Flag manifold σ-models and Ricci flow

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Based on several papers of the author,

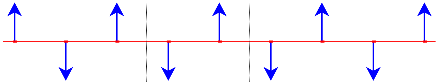
including a recent one with D. Lüst and an upcoming one

Flag manifolds.

In this talk we will consider relativistic σ-models with worldsheet $\mathbb{R}^2$ and target space $\mathscr{M}$. We will be interested in the case when $\mathscr{M}$ is a flag manifold:

$$\mathscr{F}_{n_1,\ldots,n_m} = \frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_S))}, \qquad \sum_{i=1}^{S} n_i = N$$

There are several encounters of such sigma-models in mathematical physics.
First, they arise as effective continuum theories of spin chains with $SU(N)$-symmetry. ($SU(2)$-case: [Haldane '83])
The idea is that the flag manifold is the space of Néel vacua of the classical chain:



– Geometric theory: [DB '11-'12]
– Analysis of spin chains: [Affleck et.al. '17 ($SU(3)$), '19 ($SU(N)$)]
– Discrete 't Hooft anomalies [Tanizaki & Sulejmanpasic '18, Seiberg et.al. '18]

– Flag manifold sigma-models arise in the [Cho '80]-[Faddeev-Niemi '97-'99] approach to $SU(N)$ gauge theories, and in the case of complete flags these admit Hopfion solutions, since $\pi_3\left(\frac{U(N)}{U(1)^N}\right) \simeq \mathbb{Z}$ [Amari, Sawado '18]

– Describe the worldsheet theory of non-Abelian vortices in certain 4D (SUSY) gauge theories [Bolokhov, Shifman, Yung, '09] [Ireson '19]

– In this talk I will discuss mostly the sigma-models that are (conjecturally) integrable. These were considered in:

[Young '06, Beisert & Lücker '12] – $\mathbb{Z}_m$-graded spaces,

[DB '14-19] – General complex homogeneous spaces,

[Costello & Yamazaki '19] – Chern-Simons theory,

[Bytsko '94, Brodbeck & Zagermann '00, Delduc, Kameyama, Lacroix, Magro, Vicedo '19] – Ultralocality of Poisson brackets: $\{\mathcal{L}_\lambda(x), \mathcal{L}_\mu(y)\} \sim [\mathbf{r}(\lambda-\mu), \mathcal{L}_\lambda \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_\mu]\, \delta(x-y)$

Let $g \in SU(N)$, $g^{-1}dg = i\,(\Phi\, dz + \overline{\Phi}\, d\overline{z})$.
$\overline{\partial}\Phi + i\,[\overline{\Phi}, \Phi] = 0\,.$ Flatness of Noether current, Principal Chiral Model
Impose the condition $\Phi^S = 0$ (closure of nilpotent orbit in $\mathfrak{gl}_N$)

Assume

$$\Phi^S = 0, \qquad \text{and} \qquad \Phi^{S-1} \neq 0\,.$$

The map $\Phi(z, \overline{z})$ satisfying the E.O.M. defines a flag

$$0 \subset \mathrm{Ker}(\Phi) \subset \mathrm{Ker}(\Phi^2) \subset \cdots \subset \mathrm{Ker}(\Phi^S) \simeq \mathbb{C}^N$$

This is a point in

$$\mathscr{F} := \frac{U(N)}{U(\kappa_1) \times \cdots \times U(\kappa_S)}, \qquad \text{where}$$

$$\kappa_j = \dim \mathrm{Ker}(\Phi^j)/\mathrm{Ker}(\Phi^{j-1}) \quad = \text{number of Jordan blocks of size at least } i\,.$$

Claim: the map $\Sigma \to \mathscr{F}$ is a solution of a flag manifold sigma-model
This is a map to a single orbit: type does not change due to e.o.m. [DB, 2019]
I will now describe the model.

Complex structures.

To this end one needs a complex structure $\mathscr{J}$ on $\mathscr{F}$.
It is defined by an ordering of the factors in the denominator $\frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_S))}$ [Borel & Hirzebruch '58].
$\mathscr{F}$ may then be interpreted as the manifold of embedded linear subspaces:

$$0 \in V_1 \subset \ldots \subset V_S = \mathbb{C}^N, \qquad \dim_{\mathbb{C}} V_k := d_k = \sum_{i=1}^{k} n_i \,.$$

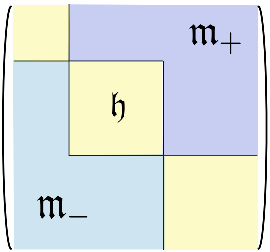
$\mathscr{F} = G/H$, the Lie algebra $\mathfrak{g}$ admits the standard decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

In the presence of $\mathscr{J}$ one has a more detailed decomposition of the Lie algebra:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-, \qquad \mathscr{J} \circ \mathfrak{m}_\pm = \pm i\, \mathfrak{m}_\pm \,.$$

$$[\mathfrak{h}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm \,, \qquad \text{(Homogeneity of } \mathscr{J}\text{)}$$
$$[\mathfrak{m}_\pm, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm \qquad \text{(Integrability of } \mathscr{J}\text{)}.$$



The decomposition of the Lie algebra.

The metric and B-field.

Decompose $\mathfrak{m}_+$ into irreps of $\mathfrak{h}$: $\mathfrak{m}_+ = \oplus_{1\leq i<j\leq S}(\mathfrak{m}_+)_{ij}$,

Maurer-Cartan one-form $J := -g^{-1}dg = \sum\limits_{i,j=1}^{S} J_{ij}$.

$$ds^2 = \mathscr{G}_{ij}dX^i dX^j = \sum_{1\leq i<j\leq S} a_{ij}\,\mathrm{tr}(J_{ij}J_{ji})\,, \qquad a_{ij}>0$$

$$B = \sum_{1\leq i<j\leq S} b_{ij}\,\mathrm{tr}(J_{ij}\wedge J_{ji})$$

If $b_{ij}=a_{ij}$, B is called the fundamental Hermitian form of the metric $\mathscr{G}$ w.r.t. one of the complex structures $\mathscr{J}$ on $\mathscr{F}$: $B=\mathscr{G}\circ\mathscr{J}$.

- Kähler metric: $a_{ij}=z_i-z_j$ (Kirillov-Kostant-Souriau form $B=\mathrm{Tr}(z\,J\wedge J)$).
- Normal metric: $a_{ij}=1$, $\quad ds^2=\mathrm{Tr}(J_{\mathfrak{m}}J_{\mathfrak{m}})$ ('Killing metric')
  Geodesics are homogeneous [Alekseevsky & Arvanitoyeorgos '07].

Action: $$\mathcal{S}[\mathscr{G},\mathscr{I}] := \int_\Sigma d^2z\,\|\partial X\|^2_{\mathscr{G}} + \int_\Sigma X^*B \quad\sim\quad \int_\Sigma d^2z\,G_{m\bar{n}}\,\partial U^m\overline{\partial U^n}$$

Integrability.

The conjecture of integrability of the models is based on the following evidence:

- The zero-curvature representation

$$A_u = \frac{1+u}{2}\, K_z dz + \frac{1+u^{-1}}{2}\, K_{\overline{z}} d\overline{z}, \qquad u \in \mathbb{C}^*\,,$$

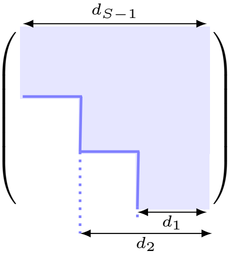
  where K = Noether current (*flat*).
  A more general class – complex hom. spaces [DB '16]:
  toric bundles over flag manifolds [Wang '54].
- Involutivity of the integrals of motion [Delduc et. al. '19]
- Explicit classical solutions $\left(\frac{U(3)}{U(1)^3}\right)$ [DB '16], generalizing [Din, Zakrzewski '80]
- Analogy with the case of symmetric spaces (review: [Zarembo '17])
- Explicit form of anomaly in non-local charge:
  similar to Grassmannian case [DB '19]

Complex symmetric spaces: $[\mathfrak{m}_+, \mathfrak{m}_+] = 0$ ($\Rightarrow [\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{h}$).
Symmetric spaces of $SU(N)$: Grassmannians $G(m,N) := \frac{SU(N)}{S(U(m)\times U(N-m))}$

Kähler case: GLSM $\leftrightarrow$ Kähler quotients.
Grassmannian: $G(m,N) = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^N)//U(m)$. Lagrangian

$$\mathscr{L} = \mathrm{Tr}((\overline{D}U)^\dagger(\overline{D}U)), \qquad \overline{D}U := \overline{\partial}U - i\,U\,\overline{\mathcal{A}}\,, \qquad U^\dagger U = \mathbb{1}_m\,.$$

[Cremmer, Scherk '78, D'Adda, Lüscher, di Vecchia '78].
Flag manifold with Kähler metric: GLSM $\leftrightarrow$ Nakajima (quiver) varieties
[Nakajima '94, Nitta '03, Donagi & Sharpe '08].
Flag manifold with 'Killing metric' (not Kähler for $S > 2$): a 'gauge field' [DB '17]

$\mathcal{A} =$ 

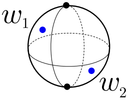
, $\qquad \overline{\mathcal{A}} = (\mathcal{A})^\dagger\,, \qquad$ A 'reduced' gauge field!

The Costello-Yamazaki approach.

[Costello-Yamazaki '19]: a semi-holomorphic 4D Chern-Simons theory on $\Sigma \times \mathscr{C}$, where
Σ = 'topological plane' $(z, \overline{z}) \to$ worldsheet to-be
$\mathscr{C}$ = complex curve $(w, \overline{w})$ with a holomorphic differential $\omega = dw \neq 0$.
$K_{\mathscr{C}} = 0$ implies $\mathscr{C} \simeq \mathbb{C}, \mathbb{C}^*, E_\tau$. The action:

$$S_{\text{CS}} = \frac{1}{\hbar} \int_{\Sigma \times \mathscr{C}} \omega \wedge \text{Tr} \left(A \wedge (dA + \frac{2}{3} A \wedge A) \right) ,$$

where $A = A_z dz + A_{\overline{z}} d\overline{z} + A_{\overline{w}} d\overline{w}$. One couples this theory to two $\beta\gamma$ systems, with target space $T^*\mathcal{M}$, where $\mathcal{M}$ is a complex homogeneous space:



$\mathscr{C} \approx \mathbb{C}^*$

$$S_{def} = \int_{\Sigma} d^2 z \, (p_i D_{\overline{z}}^{(w_1)} q^i + \overline{p}_i D_z^{(w_2)} \overline{q}^i) ,$$

where $\quad D_{\overline{z}}^{(w_1)} q^i = \partial_{\overline{z}} q^i - \sum_a (A_{\overline{z}}^{(w_1)})_a v_a^i \, .$

'Light-cone' gauge $A_{\overline{w}} = 0$, solve for $A_z, A_{\overline{z}}$. In this gauge the equations are

$$\partial_{\overline{z}} A_z - \partial_z A_{\overline{z}} + [A_z, A_{\overline{z}}] = 0,$$
$$\partial_{\overline{w}} A_z = \delta^{(2)}(w - w_1) \sum_a p_i v_a^i \, \tau_a$$
$$\partial_{\overline{w}} A_{\overline{z}} = \delta^{(2)}(w - w_2) \sum_a \overline{p}_{\overline{i}} v_a^{\overline{i}} \, \tau_a \, .$$

Family $A_z(w), A_{\overline{z}}(w)$ of flat connections, depending meromorphically on w!

Key observation: Green's function $\overline{\partial}_{\overline{w}}^{-1}$ = classical r-matrix [Belavin, Drinfeld '80]

Rational case: $r(w) = \frac{\sum \tau_a \otimes \tau_a}{w} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r(w) = \frac{\mathbb{1}}{w} \in \mathrm{End}(\mathfrak{g})$.

Trigonometric case: let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (complex structure on G, Manin triple, etc.), then $r_{\mathrm{comp.}}(u) = \frac{\sum \tau_a^+ \otimes \tau_a^-}{1-u} - \frac{\sum \tau_a^- \otimes \tau_a^+}{1-u^{-1}} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r_{\mathrm{comp.}}(z) = \frac{\Pi_+}{1-u} - \frac{\Pi_-}{1-u^{-1}} \in \mathrm{End}(\mathfrak{g})$.

Upon integrating out $A_z, A_{\overline{z}}$, we get (rational case)

$$S = \int d^2z \, \left(p_i \partial_{\overline{z}} q^i + \overline{p}_i \partial_z \overline{q}^i + \mathbf{r}_{w_1 - w_2}\left(p_i v_a^i \, \tau_a, \overline{p}_{\overline{i}} v_a^{\overline{i}} \, \tau_a\right)\right) =$$

$$= \int d^2z \, \left(p_i \partial_{\overline{z}} q^i + \overline{p}_i \partial_z \overline{q}^i + \frac{1}{w_1 - w_2} \sum |p_i v_a^i|^2\right) =$$

$$= \text{integrate out} \quad p, \overline{p} \quad (\text{the fiber of} \quad T^*\mathcal{M}) \sim$$

$$\sim \int d^2z \, \left(G_{i\overline{j}} \partial_{\overline{z}} q^i \partial_z \overline{q}^{\overline{j}}\right) \qquad \text{with} \qquad G_{i\overline{j}} = \left(\sum_a v_a^i v_a^{\overline{j}}\right)^{-1}$$

Invertibility $\leftrightarrow$ Homogeneous space
Rational case: the flag manifold σ-model described earlier. [DB, 2019]

But this also provides deformations of those models, trigonometric and elliptic. We pass over to this topic, starting with deformations of the $\mathbb{CP}^{n-1}$ model.

The sausage model.

Simplest deformation: 2D target space with a $U(1)$-isometry [DB-Lüst, 2020]

$$ds^2 = \sum_{i,j=1}^{2} G_{ij}\, dX^i dX^j = \frac{1}{4\, g(\mu)}\, d\mu^2 + g(\mu)\, d\phi^2\,.$$

For what $g(\mu)$ is the model integrable?
Mechanical reduction always integrable (2 integrals of motion).
Generalized Pohlmeyer map: set $G_{ij}\, \partial X^i \overline{\partial} X^j = \cosh\chi$.

Sinh-Gordon equation replaced by $\quad \overline{\partial}\partial\chi - 2g''(\mu)\,\sinh\chi = 0\,.$
Add the equation for μ: $\quad \partial\overline{\partial}\mu - 2\, g'(\mu)\,\cosh\chi = 0\,.$

The two eqs. follow from a single Lagrangian if $g(\mu) = b + a\,\cosh\mu$:

$$\mathscr{L} = \frac{1}{2}\partial\mu\overline{\partial}\mu + \frac{1}{2}\partial\chi\overline{\partial}\chi + 2a\,\cosh\mu\,\cosh\chi\; =$$
$$= \left(\frac{1}{2}\partial\widetilde{\mu}\,\overline{\partial}\widetilde{\mu} + a\,\cosh(\sqrt{2}\widetilde{\mu})\right) + \left(\frac{1}{2}\partial\widetilde{\chi}\,\overline{\partial}\widetilde{\chi} + a\,\cosh(\sqrt{2}\widetilde{\chi})\right)$$

The (Kähler) metric of the 'sausage' [Fateev, Onofri, Zamolodchikov, 1994]

$$ds^2 = \left(\frac{1}{s} - s\right) \frac{|dW|^2}{(s+|W|^2)(\frac{1}{s}+|W|^2)}\,, \qquad 0 < s < 1\,.$$

$\mathbb{CP}^{N-1}$ also has a generalized Kähler deformation [Demulder et.al. 2020], constructed along the lines of [Delduc, Magro, Vicedo 2013]. The B-field has the form $B = \sum_i b_i \wedge d\phi_i$, so T-dualizing all angles we get rid of it.

T-dual geometry is Kähler with potential [DB-Lüst, 2020]

$$\mathcal{K} = \sum_{j=1}^{n} (Z_j \overline{Z}_{j-1} - \overline{Z}_j Z_{j-1}) + 2\sum_{j=1}^{n} P(t_j - t_{j-1} - 2\,\tau), \qquad t_j = Z_j + \overline{Z_j}\,,$$

where $P(t) = \mathrm{Li}_2(e^{-t}) + \frac{t^2}{4}$.

$\mathcal{K}$ not invariant under $Z_i \to Z_i + \delta\alpha_i$. T-duality does not preserve the Kähler property (otherwise T: chiral $\leftrightarrow$ twisted chiral [Rocek, Verlinde 1991]).

The metric satisfies the (simple) Ricci flow equation $-\frac{dg_{i\bar{j}}}{d\tau} = 4\,R_{i\bar{j}}$ with $s = e^{N\tau}$.

A new look at σ-models [DB, to appear]

The $\beta\gamma$ deformed Lagrangian ($s =$ def. parameter, $\Phi = A_z$): [DB, 2019]

$$\mathscr{L} = \mathrm{Tr}\left(V\overline{\mathscr{D}}U\right) + \mathrm{Tr}\left(V\overline{\mathscr{D}}U\right)^\dagger + \mathrm{Tr}\left(r_s^{-1}(\Phi)\overline{\Phi}\right)$$
$$\overline{\mathscr{D}}U = \overline{\partial}U + i\,U\overline{\mathcal{A}} + i\,\overline{\Phi}U\,.$$

r_s is the classical r-matrix: $r_s = \frac{s}{1-s}\,\pi_+ + \frac{1}{1-s}\,\pi_- + \frac{1}{2}\frac{1+s}{1-s}\,\pi_0$ (solution of CYBE)
Φ enters quadratically $\to$ integrate it out.
Very fruitful approach: introduce a 'Dirac boson'

$$\Psi_a = \begin{pmatrix} U_a \\ \overline{V}_a \end{pmatrix}, \qquad a = 1, \ldots, N\,.$$

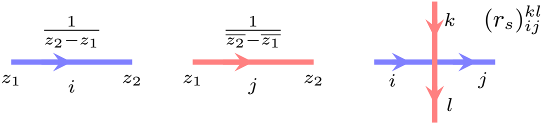
Then

$$\mathscr{L} = \overline{\Psi_a}\not{D}\Psi_a + (r_s)_{ab}^{cd}\left(\overline{\Psi}_a\frac{1+\gamma_5}{2}\Psi_c\right)\left(\overline{\Psi}_d\frac{1-\gamma_5}{2}\Psi_b\right).$$

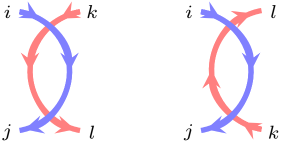
σ-model = chiral gauged Gross-Neveu model (in bosonic incarnation)!

– Chirality: integrate out V-variables $\to$ return to the geometric form of the model.
– The deformation is manifestly Hermitian

Feynman rules:



Diagrams contributing to the β-function at one loop:



β-function:

$$\beta_{ij}^{kl} = \sum_{p,q=1}^{N} \left((r_s)_{ip}^{kq} (r_s)_{pj}^{ql} - (r_s)_{ip}^{ql} (r_s)_{pj}^{kq} \right)$$

Ricci flow [DB, to appear]

The Ricci flow equation $\dot{r}_{ij}^{kl} = \beta_{ij}^{kl}$ has a remarkably simple solution $s = e^{N\tau}$ (was conjectured in [Costello-Yamazaki 2019]).

Alternatively, return to the σ-model and solve the geometric Ricci flow equations

$$-\dot{g_{ij}} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + 2\,\nabla_i \nabla_j \Phi\,,$$

$$-\dot{B_{ij}} = -\frac{1}{2}\nabla^k H_{kij} + \nabla^k \Phi\, H_{kij}\,,$$

$$-\dot{\Phi} = \text{const.} - \frac{1}{2}\nabla^k \nabla_k \Phi + \nabla^k \Phi \nabla_k \Phi + \frac{1}{24} H_{kmn} H^{kmn}$$

$\mathbb{CP}^1$: the solution $s = e^{2\tau}$ $(N = 2)$ found in [Fateev, Onofri, Zamolodchikov, 1994]
$\mathbb{CP}^{N-1}$: Ricci flow interpolates between a cylinder $(\mathbb{C}^\times)^{N-1}$ in the UV (asymptotic freedom) and a 'round' projective space of vanishing radius in the IR.

Consider the Ricci flow for the metric

$$-\dot{g_{ij}} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + \nabla_i \mathcal{D}_j \Phi + \nabla_j \mathcal{D}_i \Phi \,, \qquad \mathcal{D}\Phi = d\Phi - \mathcal{E}\,.$$

The structure of the blow-up (strong coupling) is as follows: $g_{ij} \sim (1-s)\,(g_{\text{hom.}})_{ij}$.

For the homogeneous metric $\mathcal{D}_i\Phi = 0$. Since $s = e^{N\tau}$, in the limit $-\dot{g_{ij}} \to N\,(g_{\text{hom.}})_{ij}$
Homogeneous (Killing) metric $g_{\text{hom.}}$ satisfies a generalized Einstein condition

$$R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g_{\text{hom.}}^{mm'} g_{\text{hom.}}^{nn'} = N\,(g_{\text{hom.}})_{ij}\,.$$

For Grassmannians $H = 0$, i.e. $R_{ij} = N\,(g_{\text{hom.}})_{ij}$.
N= first Chern number of the tangent bundle
$c_1(G(m,N)) = N\,[\mathscr{C}]$ ($\mathscr{C}$=generator) – independent of m!

β-function for a symmetric space = dual Coxeter number (independent of H in $\frac{G}{H}$).
We observe this for non-symmetric spaces – can be proven directly!

Anomalies [DB, to appear]

Euclidean chiral $\mathbb{C}^\times$-symmetry [Zumino, 1977; Mehta 1990] is crucial:

$$U \to \lambda U, \quad V \to \lambda^{-1} V\,, \qquad \text{where} \quad \lambda \in \mathbb{C}^\times\,.$$

Remember $U \in \mathbb{C}^N$, so to pass to $\mathbb{CP}^{N-1}$ we need to gauge the chiral symmetry. However the symmetry is typically anomalous: recall Schwinger's effective action

$$\mathscr{S}_{\text{eff.}} = \frac{\xi}{2} \int dz\, d\bar{z}\, F_{z\bar{z}} \frac{1}{\triangle} F_{z\bar{z}}\,, \qquad F_{z\bar{z}} = i\,(\partial \overline{\mathcal{A}} - \overline{\partial} \mathcal{A})$$

Not invariant under the complexified gauge transformations $\mathcal{A} \to \mathcal{A} + \partial\alpha$, $\overline{\mathcal{A}} \to \overline{\mathcal{A}} + \overline{\partial}\overline{\alpha}$. To cancel the anomaly one can add fermions minimally:

$$\mathscr{L} = \overline{\Psi}_a \not{D} \Psi_a + (r_s)_{ab}^{cd} \left(\overline{\Psi}_a \frac{1+\gamma_5}{2} \Psi_c \right) \left(\overline{\Psi}_d \frac{1-\gamma_5}{2} \Psi_b \right) + \overline{\Theta}_a \not{D} \Theta_a\,,$$

$$\Psi, \Theta \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2 \otimes \mathbb{C}^N)\,.$$

Incidentally these are the same fermions that cancel the anomaly in Lüscher's nonlocal charge [Abdalla et.al., 1981-84]!

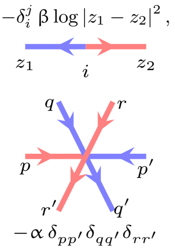
Conjecture: all such flag manifold models with fermions are quantum integrable

Instead of working in a standard gauge like $\overline{U}U = 1$, we can choose

$$U_N = 1 \qquad \text{(inhomogeneous gauge)}$$

Varying the action w.r.t. $\overline{\mathcal{A}}$, we get $UV = 0$, i.e. $V_N = -\sum_{j=1}^{N-1} U_j V_j$.
Substituting in the Lagrangian, we get

$$\mathscr{L} = \sum_{k=1}^{N-1} \left(V_k \overline{\partial} U_k - \overline{V}_k \partial \overline{U}_k + \beta |V_k|^2\right) + \underbrace{+ \sum_{l,m=1}^{N-1} a_{lm} \, |U_l|^2 |V_m|^2 + \gamma \left| \sum_{p=1}^{N-1} U_p V_p \right|^2}_{\text{quartic vertices}} + \underbrace{\alpha \left(\sum_{k=1}^{N-1} |U_k|^2 \right) \left| \sum_{p=1}^{N-1} U_p V_p \right|^2}_{\text{sextic vertices}}$$



Instead of a σ-model we obtained a theory with polynomial interactions! Parallel with Ashtekar variables:

- Direct derivation for $\frac{SL(2,\mathbb{R})}{SO(2)}$ [Brodbeck & Zagermann '00]
- Interactions are polynomial
- Degenerations are allowed (compare with nilpotent orbits)

- Integrable sigma-models beyond symmetric target spaces [DB '14$^+$, Costello-Yamazaki 2019]
- Related to PCM through nilpotent orbits [DB '19]
- GLSM formulation beyond Kähler target spaces [DB '17]
- Hermitian deformations of these models [Costello-Yamazaki 2019, DB - to appear]
- The anomaly of the bosonic model similar to the one for symmetric spaces [Abdalla, Abdalla, Gomes '81-'84, DB '19]
- Cancel it by adding fermions (minimally/ supersymmetrically?), bosonized version [Basso & Rej '12]
- Pohlmeyer reduction [Pohlmeyer '76, Grigoriev-Tseytlin '08, DB-Lüst '20] Toda theories with additional linear fields. Example: $\frac{U(3)}{U(1)^3}$

$$2\,\partial\overline{\partial}X_1 + e^{2(X_1-X_2)} - e^{2(X_3-X_1)} + e^{2(X_1-X_3)}\|(U_{13})_z\|^2 - e^{2(X_2-X_1)}\|(U_{21})_z\|^2 = 0$$

$$\ldots$$

$$\overline{\partial}(U_{13})_z + e^{2(X_3-X_2)}\,\overline{(U_{32})_z} - e^{2(X_2-X_1)}\,\overline{(U_{21})_z} = 0$$

$$\ldots$$

General case?

- σ-models = gauged chiral Gross-Neveu models [DB, to appear]
- The one-loop β-function is universal for all of these models (one-loop exact?)
- (Complicated) Ricci flow eqs. have a simple (ancient) solution, presumably interpolating between the homogeneous metric and a cylinder

- Relation to deformations of $\mathbb{Z}_k$-graded spaces, i.e. analogues of symmetric space deformations of [Fateev '96, Klimčík '02$^+$, Delduc, Magro, Vicedo '13]
- Dual Toda-like theories, $\mathfrak{gl}(N|N)$ symmetry [Fateev '17], [Litvinov '19]
- Construct the full quantum theory: S/R-matrix, thermodynamic Bethe ansatz. Possibly using the ODE/IQFT approach [Bazhanov, Lukyanov, Zamolodchikov 98$^+$, Bazhanov, Kotousov, Lukyanov '17]
- $\cdots$