metric, the GLSM representation follows from the theory of Nakajima (quiver) varieties
[Donagi & Sharpe '08, Ginzburg '12]

The gauged linear sigma-model (GLSM).

$m > 2$: $\mathcal{F}$ not a symmetric space, $dB \neq 0$, normal metric not Kähler.
How can one construct a GLSM representation?

The model depends on the complex structure $\mathscr{J}$, i.e. we fix the flag

$$0 \in V_1 \subset \ldots \subset V_{m-1} \subset V_m = \mathbb{C}^N \,,$$
$$\text{and} \quad M := \dim V_{m-1} \,.$$

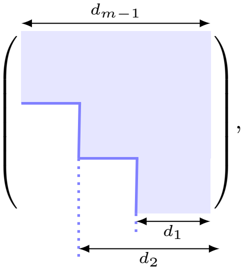
Construct a matrix $U \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)$, whose columns parametrize M vectors that define the flag. They are orthonormal:

$$U^\dagger U = \mathbb{1}_M \qquad \text{(Analog of the moment map constraint)}$$

The $\frac{1}{N}$-expansion: $\quad N \to \infty \,, \quad M$ fixed.

Now we introduce an analogue of gauge field [DB '17]

$\mathcal{A}_z =$



$, \qquad \mathcal{A}_{\bar z} = (\mathcal{A}_z)^\dagger\,.$

and the covariant derivative $\mathscr{D}_{\bar z} U := \partial_{\bar z} U - i\, U\, \mathscr{A}_{\bar z}$. The Lagrangian has the form:

$$\mathscr{L} = \mathrm{Tr}((D_{\bar z} U)^\dagger (D_{\bar z} U)), \qquad \text{where} \qquad U^\dagger U = \mathbb{1}_M\,.$$

– Similar to the Grassmannian $G(M, N)$ model, but with a 'reduced' gauge field.

The gauged linear sigma-model (GLSM).

The representation in fact applies to *any* complex structure $\mathscr{J}$.
For some complex structures $\mathscr{J}_1$, $\mathscr{J}_2$:

$$\mathcal{S}[\mathscr{J}_1] - \mathcal{S}[\mathscr{J}_2] = \int\limits_{\Sigma} \mathscr{O}_{12}, \qquad d\mathscr{O}_{12} = 0\,.$$

Recall that the complex structures on $\mathscr{F}$ are in a one-to-one correspondence with an ordering of the mutually orthogonal spaces $\mathbb{C}^{n_1}, \ldots \mathbb{C}^{n_m}$.

Proposition. [DB '19] $\mathcal{S}[\mathscr{J}_1]$ and $\mathcal{S}[\mathscr{J}_2]$ differ by a topological term if and only if the corresponding sequences of spaces $\{\mathbb{C}^{n_1}, \ldots \mathbb{C}^{n_m}\}$ differ by a cyclic permutation.

Therefore we can always cyclically permute the subspaces to make sure $\mathbb{C}^{N-M}$ is the largest subspace in the ordering.

The non-local conserved charge.

Consider the Wilson loop $\mathrm{P}\, e^{-\int_\Gamma \mathscr{A}_u}$ of the flat connection $\mathscr{A}_u$ along an infinite contour $\Gamma \subset \Sigma \simeq \mathbb{R}^2$ and expand it around the point $u = 1$.

$$\mathcal{Q}_1 = \int\limits_\Gamma *K \qquad\qquad (SU(N)\ \text{symmetry charge})$$

$$\mathcal{Q}_2 = \int\limits_\Gamma K - \frac{1}{2} \int\limits_{t<s} dt\, ds\, [(*K)_t, (*K)_s] \qquad\qquad (\text{non-local charge [Lüscher '78]})$$

These charges generate the Yangian algebra, which underlies the integrable structure of the theory [Bernard '91].

The charges are "conserved", i.e. they do not depend on Γ as long as it does not cross the points of insertion of the local operators (recall $d * K = 0$).

The non-local conserved charge.

To prove that $\mathcal{Q}_2 = \int\limits_{\Gamma} K - \frac{1}{2} \int\limits_{t<s} dt\, ds\, [(*K)_t, (*K)_s]$ is independent of Γ, we introduce the one-form

$$S(p) := [\left(\int\limits_{-\infty}^{p} *K \right), *K(p)].$$

Then $\mathcal{Q}_2 = \int\limits_{\Gamma} (K - \frac{1}{2} S)$. Since $dS = 2K \wedge K$, we get

$$\mathcal{Q}_2(\Gamma_2) - \mathcal{Q}_2(\Gamma_1) = \int\limits_{D} (dK - K \wedge K) = 0$$

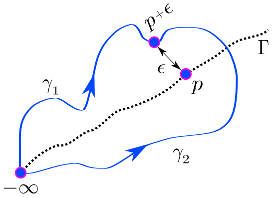
$$\partial D = \Gamma_2 - \Gamma_1\,.$$

In the quantum theory the one-form S is not well-defined. Consider a regularized version (ϵ fixed)

$$S_\epsilon(p) := \left[\left(\int_{-\infty}^{p+\epsilon} *K \right), *K(p) \right].$$

The one-form S_ϵ has an ambiguity under $\epsilon \to e^{2\pi i}\epsilon$. Indeed,

$$S_{e^{2\pi i}\epsilon} - S_\epsilon = \left[\left(\oint *K \right), *K(p) \right].$$



The regularized non-local conserved charge.

Using the Ward identity

$$[\left(\oint *K\right), *K(p)] = 2 * K(p)\,,$$

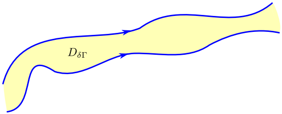
we see that the following operator is ambiguity-free [DB '18]

$$\mathcal{Q}_\epsilon(\Gamma) := \int\limits_\Gamma \left(\left[ia + \frac{1}{2\pi}\log(\epsilon) \right] K_z dz + \left[-ia + \frac{1}{2\pi}\log(\bar{\epsilon}) \right] K_{\bar{z}} d\bar{z} - \frac{1}{2} S_\epsilon \right)\,,$$

This is similar, but not identical to the original definition of Lüscher.

- There exists a limit $\lim\limits_{\epsilon\to 0} \mathcal{Q}_\epsilon$
- The limit depends on the curve Γ through an anomaly 2-form Ω_A, namely

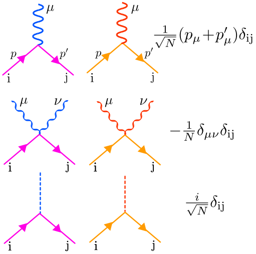
$$\delta_\Gamma \left(\lim_{\epsilon\to 0} \mathcal{Q}_\epsilon \right) = \int\limits_{D_{\delta\Gamma}} \Omega_A\,,$$

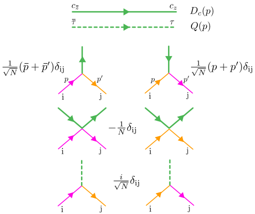


The simplest Lagrangian.

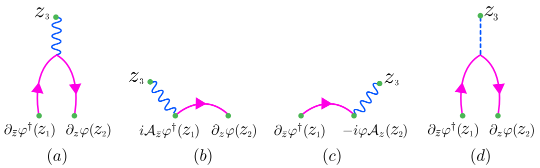
Anomalies in the $\mathbb{CP}^{N-1}$-model:
[Polyakov '77], [Goldschmidt, Witten '80] (local),
[Abdalla, Abdalla, Gomes '81-'84] (non-local).

To compute Ω_A one introduces the Feynman rules of the $\frac{1}{N}$-expansion [DB, 2017]. To this end we restrict to the target space $\mathscr{F} = \frac{U(N)}{U(1)\times U(1)\times U(N-2)}$.

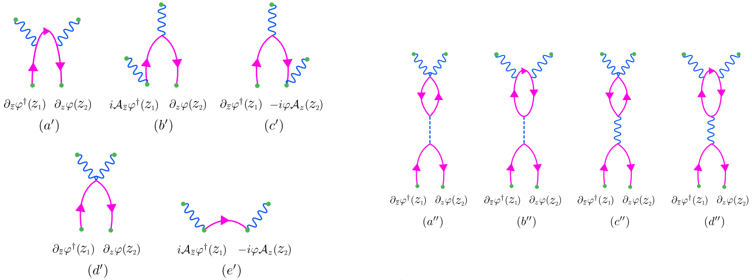




The OPE $[K_z(p+\epsilon), K_{\bar{z}}(p)]$ is given by the following diagrams ($\varphi = U$):



Order $\frac{1}{\sqrt{N}}$.



Order $\frac{1}{N}$.

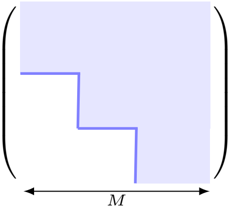
The anomaly 2-form.

The final result for the anomaly 2-form [DB '19]:

$$\Omega_A = \frac{1}{4\pi}\, U F U^\dagger, \qquad \text{where} \qquad F = d\mathcal{A} - \mathcal{A}\wedge\mathcal{A}\,.$$

Here $U \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)$. The auxiliary 'gauge field' $\mathcal{A} \in \mathfrak{gl}_M$ has restricted form, as compared to the gauge field of the Grassmannian $G(M, N)$:

$\mathcal{A}_z =$

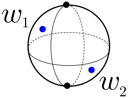


$, \qquad \mathcal{A}_{\bar z} = (\mathcal{A}_z)^\dagger\,.$

[Costello-Yamazaki '19]: a semi-holomorphic 4D Chern-Simons theory on $\Sigma \times \mathscr{C}$, where Σ = 'topological plane' $(z, \bar{z})$,
$\mathscr{C}$ = complex curve $(w, \bar{w})$ with a holomorphic differential $\omega = dw \neq 0$.
$K_{\mathscr{C}} = 0$ implies $\mathscr{C} \simeq \mathbb{C}, \mathbb{C}^*, E_\tau$. The action:

$$S_{\mathrm{CS}} = \frac{1}{\hbar} \int\limits_{\Sigma \times \mathscr{C}} \omega \wedge \mathrm{Tr} \left(A \wedge (dA + \frac{2}{3} A \wedge A) \right) ,$$

where $A = A_z dz + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w}$. One couples this theory to two $\beta\gamma$ systems, with target space $T^*\mathcal{M}$, where $\mathcal{M}$ is a complex homogeneous space:



$\mathscr{C} \approx \mathbb{C}^*$

$$S_{def} = \int\limits_{\Sigma} i\, dz \wedge d\bar{z} \left(p_i D_{\bar{z}}^{(w_1)} q^i + \bar{p}_i D_z^{(w_2)} \bar{q}^i \right),$$

where $$D_{\bar{z}}^{(w_1)} q^i = \partial_{\bar{z}} q^i - \sum_a (A_{\bar{z}}^{(w_1)})_a v_a^i \,.$$

The Costello-Yamazaki approach.

Next one sets the 'light-cone' gauge $A_{\bar{w}} = 0$ and solves for $A_z, A_{\bar{z}}$. In this gauge the equations are

$$\begin{aligned}
&\partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} + [A_z, A_{\bar{z}}] = 0, \\
&\partial_{\bar{w}} A_z = \delta^{(2)}(w - w_1) \sum_a p_i v_a^i \, \tau_a \\
&\partial_{\bar{w}} A_{\bar{z}} = \delta^{(2)}(w - w_2) \sum_a \bar{p}_{\bar{i}} v_a^{\bar{i}} \, \tau_a \, .
\end{aligned}$$

This is a family $A_z(w), A_{\bar{z}}(w)$ of flat connections, depending meromorphically on w.

Key observation: Green's function $\bar{\partial}_{\bar{w}}^{-1}$ = classical r-matrix [Belavin, Drinfeld '80]

Rational case: $r(w) = \frac{\sum \tau_a \otimes \tau_a}{w} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r(w) = \frac{\mathbb{1}}{w} \in \mathrm{End}(\mathfrak{g})$.

Trigonometric case: let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (complex structure on G, Manin triple, etc.), then $r(u) = \frac{\sum \tau_a^+ \otimes \tau_a^-}{1-u} - \frac{\sum \tau_a^- \otimes \tau_a^+}{1-u^{-1}} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r(z) = \frac{\Pi_+}{1-u} - \frac{\Pi_-}{1-u^{-1}} \in \mathrm{End}(\mathfrak{g})$.

The Costello-Yamazaki approach.

Upon integrating out $A_z, A_{\bar{z}}$, we get (rational case)

$$S = \int i\, dz \wedge d\bar{z} \left(p_i \partial_{\bar{z}} q^i + \bar{p}_i \partial_z \bar{q}^i + \mathbf{r}_{w_1 - w_2} \left(p_i v_a^i\, \tau_a, \bar{p}_{\bar{i}} v_a^{\bar{i}}\, \tau_a \right) \right) =$$

$$= \int i\, dz \wedge d\bar{z} \left(p_i \partial_{\bar{z}} q^i + \bar{p}_i \partial_z \bar{q}^i + \frac{1}{w_1 - w_2} \sum |p_i v_a^i|^2 \right) =$$

$$= \text{integrate out} \quad p, \bar{p} \quad (\text{the fiber of} \quad T^* \mathcal{M}) \sim$$

$$\sim \int i\, dz \wedge d\bar{z} \left(G_{i\bar{j}} \partial_{\bar{z}} q^i \partial_z \bar{q}^{\bar{j}} \right) \qquad \text{with} \qquad G_{i\bar{j}} = \left(\sum_a v_a^i v_a^{\bar{j}} \right)^{-1}$$

Now I will prove that this is the same flag manifold σ-model that was described earlier. [DB, to appear]

The Costello-Yamazaki approach.

To this end let us return to the GLSM formulation:

$$\mathscr{L} = \mathrm{Tr}((D_{\bar z}U)^\dagger (D_{\bar z}U)), \qquad (U^\dagger U = \mathbb{1}_k\,.)$$
$$\to \mathrm{Tr}\left((D_{\bar z}U)^\dagger\, D_{\bar z}U\, \frac{1}{U^\dagger U}\right), \qquad (\mathrm{rk}(U^\dagger U) = k\,.)$$

The last line shows that it is a non-reductive GIT quotient.

$$\to \mathrm{Tr}\,(V D_{\bar z}U) + \mathrm{Tr}\,(V D_{\bar z}U)^\dagger - \mathrm{Tr}\left(V V^\dagger U^\dagger U\right)$$
$$\to \mathrm{Tr}\,(V \mathscr{D}_{\bar z}U) + \mathrm{Tr}\,(V \mathscr{D}_{\bar z}U)^\dagger + \mathrm{Tr}\,(\Phi_z \Phi_{\bar z})$$

where $\mathscr{D}_{\bar z}U = \partial_{\bar z}U + i\,U\mathcal{A}_{\bar z} + i\,\Phi_{\bar z}U$.

$\Phi_{\bar z}$ should be interpreted as the Chern-Simons gauge field of Costello-Yamazaki (and the classical r-matrix $\mathbf{r} = \mathbb{1}$). The vector fields v_a^i linearize: $v_a = \mathrm{Tr}(\tau_a\, U\, \frac{\partial}{\partial U})\,.$

The $\beta\gamma$ Lagrangian:

$$\mathscr{L} = \mathrm{Tr}\,(V\mathscr{D}_{\bar z}U) + \mathrm{Tr}\,(V\mathscr{D}_{\bar z}U)^\dagger + \mathrm{Tr}\,(\Phi_z\Phi_{\bar z})$$
$$\mathscr{D}_{\bar z}U = \partial_{\bar z}U + i\,U\mathcal{A}_{\bar z} + i\,\Phi_{\bar z}U\,.$$

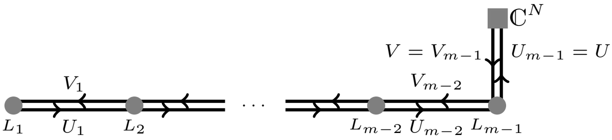
E.O.M.:

$VU\big|_{\mathfrak{k}^*} = 0,\qquad \mathfrak{k} =$ parabolic subalgebra of $\mathfrak{gl}_M,\quad$ (Moment Map)

$\Phi_z = -i\,UV\quad =\quad$ Noether current .

$\mathscr{D}_{\bar z}U = 0,\qquad \mathscr{D}_{\bar z}V = 0$

[Nakajima '94] quiver for $T^*\mathscr{F}$:



Nilpotent orbits

Implications of the E.O.M.:

$\partial_{\bar{z}}\Phi_z + i\,[\Phi_{\bar{z}}, \Phi_z] = 0\,.$ Flatness of Noether current, Principal Chiral Model

$(\Phi_z)^m = 0$ Nilpotent orbit in $\mathfrak{gl}_N$.

Assume

$$(\Phi_z)^m = 0, \qquad \text{and} \qquad (\Phi_z)^{m-1} \neq 0\,.$$

The map $\Phi_z(z, \bar{z})$ satisfying the E.O.M. defines a flag

$$0 \subset \mathrm{Ker}(\Phi_z) \subset \mathrm{Ker}(\Phi_z^2) \subset \cdots \subset \mathrm{Ker}(\Phi_z^m) \simeq \mathbb{C}^N$$

This is a point in

$$\mathscr{F} := \frac{U(N)}{U(\kappa_1) \times \cdots \times U(\kappa_m)}, \qquad \text{where}$$

$$\kappa_j = \dim \mathrm{Ker}(\Phi_z^j)/\mathrm{Ker}(\Phi_z^{j-1}) \qquad = \text{number of Jordan blocks of size at least } i\,.$$

The map $\Sigma \to \mathscr{F}$ is a solution of the flag manifold sigma-model. [DB, to appear]

Conclusion and outlook.

- Integrable sigma-models beyond symmetric target spaces [DB '14$^+$]
- Relation to η-deformed models [Fateev '96, Klimcik '09, Delduc, Magro, Vicedo '13$^+$, DB '16]
- GLSM formulation beyond Kähler target spaces [DB '17]
- The anomaly has a form, similar to the case of symmetric spaces [Abdalla, Abdalla, Gomes '81-'84, DB '19]
- Potentially possible to cancel it by introducing additional degrees of freedom (bosons/fermions) [DB '10, Basso & Rej '12]
- Nilpotent orbits – possibly a new look at classical integrable σ-models

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- Pohlmeyer reduction [Pohlmeyer '76]
- Quantum version
- 'Sausage'-like deformations [Fateev, Onofri, Zamolodchikov '94], [Bazhanov, Kotousov, Lukyanov '17]
- Dual Toda-like theories [Fateev '17], [Litvinov, Spodyneiko '18]