Sigma-models with non-symmetric homogeneous target spaces

Dmitri Bykov

Steklov Mathematical Institute (Moscow) &

Max-Planck-Institut für Gravitationsphysik (Potsdam)

Based on arXiv:1412.3746 and arXiv:1506.08156

SQS 2015, August 5, Dubna

σ-models

The action of a σ-model describing maps X from a 2D worldsheet $\mathscr{C}$ to a target space $\mathscr{M}$ with metric h is given by

$$\mathcal{S} = \frac{1}{2} \int\limits_{\mathscr{C}} d^2 z \, h_{ij}(X) \, \partial_\mu X^i \, \partial_\mu X^j \tag{1}$$

Its critical points $X(z, \bar{z})$ are called *harmonic maps*.

We will be interested in the case when the target space $\mathscr{M}$ is homogeneous: $\mathscr{M} = G/H$, $\quad G$ compact and simple. We will use the following standard decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2}$$

where $\mathfrak{m} \perp \mathfrak{h}$ with respect to the Killing metric on $\mathfrak{g}$.

Symmetric target spaces

For *any* homogeneous space one has the following relations:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \qquad \Rightarrow \qquad \mathfrak{h} \text{ is a subalgebra}$$
$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \qquad \Rightarrow \qquad \mathfrak{m} \text{ is a representation of } \mathfrak{h}$$

A homogeneous space G/H is called *symmetric* if

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \tag{3}$$

Equivalently, there exists a $\mathbb{Z}_2$-grading on $\mathfrak{g}$, i.e. a Lie algebra homomorphism σ of $\mathfrak{g}$, such that $\sigma(\mathfrak{h}) = \mathfrak{h}$ and $\sigma(\mathfrak{m}) = -\mathfrak{m}$.

Equations of motion. 1

The action of a σ-model with homogeneous target space G/H is globally invariant under the Lie group G. Therefore, there exists a conserved Noether current $k^\mu \in \mathfrak{g}$:

$$\partial_\mu k^\mu = 0 \tag{4}$$

Since the group G acts transitively on its quotient space G/H, the equations of motion are in fact *equivalent* to the conservation of the current.

Equations of motion. 2

It was observed by Pohlmeyer ('76) that in the case when the target space is *symmetric*, the current k is, moreover, flat (with proper normalization):

$$dk - k \wedge k = 0 \tag{5}$$

To get an idea, why this can be the case, recall that the Maurer-Cartan equation has the solution

$$k = -g^{-1}dg, \qquad g \in G \tag{6}$$

What is the relation between g and a point in the configuration space $[\tilde{g}] \in G/H$? The answer is given by Cartan's embedding $G/H \hookrightarrow G$:

$$g = \widehat{\sigma}(\tilde{g})\tilde{g}^{-1} \tag{7}$$

$\widehat{\sigma}$ is a Lie group homomorphism induced by the Lie algebra involution σ.

Equations of motion. 3

Another observation of Pohlmeyer was that the two conditions

$$d * k = 0 \qquad \text{(Conservation)} \qquad (8)$$

$$dk - k \wedge k = 0 \qquad \text{(Flatness)}$$

may be rewritten as an equation of flatness of a connection

$$A_u = \frac{1+u}{2}\, k_z dz + \frac{1+u^{-1}}{2}\, k_{\bar{z}} d\bar{z}, \qquad (9)$$

where we have decomposed the current k as $k = k_z dz + k_{\bar{z}} d\bar{z}$. We have

$$dA_u - A_u \wedge A_u = 0 \qquad (10)$$

This leads to an associated linear system (Lax pair)

$$(d - A_u)\Psi = 0 \qquad (11)$$

Integrability

The existence of a linear system described above is often a sufficient condition for the classical integrability of the model.

The linear system was used by Zakharov & Mikhaylov ('79) to solve the equations of motion for the principal chiral model (target space G). A more rigorous approach was developed by Uhlenbeck ('89). Solutions of the e.o.m. for σ-models with symmetric target spaces may be obtained by restricting the solutions of the principal chiral model.

For homogeneous, but not symmetric target spaces, no linear system is known (no Cartan involution). Hence the models are believed to be non-integrable.

A different model

We will consider the simplest homogeneous, but non-symmetric target space – the flag manifold

$$\mathcal{F}_3 = \frac{U(3)}{U(1)^3} \qquad (12)$$

It is the space of ordered triples of lines through the origin in $\mathbb{C}^3$, and can be parametrized by three orthonormal vectors

$u_i, \quad i = 1, 2, 3$
$\bar{u}_i \circ u_j = \delta_{ij}$, modulo phase rotations: $u_k \sim e^{i\alpha_k} u_k$.

Complex structures on the flag manifold

To formulate the model, we need to pick a particular complex structure on $\mathcal{F}_3$. The (co)tangent space to $\mathcal{F}_3$ is spanned at each point by the one-forms

$$J_{ij} := u_i \circ d\bar{u}_j, \quad i \neq j \tag{13}$$

One can pick any three non-mutually conjugate one-forms and *define* the action of the complex structure operator I on them:

$$I \circ J_{12} = \pm i J_{12}, \quad I \circ J_{23} = \pm i J_{23}, \quad I \circ J_{31} = \pm i J_{31} \tag{14}$$

Altogether there are $2^3 = 8$ possible choices, so that there are 8 invariant almost complex structures. However, only 6 of them are *integrable*.

The action

Pick any integrable complex structure $\mathscr{I}$, and the metric h on $\mathcal{F}_3$ induced from the Killing metric on the Lie algebra $su(3)$. The proposed model has the action

$$\mathcal{S} = \int\limits_{\mathscr{C}} d^2z \, \|\partial X\|^2 + \int\limits_{\mathscr{C}} \omega = \tag{15}$$

$$= \int\limits_{\mathscr{C}} d^2z \, \left(h_{ij} \partial_\mu X^i \partial_\mu X^j + \epsilon_{\mu\nu} \omega_{ij} \partial_\mu X^i \partial_\nu X^j \right),$$

where $\omega = h \circ \mathscr{I}$ is the Kähler form. Note, however, that the metric h is not Kähler, hence the form ω is not closed: $d\omega \neq 0$. Therefore the second term in (15) contributes to the e.o.m.!

The action simplified

Pick the integrable complex structure $\mathscr{I}$, in which J_{12}, J_{13}, J_{23} are holomorphic one-forms. Then the action can be written as (DB '14)

$$\mathcal{S} = \int d^2z \, \left(|(J_{12})_{\bar{z}}|^2 + |(J_{13})_{\bar{z}}|^2 + |(J_{23})_{\bar{z}}|^2 \right) \tag{16}$$

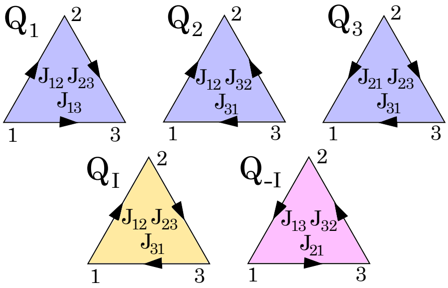
The e.o.m. are:

$$\mathscr{D}_z (J_{12})_{\bar{z}} = 0, \qquad \mathscr{D}_z (J_{31})_{\bar{z}} = 0, \qquad \mathscr{D}_z (J_{23})_{\bar{z}} = 0 \tag{17}$$

From the action (16) it is clear that the holomorphic curves defined by $(J_{12})_{\bar{z}} = (J_{13})_{\bar{z}} = (J_{23})_{\bar{z}} = 0$ minimize the action, hence are solutions of the e.o.m. From (17) it follows that $(J_{12})_{\bar{z}} = (J_{31})_{\bar{z}} = (J_{23})_{\bar{z}} = 0$ is a solution as well. This defines a curve, holomorphic in a different, non-integrable almost complex structure I.

We have seen that the curves, holomorphic in at least two different almost complex structures, satisfy the e.o.m. As we discussed, there are 8 almost complex structures on the flag manifold. Are there any other holomorphic curves that still solve the e.o.m.?

The answer is YES. The relevant complex structures are:



We have already discussed why the $\mathcal{Q}_I$-holomorphic curves and $\mathcal{Q}_1$-holomorphic curves satisfy the e.o.m.

To see why the $\mathcal{Q}_2$- and $\mathcal{Q}_3$-holomorphic curves satisfy the e.o.m., one should note that the differences between the respective Kähler forms are closed forms, i.e. for example $\omega_1 - \omega_2 = \Omega_{top}$ with $d\Omega_{top} = 0$. Therefore the two actions $\mathcal{S}_1$ and $\mathcal{S}_2$ differ by a topological term:

$$\mathcal{S}_1 - \mathcal{S}_2 = \int\limits_{\mathscr{C}} \Omega_{top} \tag{18}$$

This leads to an interesting bound on the instanton numbers of the holomorphic curves. To see this, note that the flag manifold may be embedded as

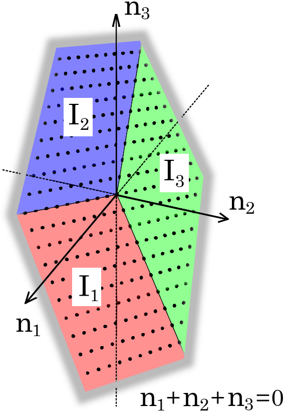
$$i : \mathcal{F}_3 \hookrightarrow \mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2 \tag{19}$$

The second cohomology $H^2(\mathcal{F}_3, \mathbb{R}) = \mathbb{R}^2$ can be described via the pull-backs of the Fubini-Study forms of the $\mathbb{CP}^2$'s, and the corresponding instanton numbers are $n_i = \int\limits_{\mathscr{C}} i^*(\Omega^{(i)}_{FS}),\ i = 1, 2, 3.$

These are subject to the condition

$$n_1 + n_2 + n_3 = 0. \tag{20}$$

The bounds on the topological numbers n_i for the holomorphic curves, which follow from the non-negativity of the actions $\mathcal{S}_i$, are:



Solutions for $\mathscr{C} = \mathbb{CP}^1$

The main point of introducing the action (16) is that, as it turns out, the corresponding Noether current is flat, in full analogy with what happens for σ-models with *symmetric* target-spaces.

The full consequences of this fact still remain to be investigated, but for the moment we can provide a complete description of the solutions of the e.o.m. for the case when the worldsheet $\mathscr{C} = \mathbb{CP}^1$. To describe these solutions, one should recall that there exist three fibrations

$$\pi_i : \mathcal{F}_3 \to (\mathbb{CP}^2)_i, \quad i = 1, 2, 3, \tag{21}$$

each with fiber $\mathbb{CP}^1$.

Solutions for $\mathscr{C} = \mathbb{CP}^1$. 2

All solutions to the e.o.m. are parametrized by the following data:

- One of the projections $\pi_i : \mathcal{F}_3 \to (\mathbb{CP}^2)_i, \quad i = 1, 2, 3$
- A harmonic map $v_{har} : \mathbb{CP}^1 \to (\mathbb{CP}^2)_i$ to the base of the projection
- A holomorphic map $w_{hol} : \mathbb{CP}^1 \to \mathbb{CP}^1$ to the fiber of the projection,
- .

For every triple (i, v_{har}, w_{hol}) there exists a solution of the e.o.m., and all solutions are obtained in this way. (DB '15)

The crucial point is that the harmonic maps to the base manifold $\mathbb{CP}^2$ are known explicitly (Din, Zakrzewski '80) (and the holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are just rational functions).

Outlook

- σ-models with non-symmetric target spaces are believed to be non-integrable
- We have proposed a modified σ-model with a non-symmetric target space, but a non-zero B-field, for which there exists a Lax pair
- For the case when the worldsheet is a sphere, $\mathscr{C} = \mathbb{CP}^1$, we have constructed *all* solutions of the e.o.m.
- Crucial test of integrability: construct solutions for the cylinder worldsheet, $\mathscr{C} = S^1 \times \mathbb{R}$