Some integrability properties of the AdS/CFT correspondence

Dmitri Bykov

Steklov Mathematical Institute, Moscow

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Part I. The overview

The AdS/CFT correspondence

N=4 SYM vs. AdS_5 x S^5 superstring

Maldacena, 1997

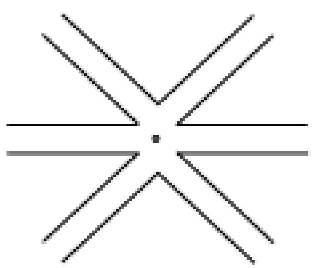
$\mathrm{PSU}(2,2 \mid 4)$

Conformal group $\mathrm{SO}(2,4)$

R-symmetry group $\mathrm{SU}(4)$

The dilatation operator hierarchy

Minahan and Zarembo, 2002
Beisert and Staudacher, 2005



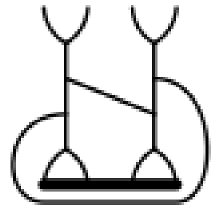
1-loop chiral SU(2) sector ----> XXX spin chain

The Lax pair for the superstring

Bena, Polchinski and Roiban, 2003

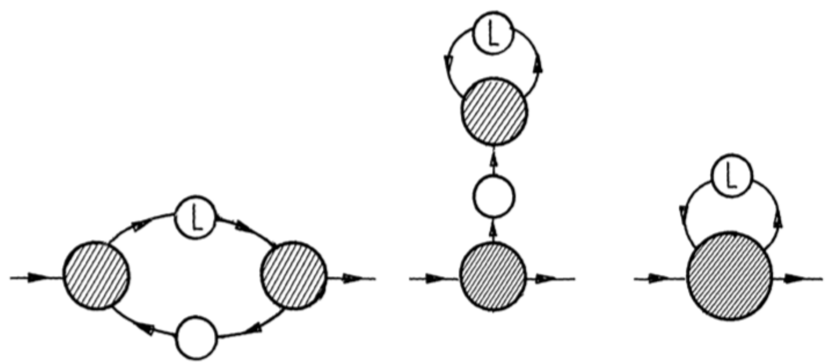
Recent advances

Wrapping effects



Ambjorn, Janik and Kristjansen, 2005

The Luescher formula



Luescher, 1986
Janik, 2008

On the way to the thermodynamic Bethe ansatz

Gromov, Kazakov and Vieira, 2008, 2009
Arutyunov and Frolov, 2008, 2009

Part II. The AdS_4 x CP^3 superstring

AdS_4 x CP^3

Aharony, Bergmann, Jafferis, Maldacena, 2008

$\mathcal{N}=6$ superconformal Chern-Simons-matter theories (N,k)

Discrete values of the coupling constant

$\Updownarrow$

$\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ solution of 11D supergravity

$$\mathrm{S}^7 \xrightarrow{\pi} \mathbb{C}\mathrm{P}^3; \quad \pi^{-1}(x) \sim S^1, \quad x \in \mathbb{C}\mathrm{P}^3$$

't Hooft limit $N, k \to \infty, \quad \lambda \sim \dfrac{N}{k}$ fixed and real

IIA superstring theory on $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$

$$S^{\mathcal{N}=2} = \int \frac{k}{4\pi}\mathrm{Tr}(A\wedge dA + \frac{2}{3}A^3) + D_\mu\bar\phi_i D^\mu\phi_i + i\bar\psi_i\gamma^\mu D_\mu\psi_i$$
$$- \frac{16\pi^2}{k^2}(\bar\phi_i T^a_{R_i}\phi_i)(\bar\phi_j T^b_{R_j}\phi_j)(\bar\phi_k T^a_{R_k}T^b_{R_k}\phi_k) - \frac{4\pi}{k}(\bar\phi_i T^a_{R_i}\phi_i)(\bar\psi_j T^a_{R_j}\psi_j)$$
$$- \frac{8\pi}{k}(\bar\psi_i T^a_{R_i}\phi_i)(\bar\phi_j T^a_{R_j}\psi_j).$$

The coset

Arutyunov and Frolov, 2008

String theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$
as a sigma model on the coset $\dfrac{\mathrm{OSP}(2,2|6)}{\mathrm{SO}(3,1) \times \mathrm{U}(3)}$

maximally symmetric space

Maximal bosonic subgroup $\mathrm{USP}(2,2) \times \mathrm{SO}(6)$

24 real fermions

↓ κ -symmetry fixing

De Azcarraga, Lukierski, 1982
Siegel, 1983
Green, Schwarz, 1984

16 real fermions = # of physical bosonic d.o.f.

There exists a $\mathbb{Z}_4$ -grading of the Lie algebra $\mathrm{osp}(2,2|6)$

Let $g \in OSP(2,2|6)$

Define a left-invariant $\mathrm{osp}(2,2|6)$ -valued 1-form $\mathrm{A} \equiv g^{-1}dg$

Its decomposition under the grading gives

$\mathrm{A}^{(2)}$ - vielbein (zehnbein)

$\mathrm{A}^{(0)}$ - spin connection

$\mathrm{A}^{(1)},\ \mathrm{A}^{(3)}$ - fermionic components

The Lagrangian

$$\mathcal{L} = \gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\epsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)$$

$\kappa = \pm 1$ by κ-symmetry

Coset parametrization

$$g = g_{\mathrm{A}} g_{\chi} g_{\mathrm{B}}$$

fermionic elements

Global symmetry group acts

from the left $g \to g_0 g, \; g_0 \in \mathrm{OSP}(2,2|6)$

whereas local κ -symmetry acts from the right

$g \to g e^{\epsilon}, \; \epsilon$ constrained

A closer look at $\mathbb{CP}^3$

Orthogonal complex structures in $\mathbb{R}^6$

Pick the simplest one $K_6 = I_3 \otimes i\sigma_2$

$\omega K_6 \omega^{-1}$ again a complex structure

$\omega = 1 + \epsilon + \ldots$

$$\{K_6, [\epsilon, K_6]\} = 0$$

$$f(a) = [a, K_6] \qquad g(b) \equiv \{K_6, b\}$$

$$\boxed{0 \longrightarrow u(3) \xrightarrow{i} o(6) \xrightarrow{f} o(6) \xrightarrow{g} \mathbb{R}^N}$$

Further integrability properties

Two-loop anomalous dimensions

Minahan, Zarembo, 2008
Minahan, Schulgin, Zarembo, 2009

All-loop asymptotic Bethe ansatz

Gromov, Vieira, 2008

Conjectured thermodynamic Bethe ansatz

Gromov, 2009

Part III. The low-energy limit of the spinning string

The large spin operators

Gross and Wilczek, 1974
Mueller, 1979

$$ {}^{n}O^{V}_{\mu_1\cdots\mu_n} = i^{n-2}\, S\, \mathrm{Tr}\, F_{\mu_1\alpha} \nabla_{\mu_2} \cdots \nabla_{\mu_{n-1}} F^{\alpha}{}_{\mu_n} - \text{trace terms}, \quad (33) $$

$$ {}^{n}O^{F\pm,0}_{\mu_1\cdots\mu_n} = \tfrac{1}{2}\, i^{n-1}\, S\, \overline{\psi}\, \gamma_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} (1 \pm \gamma_5)\, \psi - \text{trace terms}, \quad (34) $$

$$ {}^{n}O^{F\pm,a}_{\mu_1\cdots\mu_n} = \tfrac{1}{2}\, i^{n-1}\, S\, \overline{\psi}\, \gamma_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} (1 \pm \gamma_5)\, \tfrac{1}{2} \lambda^{a}\, \psi - \text{trace terms}, \quad (35) $$

The anomalous dimension

$$ \Delta = f(\lambda)\ \log(S) $$

The spinning string

Gubser, Klebanov and Polyakov, 2002

AdS metric in global coordinates

$$(ds^2)_{AdS_3} = R^2\left(-\cosh^2(\rho)dT^2 + d\rho^2 + \sinh^2(\rho)d\phi^2\right)$$

The solution

$$T = \kappa\tau,\ \phi = \kappa\tau,\ \rho = \pm\kappa\sigma + \rho_0$$

Relation between charges

$$E - S = \frac{\sqrt{\lambda}}{\pi}\log(S)$$

The Alday-Maldacena limit

Parameter $u = \frac{J}{\sqrt{\lambda}\,\log(S)}$

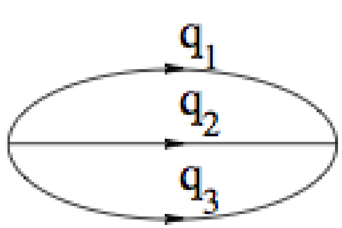
Alday and Maldacena, 2007

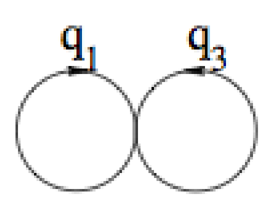
$\varphi = \omega_2 \tau$ $\omega_2 = u\kappa$

Energy (anomalous dimension) $\Delta = (f(\lambda) + \epsilon(\lambda, u))\,\log(S)$

$$\epsilon(\lambda, u) = u^2 \sum \frac{a_n}{(\sqrt{\lambda})^n} (\log(u))^{n-1}, \text{ as } u \to 0$$

SO(6) sigma model





Roiban and Tseytlin, 2007

2-loop sigma model calculation

The spectrum of fluctuations

Frolov and Tseytlin, 2002
Frolov, Tirziu and Tseytlin, 2005

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + a_1 + \frac{a_2}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \qquad (u \neq 0)$$

$$\langle H \rangle = \sum_{n \in \mathrm{Z}^+} \left(\sum_{i \in \mathrm{Bosons}} \sqrt{n^2 + m_i^2} - \sum_{i \in \mathrm{Fermions}} \sqrt{n^2 + m_i^2} \right)$$

$AdS_5 \times S^5$ case: $a_1 \approx -\frac{3 \ln 2}{\pi}$

The spectrum of fluctuations. $AdS_4 \times \mathbb{C}P^3$ case

6 massless modes from $\mathbb{C}P^3$

Alday, Arutyunov and DB, 2007
McLoughlin, Roiban and Tseytlin, 2007

$$\mathscr{D} = 2^8\omega_2^{16}\left[(2k_0-\omega_2)^2-4(k_1^2+\varkappa^2)\right]^2\left[(2k_0+\omega_2)^2-4(k_1^2+\varkappa^2)\right]^2 \times$$
$$\times\left[k_0^4-k_0^2(2k_1^2+\varkappa^2)+k_1^2(k_1^2-\omega_2^2+\varkappa^2)\right]^2$$

- 2 fermions with frequency $\frac{\omega_2}{2}+\sqrt{n^2+\varkappa^2}$
- 2 fermions with frequency $-\frac{\omega_2}{2}+\sqrt{n^2+\varkappa^2}$
- 2 fermions with frequency $\sqrt{n^2+\frac{1}{2}\varkappa^2+\frac{1}{2}\sqrt{\varkappa^4+4\omega_2^2n^2}}$
- 2 fermions with frequency $\sqrt{n^2+\frac{1}{2}\varkappa^2-\frac{1}{2}\sqrt{\varkappa^4+4\omega_2^2n^2}}$

The rank of kappa-symmetry increases when $\omega_2\to 0$.
Use of the coset is problematic!

The full type IIA Green-Schwarz action for AdS_4 x CP^3

$AdS_5 \times S^5$ case — Metsaev and Tseytlin, 1998

$\frac{OSP(6|4)}{U(3)\times SO(1,3)}$ coset — Arutyunov and Frolov, 2008

$\frac{OSP(8|4)}{SO(7)\times SO(1,3)}$ coset — Gomis, Sorokin and Wulff, 2009

M2 brane worldsheet action — Bergshoeff, Sezgin and Townsend, 1987

The dimensional reduction: from 11D SUGRA to IIA

$$g_{\mu\nu},\ \psi^a,\ H_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

Duff, Howe, Inami and Stelle, 1987

Howe, and Sezgin, 2005

«The reduction of supergravity formulated in D=11 superspace to ten dimensions was outlined in [DHIS], but the full details were not given there. Here we provide them.»

$$\mathcal{F} = \mathcal{K}^{(3)} \wedge d\varphi + \mathcal{G}^{(4)} \qquad d\mathcal{K}^{(3)} = d\mathcal{G}^{(4)} = 0$$

$$\mathcal{K}^{(3)} = d\mathcal{B}^{(2)} \qquad \mathcal{D}^{(4)} = \mathcal{G}^{(4)} - \mathcal{A}^{(1)} \wedge \mathcal{K}^{(3)}$$

The Hopf fibration of 11D supergravity   Nilsson and Pope, 1984

The Hopf fibre bundle

$$\pi : S^{2n+1} \to \mathbb{C}P^n$$

n=1: $|z_1|^2 + |z_2|^2 = 1$ $\qquad \pi(z_1, z_2) = \frac{z_2}{z_1}$

$$z_1 = \frac{y_1}{\sqrt{|y_1|^2+|y_2|^2}},\ z_2 = \frac{y_2}{\sqrt{|y_1|^2+|y_2|^2}}$$

The metric:

$$(ds^2)_{S^3} = \frac{dy_i d\bar{y}_i}{\rho^2} - \frac{|dy_i\bar{y}_i|^2}{\rho^4} + \left(i\frac{dy_i\bar{y}_i - y_i d\bar{y}_i}{2\rho^2}\right)^2, \text{ where } \rho^2 = |y_i|^2$$

$$(ds^2)_{S^3} = \frac{dZ d\bar{Z}}{(1+Z\bar{Z})^2} + (d\varphi - A)^2$$

The low-energy limit

DB, 2010, paper to appear

Dropping terms of dimension > 2

$$\mathcal{L} = G_{a\bar{b}}(w,\bar{w})\left(\partial_+ w_a \partial_- \bar{w}_b + \partial_- w_a \partial_+ \bar{w}_b\right) + \frac{i}{2}(\chi \mathcal{D}_+ \chi - \psi \mathcal{D}_- \psi)$$

where

$$\mathcal{D}_\pm = \partial_\pm + \frac{\bar{w}_i \partial_\pm w_i - w_i \partial_\pm \bar{w}_i}{1+|w_i|^2}\, e$$

No supersymmetry

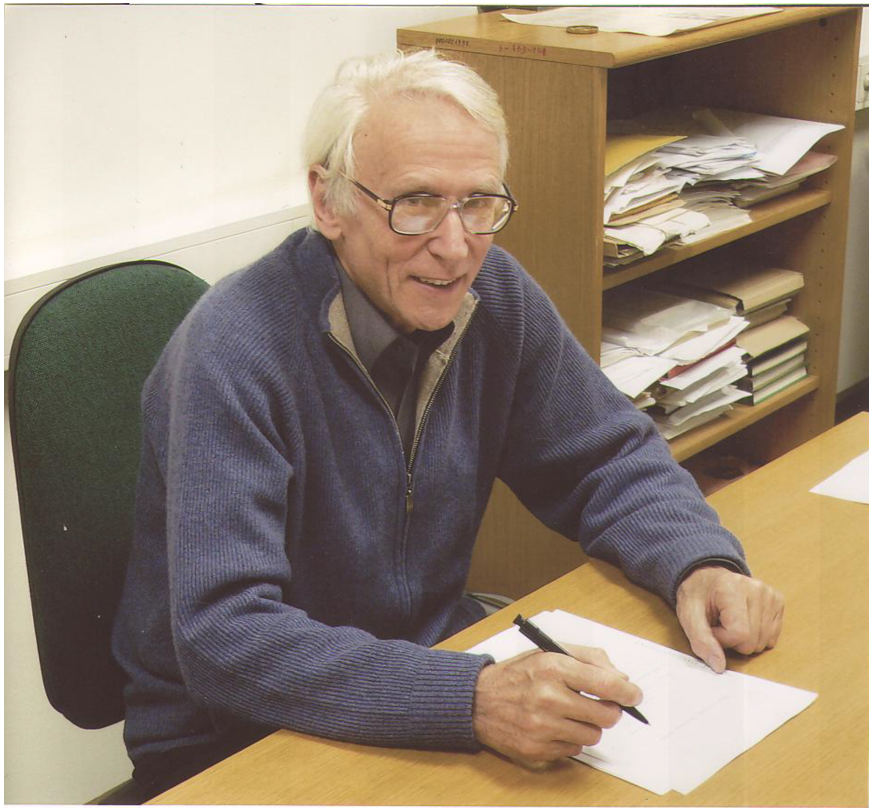
Comparison with SUSY CP^1 model

Di Vecchia, Ferrara, 1977
Zumino, 1979
Witten, 1981

Open issues

- Is the theory integrable?

- It might not be crucial, since one should be able to find the coefficients of the leading logarithms using a renormalization group argument



My best wishes to Andrey Alekseevich!