The worldsheet low-energy limit of the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ superstring

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The most prominent example of the AdS/CFT correspondence is the $\mathbf{AdS_5 \times S^5}$ vs. maximally supersymmetric Yang-Mills theory in D=4

The example we will focus on is the $\mathbf{AdS_4 \times \mathbb{C}P^3}$ versus a Chern-Simons theory with N=6 (out of a maximal number of 8) complex susy's in D=3

$$S^{\mathcal{N}=2} = \int \frac{k}{4\pi} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A^3) + D_\mu \bar{\phi}_i D^\mu \phi_i + i\bar{\psi}_i \gamma^\mu D_\mu \psi_i$$
$$- \frac{16\pi^2}{k^2} (\bar{\phi}_i T^a_{R_i} \phi_i)(\bar{\phi}_j T^b_{R_j} \phi_j)(\bar{\phi}_k T^a_{R_k} T^b_{R_k} \phi_k) - \frac{4\pi}{k} (\bar{\phi}_i T^a_{R_i} \phi_i)(\bar{\psi}_j T^a_{R_j} \psi_j)$$
$$- \frac{8\pi}{k} (\bar{\psi}_i T^a_{R_i} \phi_i)(\bar{\phi}_j T^a_{R_j} \psi_j).$$

k is the coupling constant

In the planar limit the ratio $\mathbf{\frac{N}{k}}$ is kept fixed

The coset

Arutyunov and Frolov, 2008

String theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$
as a sigma model on the coset $\frac{OSP(6|4)}{U(3)\times SO(1,3)}$ — maximally symmetric space

Maximal bosonic subgroup $USP(4) \times SO(6)$

24 real fermions

↓ κ-symmetry fixing

16 real fermions

De Azcarraga, Lukierski, 1982
Siegel, 1983
Green, Schwarz, 1984

There exists a $\mathbb{Z}_4$ -grading of the Lie algebra $osp(6|4)$

Let $g \in osp(6|4)$

Define a left-invariant $osp(6|4)$-valued 1-form $\mathrm{A} \equiv g^{-1}dg$

Its decomposition under the grading gives

$\mathrm{A}^{(2)}$ - vielbein (zehnbein)

$\mathrm{A}^{(0)}$ - spin connection

$\mathrm{A}^{(1)}, \mathrm{A}^{(3)}$ - fermionic components

The Lagrangian

$$\mathcal{L} = \gamma^{\alpha\beta}\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) + \kappa\epsilon^{\alpha\beta}\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big)$$

$\kappa = \pm 1$ by κ-symmetry

Coset parametrization

$$g = g_{\mathrm{A}} g_{\chi} g_{\mathrm{B}}$$

↑

fermionic elements

Global symmetry group acts

from the left $g \to g_0 g, \quad g_0 \in OSP(6|4)$

whereas local κ-symmetry acts from the right

$g \to g e^{\epsilon}, \ \epsilon$ constrained

A closer look at $\mathbb{CP}^3$

Orthogonal complex structures in $\mathbb{R}^6$

Pick the simplest one $K_6 = I_3 \otimes i\sigma_2$

$\omega K_6 \omega^{-1}$ again a complex structure

$\omega = 1 + \epsilon + ...$

$$\{K_6, [\epsilon, K_6]\} = 0$$

$$f(a) = [a, K_6] \qquad g(b) \equiv \{K_6, b\}$$

$$\boxed{0 \longrightarrow u(3) \xrightarrow{i} o(6) \xrightarrow{f} o(6) \xrightarrow{g} \mathbb{R}^N}$$

In a generic gauge theory there's a class of the so-called large-spin operators

$$^{n}O^{V}_{\mu_1 \cdots \mu_n} = i^{n-2} S \,\mathrm{Tr}\, F_{\mu_1 \alpha} \nabla_{\mu_2} \cdots \nabla_{\mu_{n-1}} F^{\alpha}{}_{\mu_n} - \text{trace terms},$$

Gross and Wilczek, 1974

$$^{n}O^{F\pm,0}_{\mu_1 \cdots \mu_n} = \tfrac{1}{2} i^{n-1} S \bar{\psi} \gamma_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} (1 \pm \gamma_5) \psi - \text{trace terms},$$

$$^{n}O^{F\pm,a}_{\mu_1 \cdots \mu_n} = \tfrac{1}{2} i^{n-1} S \bar{\psi} \gamma_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} (1 \pm \gamma_5) \tfrac{1}{2} \lambda^a \psi - \text{trace terms},$$

Their anomalous dimension
has the following characteristic behaviour

$$\Delta = f(\lambda) \; \log(S)$$

Anomalous dimensions are observable

Consider $(p,q) \to (p,q)$ forward scattering

Introduce the spectral function

$$Q^2 = -q^2, \quad p^2 = 1, \quad \nu = p \cdot q, \quad \omega = \frac{2\nu}{Q^2}$$

$$W(\nu > 0, Q^2 > 0) = \mathrm{Im}(T(\nu, Q^2))$$

$$\mathrm{T}(\nu, Q^2) \sim \int dx\, e^{iqx} \langle p | T(j(x) j(0)) | p \rangle$$

The moments are defined as follows

$$\mu_n(Q^2) = \int_1^\infty d\omega \, \frac{W(\nu, Q^2)}{\omega^{n+1}}$$

Nachtmann, 1973

Expansion in spin operators equivalent to spherical harmonic expansion

$$T(\nu, Q^2) = \frac{1}{\pi} \sum_{n \text{ even}} C_n\left(\frac{i\nu}{Q}\right) \sin\left(\frac{\pi}{2}(n+1)\right) Q^{-n} \mu_n(Q^2)$$

$C_n(x)$ being the Chebyshev polynomials

In particular, the logarithmic scaling should allow to make predictions about physical processes in a relevant kinematic regime (small angular correlations)

The moments behave for large Q^2 as

$$\mu_n(Q^2) \sim (Q^2)^{-\frac{1}{2}D_n}$$

where

$$D_n = (d_n - n - 2d_j + 4 + \Delta_n)$$

There is a unitarity bound on the behavior of the anomalous dimension Δ (which follows from the positivity of W)

Nachtmann, 1973

$$\Delta(S) \lesssim S$$

One can reproduce the characteristic behaviour of anomalous dimensions using the spinning string solution of the sigma-model equations of motion

Gubser, Klebanov and Polyakov, 2002

AdS metric in global coordinates

$$(ds^2)_{AdS_3} = R^2\left(-\cosh^2(\rho)dT^2 + d\rho^2 + \sinh^2(\rho)d\phi^2\right)$$

The solution

$$T = \kappa\tau,\ \phi = \kappa\tau,\ \rho = \pm\kappa\sigma + \rho_0$$

Relation between charges

$$E - S = \frac{\sqrt{\lambda}}{\pi}\log(S)$$

To consider the low-energy limit we first introduce additional rotation along a circle in the compact space

$$\varphi = \omega_2 \tau \qquad \omega_2 = u\kappa$$

Parameter describing the rotation is $u = \frac{J}{\sqrt{\lambda}\,\log(S)}$

Energy (anomalous dimension) of large-spin operators with an additional U(1) charge

$$\Delta = (f(\lambda) + \epsilon(\lambda, u))\,\log(S)$$

Freyhult, Rej, Staudacher, 2008

$$\epsilon(\lambda, u) = u^2 \sum \frac{a_n}{(\sqrt{\lambda})^n} (\log(u))^{n-1}, \text{ as } u \to 0$$

In the AdS_5 x S^5 case the leading logs are described by the

$\mathrm{SO}(6)$ sigma model

Alday and Maldacena, 2007

On the sigma-model side one needs to build a type IIA Green-Schwarz action for the AdS_4 x CP^3 string

$AdS_5 \times S^5$ case Metsaev and Tseytlin, 1998

$\frac{OSP(6|4)}{U(3)\times SO(1,3)}$ coset Arutyunov and Frolov, 2008

The coset action has 24 fermions and represents a partially kappa-gauge-fixed Green-Schwarz action

The spectrum of fluctuations

Frolov and Tseytlin, 2002
Frolov, Tirziu and Tseytlin, 2005

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + a_1 + \frac{a_2}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \qquad (u \neq 0)$$

$$\langle H \rangle = \sum_{n \in \mathrm{Z}^+} \left(\sum_{i \in \mathrm{Bosons}} \sqrt{n^2 + m_i^2} - \sum_{i \in \mathrm{Fermions}} \sqrt{n^2 + m_i^2} \right)$$

$AdS_5 \times S^5$ case: $a_1 \approx -\frac{3 \ln 2}{\pi}$

The spectrum of fluctuations. $AdS_4 \times \mathbb{C}P^3$ case

6 massless modes from $\mathbb{C}P^3$

Alday, Arutyunov and DB, 2007
McLoughlin, Roiban and Tseytlin, 2007

$$\mathscr{D} = 2^8 \omega_2^{16} \left[(2k_0 - \omega_2)^2 - 4(k_1^2 + \varkappa^2)\right]^2 \left[(2k_0 + \omega_2)^2 - 4(k_1^2 + \varkappa^2)\right]^2 \times$$
$$\times \left[k_0^4 - k_0^2(2k_1^2 + \varkappa^2) + k_1^2(k_1^2 - \omega_2^2 + \varkappa^2)\right]^2$$

- 2 fermions with frequency $\frac{\omega_2}{2} + \sqrt{n^2 + \varkappa^2}$
- 2 fermions with frequency $-\frac{\omega_2}{2} + \sqrt{n^2 + \varkappa^2}$
- 2 fermions with frequency $\sqrt{n^2 + \frac{1}{2}\varkappa^2 + \frac{1}{2}\sqrt{\varkappa^4 + 4\omega_2^2 n^2}}$
- 2 fermions with frequency $\sqrt{n^2 + \frac{1}{2}\varkappa^2 - \frac{1}{2}\sqrt{\varkappa^4 + 4\omega_2^2 n^2}}$

The rank of kappa-symmetry increases when $\omega_2 \to 0$. Use of the coset is problematic!

Singular gauges

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{D}^{\mu}\phi^* D_\mu\phi - \frac{\widehat{g}}{4}(\phi^*\phi - v^2)^2$$

Impose unitarity gauge: set ϕ to be real

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu\phi)^2 + g^2 A^2\phi^2 - \frac{\widehat{g}}{4}(\phi^2 - v^2)^2$$

Expand around minimum of the potential

$$\phi = v + \varphi$$

To quadratic order one gets the following action

$$\mathcal{L}_{\text{Higgs}}^{(2)} = -\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu\varphi)^2 + g^2 v^2 A^2 - \widehat{g}\, v^2\, \varphi^2$$

Singular gauges: continued

The spectrum:

3 particles of mass $m_1 = gv$ from A_μ

1 particle of mass $m_2 = \widehat{g}^{1/2} v$ from φ

$$D_{\mu\nu}(k) = \frac{1}{k^2 - g^2 v^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{g^2 v^2} \right)$$

Singular limit $v \to 0$

Therefore — change the gauge!

How can one build the complete action?

There exists an M2 brane worldsheet action with kappa-symmetry

Bergshoeff, Sezgin and Townsend, 1987

Under a Hopf reduction the $\mathbf{AdS_4} \times \mathbf{S^7}$ background of 11D SUGRA described by the

$$\frac{OSP(8|4)}{SO(7)\times SO(1,3)}$$ coset

transforms into the AdS_4 x CP^3 IIA background

Gomis, Sorokin and Wulff, 2009

M2 brane action

Peeters, Plefka, Sevrin, de Wit, 1998

Take coset representative $g \in OSP(8|4)$

The current

$$J = -g^{-1}\, dg = E^a T_a + E^\alpha Q_\alpha + A^{ab}\Omega_{ab}$$

is flat $dJ - J \wedge J = 0$

The action takes the following form

$$S = \frac{g}{2\pi} \int d\sigma\, d\tau\, dy\, \left(\eta_{ab}\, E^a E^b + \varkappa\, \epsilon^{abc}\, \partial_a X^m\, \partial_b X^n\, \partial_c X^p\, \mathcal{H}_{mnp}\right)$$

where H is the potential for the 4-form field strength

$$\mathcal{F} = \frac{1}{8}\epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} E^{\bar{a}} \wedge E^{\bar{b}} \wedge E^{\bar{c}} \wedge E^{\bar{d}} + \widehat{\lambda}\, E^\alpha \wedge [\mathbf{\Gamma}_A, \mathbf{\Gamma}_B]^\beta_\alpha\, E_\beta \wedge E^A \wedge E^B$$

The dimensional reduction: from 11D SUGRA to IIA

$$g_{\mu\nu}, \ \psi^a, \ H_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

Duff, Howe, Inami and Stelle, 1987

Howe, and Sezgin, 2005

«The reduction of supergravity formulated in D=11 superspace to ten dimensions was outlined in [DHIS], but the full details were not given there. Here we provide them.»

$$\mathcal{F} = \mathcal{K}^{(3)} \wedge d\varphi + \mathcal{G}^{(4)} \qquad d\mathcal{K}^{(3)} = d\mathcal{G}^{(4)} = 0$$

$$\mathcal{K}^{(3)} = d\mathcal{B}^{(2)} \qquad \mathcal{D}^{(4)} = \mathcal{G}^{(4)} - \mathcal{A}^{(1)} \wedge \mathcal{K}^{(3)}$$

The Hopf fibration of 11D supergravity

Nilsson and Pope, 1984
Sorokin, Tkach and Volkov, 1984

The Hopf fibre bundle

$$\pi : S^{2n+1} \to \mathbb{C}P^n$$

n=1: $|z_1|^2 + |z_2|^2 = 1$ $\quad\quad$ $\pi(z_1, z_2) = \frac{z_2}{z_1}$

$$z_1 = \frac{y_1}{\sqrt{|y_1|^2+|y_2|^2}},\ z_2 = \frac{y_2}{\sqrt{|y_1|^2+|y_2|^2}}$$

The metric:

$$(ds^2)_{S^3} = \frac{dy_i d\bar{y}_i}{\rho^2} - \frac{|dy_i \bar{y}_i|^2}{\rho^4} + \left(i\frac{dy_i\bar{y}_i - y_i d\bar{y}_i}{2\rho^2}\right)^2, \text{ where } \rho^2 = |y_i|^2$$

$$(ds^2)_{S^3} = \frac{dZd\bar{Z}}{(1+Z\bar{Z})^2} + (d\varphi - A)^2$$

To obtain the low-energy limit we need to drop the irrelevant terms, that is the ones of dimension > 2

DB, 2010

$$\mathcal{L} = \eta^{\alpha\beta}\,\overline{\mathcal{D}_\alpha z^j}\,\mathcal{D}_\beta z^j + i\overline{\Psi}\gamma^\alpha\widehat{\mathcal{D}}_\alpha\Psi + \frac{1}{4}(\overline{\Psi}\gamma^\alpha\Psi)^2$$

where $\mathcal{D}_\alpha = \partial_\alpha - i\,\mathcal{A}_\alpha$ and $\widehat{\mathcal{D}}_\alpha = \partial_\alpha + 2\,i\,\mathcal{A}_\alpha$

Besides, the z's are restricted to lie on the sphere $\sum_{j=1}^{4} |z^j|^2 = 1$

No supersymmetry

Comparison with SUSY CP^1 model

Di Vecchia, Ferrara, 1977
Zumino, 1979
Witten, 1981

The model above should be regarded as an effective theory

A cutoff Λ of the order of the first nonzero mass is in place

There are two coupling constants:

the CP^3 radius and the 4-fermion interaction

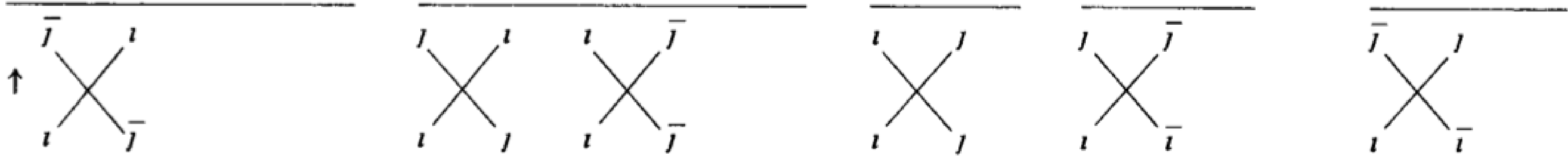
They should both be equal to $\sqrt{\lambda}$ at the cutoff scale, but in general need not be the same

In fact, it is the CP^3 coupling constant which gets renormalized and is asymptotically free, but the fermionic interaction is a marginal deformation

Minimal solutions with SU(N) symmetry:

Berg, Karowski, Weisz, Kurak, 1978

Table of minimal solutions

Classes	Parameter	$t_1(\theta)$	$u_1(\theta)$	$r_1(\theta)$	$u_2(\theta)$	$r_2(\theta)$	$t_2(\theta)$
							
I		1	1	0	0	0	0
II	$\lambda = \frac{2}{n}$	$f(\theta,\lambda)$	$t_1(i\pi-\theta)$	0	$-\lambda \frac{i\pi}{\theta} u_1(\theta)$	0	$-\lambda \frac{i\pi}{i\pi-\theta} t_1(\theta)$
III	$\lambda = \frac{1}{n-1}$	$f(\theta,\lambda)f(i\pi-\theta,\lambda)$	$t_1(\theta)$	$-\lambda \frac{i\pi}{\theta} t_1(\theta)$	$r_1(\theta)$	$-\lambda \frac{i\pi}{i\pi-\theta} t_1(\theta)$	$r_2(\theta)$
IV	$\lambda = \frac{1}{n+1}$	$f(\theta,\lambda)f(i\pi-\theta,\lambda)\, i\, \mathrm{th}\frac{1}{2}(\theta+\frac{1}{2}i\pi)$	$-t_1(\theta)$	$\lambda \frac{i\pi}{\theta} t_1(\theta)$	$r_1(\theta)$	$-\lambda \frac{i\pi}{i\pi-\theta} t_1(\theta)$	$r_2(\theta)$
V	$\mathrm{ch}\, \pi\mu = n$	0	0	$\prod_{k=-\infty}^{\infty} \frac{f(\theta,k/2\mu i)}{f(\theta,k/2\mu i+\frac{1}{2})}$	$r_1(\theta)$	$\frac{\sin\mu(i\pi-\theta)}{\sin\mu\theta}\, r_1(\theta)$	$r_2(\theta)$
VI	$e^{\pi\mu} = n$	0	0	$\prod_{-\infty}^{\infty} \frac{f(\theta,k/2\mu i)}{f(\theta,k/2\mu i+\frac{1}{2})}$	$e^{i\mu\theta}\, r_1(\theta)$	$\frac{\sin\mu(i\pi-\theta)}{\sin\mu\theta}\, r_1(\theta)$	$e^{i\mu(i\pi-\theta)}\, r_2(\theta)$

Minimally coupled fermions

Koeberle, Kurak, 1982

$$\mathcal{L} = \eta^{\alpha\beta}\overline{\mathcal{D}_\alpha z^j}\mathcal{D}_\beta z^j + i\,\overline{\Psi^j}\gamma^\alpha\mathcal{D}_\alpha\Psi^j$$

Soliton-soliton scattering:

$$^{out}\langle\beta(\theta_1)\delta(\theta_2)|\alpha(\theta_1)\gamma(\theta_2)\rangle^{in} = t_1(i\pi-(\theta_1-\theta_2))\delta_{\alpha\beta}\delta_{\gamma\delta} + t_2(i\pi-(\theta_1-\theta_2))\delta_{\alpha\delta}\delta_{\gamma\beta}$$

Soliton-antisoliton scattering:

$$^{out}\langle\beta(\theta_1)\bar{\delta}(\theta_2)|\alpha(\theta_1)\bar{\gamma}(\theta_2)\rangle^{in} = t_1(\theta_1-\theta_2)\delta_{\alpha\beta}\delta_{\gamma\delta} + t_2(\theta_1-\theta_2)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

where

$$t_1(\theta) = \frac{\Gamma(\frac{1}{2}+\frac{\theta}{2\pi i})\Gamma(\frac{1}{2}+\frac{1}{N}-\frac{\theta}{2\pi i})}{\Gamma(\frac{1}{2}-\frac{\theta}{2\pi i})\Gamma(\frac{1}{2}+\frac{1}{N}+\frac{\theta}{2\pi i})}$$

and

$$t_2(\theta) = -\frac{2\pi i}{N(i\pi-\theta)}t_1(\theta)$$

The scattering is reflectionless

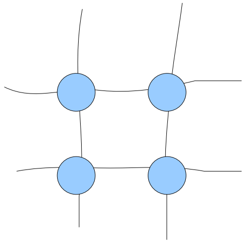
The Thirring model

$$\mathcal{L} = i\bar{\psi}\gamma^{\alpha}\partial_{\alpha}\psi + m\bar{\psi}\psi + g(\bar{\psi}\gamma^{\alpha}\psi)^2$$

$m \neq 0$ → Integrable quantum field theory (Lax operator etc.)

$m = 0$ → conformal quantum field theory with c=1

Describes the critical behaviour of various systems, for instance the Ashkin-Teller model



$$H = \sum_{\langle i,j \rangle} (s_1^i s_1^j + s_2^i s_2^j + g\, s_1^i s_1^j s_2^i s_2^j)$$

Bosonization of the model

After the standard changes

$$i\bar{\psi}\gamma^\alpha\partial_\alpha\psi \to (\partial_\alpha\phi)^2, \quad \bar{\psi}\gamma^\alpha\psi \to \epsilon^{\alpha\beta}\partial_\beta\phi, \quad \bar{\psi}\psi \to \cos(\phi)$$

one arrives at the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha\phi)^2 + \frac{1}{\sqrt{\pi+g}}\,\phi\,\epsilon^{\alpha\beta}\partial_\alpha A_\beta$$

Introduce the following unity under the path integral

$$1 = \int d\lambda_\mu\, dJ_\mu\; e^{i\int d^2x\,\lambda_\mu(J^\mu - \bar{\psi}\gamma^\mu\psi)}$$

$$\mathcal{S} = \int d^2x\,[i\bar{\psi}\gamma^\alpha\partial_\alpha\psi + iA^\alpha J_\alpha + g(J_\alpha)^2 + i\lambda_\alpha(J^\alpha - \bar{\psi}\gamma^\alpha\psi)]$$

Integrate over the fermions (using Schwinger's result) and over J to obtain $F_{\mu\nu}\frac{1}{\Box}F_{\mu\nu}$

CP^N models in the 1/N expansion

D'Adda, Di Vecchia, Luescher, 1978

$$= \frac{i}{p^2+m^2}$$

$$= iA(p;m^2)^{-1}$$

$$= i\Gamma^{-1}_{\mu\nu}$$

In Lorentz gauge we obtain the following

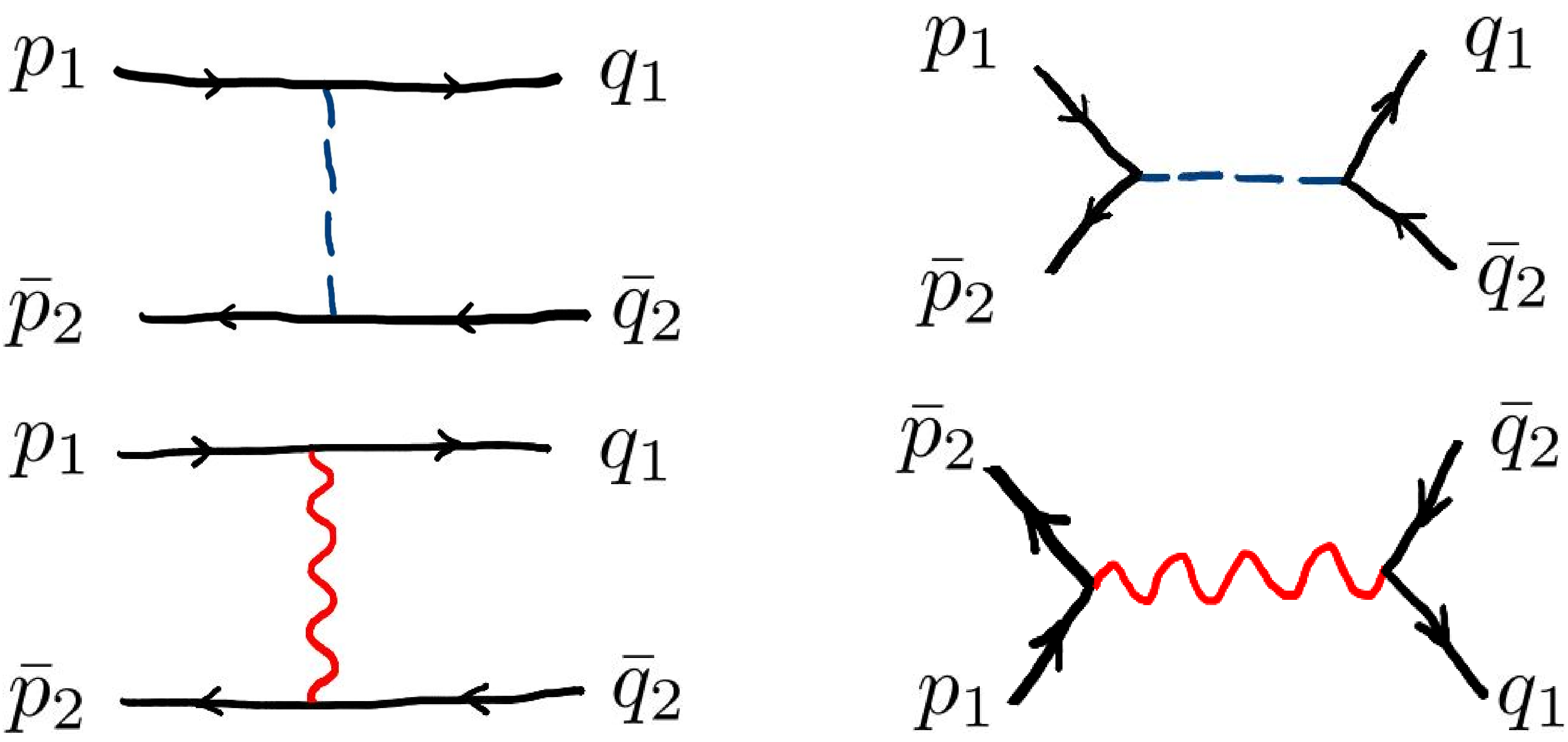
$$\Gamma_{\mu\nu} = (\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})\{(p^2+4m^2)A(p;m^2) + \frac{1}{\pi}(e^2-1)\}$$

where

$$A(p;m^2) = \frac{1}{4\pi}\frac{1}{\sqrt{p^2(p^2+4m^2)}}\log\frac{p^2+2m^2+\sqrt{(p^2+2m^2)^2-4m^4}}{p^2+2m^2-\sqrt{(p^2+2m^2)^2-4m^4}}$$

1/N difficulties of the current model

The reflection amplitude should be zero, as conjectured



As a result, the amplitude is nonzero

However, the 1/N expansion is only a sufficient condition

The chemical potential

$$H \to H - \mu Q$$

μ is positive and sufficiently large

Hasenfratz, Maggiore, Niedermayer, 1992

Hollowood, Evans, 1994

The ground state of the new Hamiltonian is a highly excited state of the original theory

The excitations in the new theory are massive → one gets rid of the infrared problems

These methods were used to calculate soliton masses in terms of perturbative normalization points

One might hope that the deformation under consideration
only modifies the soliton masses

CP^n models in D=2 with and without fermions have interesting physical properties:

- They are asymptotically free
- Have instanton solutions
- The bosonic model exhibits confinement
- The SUSY model is integrable
- Other non-SUSY integrable models of CP^n with fermions

Abdalla et al, 1981-1985
Shankar, Witten, 1977

It is an interesting question whether the model we have obtained is integrable at the quantum level. If yes, what is its S-matrix?