Two instances of fake minimal Fano 3-folds

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Abstract. We show that G-minimal Fano varieties are quantum minimal. This explains how two differential operators of type $D3$, not corresponding to any minimal Fano threefolds, indeed come from geometry Fano threefolds.

1. Introduction

In [6] for the purpose of classification of minimal Fano threefolds (i.e. with Picard group $\mathbb{Z}$, see def. 2.1) by virtue of mirror symmetry V. Golyshev introduced the notion of $D3$ differential equation — a 6-parameter class of differential equations, generalizing the construction of regularized quantum differential equations of a Fano threefold from 6 two-point Gromov–Witten invariants. The classification of primary Fano threefolds was recovered by imposing some further Picard–Fuchs and modularity conditions on the equation.

Apart from 17 quantum differential equations of minimal smooth Fano threefolds V. Golyshev [1] found two more differential equations of modular origin, satisfying these conditions, and from the point of view of differential equations hardly distinguishable from the QDEs of minimal Fanos .

Original Golyshev's construction realized these two differential equations as Picard–Fuchs equations of some modular pencils of Kummer surfaces. In this paper we give a representation of this pencils by Laurent polynomials $f_{28}^{(i)}$ and $f_{30}^{(i)}$ ($i = 1, 2, 3$, three representations for each pencil). The polynomials corresponds to some nodal toric Fano 3-folds $X_{28}^{(i)}$ and $X_{30}^{(i)}$ ($i = 1, 2, 3$) of degrees 28 and 30. These toric threefolds admits smoothings that are smooth Fano threefolds Y_{28} and Y_{30} (the same for all i). Let Q be 3-dimensional quadric, and W be a hyperplane section of bidegree $(1, 1)$ of $\mathbb{P}^2 \times \mathbb{P}^2$; then Y_{28} is the blowup of a twisted quartic on Q, Y_{30} is the blowup of a curve of bidegree $(2, 2)$ on W. By Batyrev's approach [1] these Laurent polynomials $f_d^{(i)}$ constructed from nodal toric threefolds $X_d^{(i)}$ are conjectured to be Landau–Ginzburg models mirror symmetric to the smoothings Y_d. Fano threefolds Y_{28} and Y_{30} are not minimal (their Picard groups are $\mathbb{Z}^2$ and $\mathbb{Z}^3$) and one expects their quantum differential equations to be of degree 4 and 5, but these varieties occur to be quantum minimal (see def. 2.3) — minimal differential equation vanishing I-series of these varieties has the degree 3. We seek for the systematic reason why this phenomena happened, and found the following one: these varieties Y_{28} and Y_{30} (and $X_{28}^{(i)}$, $X_{30}^{(i)}$) are G-minimal (see def. 2.2) i.e. admit a group action, and are minimal with respect to that action. Theorem 2.4 states that such kind of minimality with respect to the group action implies the quantum minimality as well. The same argument clarifies the correspondence between D. Zagier's list of 6 differential equations of type $D2$ and del Pezzo surfaces of degree $d \leqslant 6$.

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[1] also these equations were found by D. van Straten, G.Almkvist and W.Zudilin

2. Group actions

Definition 2.1. We call n-dimensional Fano variety X *minimal* if it has the same even Hodge numbers as n-dimensional projective space $\mathbb{P}^n$:

$$H^{2k}(X,\mathbb{Z}) = \mathbb{Z}$$

or equivalently $\dim H^{even}(X,\mathbb{C}) = \dim X + 1$.

Definition 2.2. A pair of a Fano variety X equipped with finite group action $G : X$ is G-*minimal* if $\dim H^{even}(X,\mathbb{C})^G = \dim X + 1$, i.e. even part of G-invariant cohomologies of X is generated by canonical class.

Definition 2.3. The Fano variety X is called *quantum minimal* if the dimension of the subring in quantum cohomology generated by the canonical class is equal to $\dim X + 1$.

Since modulo q this subring contains $\dim X + 1$ linearly independent elements $1, K_X, K_X \cup K_X, \ldots, K_X^{\dim X}$, it is enough to ask for the dimension of the anticanonical subring to be less or equal than $\dim X + 1$.

Theorem 2.4. *Let X be a Fano variety admitting some action of group G such that X is G-minimal. Then X is quantum minimal.*

This theorem holds because quantum multiplication respects the group action:

Lemma 2.5. *Let X be a Fano variety with the action of the finite cyclic group G, χ_1, χ_2 — a pair of characters of G, and $\gamma_i \in H^\bullet(X,\mathbb{C})^{\chi_i}, i = 1,2$ — a pair of G-eigenvector cohomology classes with characters χ_1, χ_2: $g\gamma_i = \chi_i(g)\gamma_i$ for $g \in G, i = 1,2$. Then $\gamma_1 \star \gamma_2 \in H^\bullet(X,\mathbb{C})^{(\chi_1\chi_2)}[[q]]$.*

Proof of the lemma. Since Gromov–Witten are well defined and are indeed invariant with respect to the isomorphims one has

$$\langle g^*\gamma_1, ..., g^*\gamma_n\rangle_\beta = \langle \gamma_1, \ldots, \gamma_n\rangle_{g_*\beta} \tag{2.6}$$

for any classes $\beta \in H_2(X)$ and $\gamma_i \in H^\bullet(X)$.

The canonical class K_X is G-invariant, so the action of G preserves anticanonical degrees of the curves in X: $(-K_X \cdot \beta) = (-K_X \cdot g_*\beta)$.

This implies the G-invariance of the correlators: $\langle g\gamma_1, ..., g\gamma_n\rangle_d = \langle \gamma_1, \ldots, \gamma_n\rangle_d$ for any n, d and $g \in G$.

Choose a basis of $H^\bullet(X,\mathbb{C})$ consisting of G-eigenvectors. Let $(\cdot,\cdot) = \langle \cdot,\cdot\rangle_0$ be the Poincare pairing. Since for a pair of eigenvectors α_1, α_2 with characters χ_1, χ_2 the pairing (α_1, α_2) is nonzero only if $\chi_1\chi_2 = 1$, we need to show $(\gamma_1\star\gamma_2, \gamma_3)$ is zero for any eigenvector γ_3 with any χ_3 different from $(\chi_1\chi_2)^{-1}$. By definition $(\gamma_1\star\gamma_2, \gamma_3) = \sum_{d\geqslant 0} q^d \langle \gamma_1, \gamma_2, \gamma_3\rangle_d$, so the vanishing $(\gamma_1\star\gamma_2, \gamma_3)$ is equivalent to the vanishing of all corellators $C_d = \langle \gamma_1, \gamma_2, \gamma_3\rangle_d$. But $C_d = \langle \gamma_1, \gamma_2, \gamma_3\rangle_d = \langle g\gamma_1, g\gamma_2, g\gamma_3\rangle = \langle \chi_1(g)\gamma_1, \chi_2(g)\gamma_2, \chi_3(g)\gamma_3\rangle = (\chi_1\chi_2\chi_3)(g) \langle \gamma_1, \gamma_2, \gamma_3\rangle_d = (\chi_1\chi_2\chi_3)(g) C_d$, so if $(\chi_1\chi_2\chi_3)(g) \neq 1$ for some g, then $C_d = 0$. □

Proof of the theorem. By lemma 2.5 the subring R of quantum cohomology generated by $-K_X$ is contained inside $H^\bullet(X,\mathbb{C})[q]$. By the proposition of the theorem the dimension of $H^\bullet(X,\mathbb{C})(q)$ over $\mathbb{C}(q)$ is $\dim X + 1$. This implies dimension of $R \otimes \mathbb{C}(q)$ over $\mathbb{C}(q)$ is $\leqslant \dim X + 1$. □

Remark 2.7. There are two frameworks for quantum cohomology — symplectic and algebraic. One may notice neither of these definitions were used in the proof. Geometrical part is hidden behind the equality 2.6 and the fact that correlators are invariant with respect to algebraic or symplectic isomorphisms.

Moreover, one can even apply the theorem in the case of non-geometric action of the Galois group (or mixed geometric and Galois action) on variety X and it's cohomologies (e.g. $H_{et}(X, \mathbb{Q}_l)$) if X is defined over $\mathbb{Q}$ (or over some number field). This is true since everything is defined over the base field of X: $M_{g,n}(X,\beta)$, evaluation map $ev : M_{g,n}(X,\beta) \to X^n$, ψ-classes and the virtual fundamental class.

Example 2.8. Let X be a del Pezzo surface different from blowup of a plane in 1 or 2 points. It is a classical result that such a surface admits some G-minimal action for some moduli (e.g. the paper [2] contains the description of all the possible minimal rational G-surfaces). Hence del Pezzo surfaces except of S_7 and S_8 are quantum minimal, and give rise to 8 equations of type $D2$.

3. Realization of D3 as Picard–Fuchs of pencils with low ramification

By the virtue of mirror symmetry quantum differential equation of n-dimensional Fano variety X coincides with Picard–Fuchs equation of some pencil w over $\mathbb{A}^1$ of $(n-1)$-dimensional Calabi–Yau varieties (the *Landau–Ginzburg model* mirror symmetric to X).

Assume for a moment that X is minimal, it's derived category of coherent sheaves $\mathcal{D}^b(X)$ admits a full exceptional collection E_i, and all singular points of w are ordinary. Critical points of w correspond to the (Lagrangian) vanishing cycles L_i, and by homological mirror symmetry these cycles correspond to the elements of the exceptional collection. Number of L_i is the number of singular points, and it should be equal to the number of E_i i.e. $\operatorname{rk} K_0(X) = \dim X + 1$. This implies number of singular fibers of w is $\leqslant \dim X + 1$.

When X is a smooth toric variety, an easy corollary from Givental's computation [5] shows that the mirror symmetric to X Landau–Ginzburg model is given by the Laurent polynomial w with Newton polytope coinciding with the fan polytope of X, with some constant term w_0 (Givental's constant, it vanishes when index of X is $\geqslant 2$) and all other coefficients equal to 1:

$$w(\Delta, w_0) = w_0 + \sum_{v \in Vertices(\Delta)} x^v$$

Batyrev states that if smooth Fano Y has a small degeneration $\pi : \mathcal{X} \to C$ to toric X (i.e. π is a flat projective morphism to a curve, X and Y are isomorphic to some fibers of π, X admits only Gorenstein terminal singularities, and the restriction map $Pic(\mathcal{X}) \to Pic(\mathcal{X}_t)$ is an isomorphism for all $t \in C$), then for some constant w_0 the Laurent polynomial $w(\Delta(X), w_0)$ is a (weak) Landau–Ginzburg model mirror symmetric to Y. Let $I_Y(t)$ be the I-series of Y and put $\Phi_w(t)$ equal to $\frac{1}{(2\pi i)^{\dim X}} \int \frac{dx}{1-tf}$ i.e. a constant (with respect to x) term of $\frac{1}{1-tf}$.

Let t be a coordinate on G_m and $D = t\frac{d}{dt}$.

Example 3.1 (example of degree 28)**.** Consider the differential operator

$$L_{28} = D^3 - tD(D+1)(2D+1) - t^2(D+1)(59(D+1)^2+5) - \\ - 68t^3(2D+3)(D+2)(D+1) - 80t^4(D+3)(D+2)(D+1) \tag{3.2}$$

The solution of this equation in coordinate q is given by

$$\eta(q)\eta(q^2)\eta(q^7)\eta(q^{14}).$$

In coordinate t after the shift of first term to zero it is

$$I_{28}(t) = 1 + 8t^2 + 24t^3 + 240t^4 + 1440t^5 + 11960t^6 + 89040t^7 + \dots$$

Consider Laurent polynomials

$$f_{28}^{(1)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{y}{z} + \frac{z}{y} + xy + xz + xyz \tag{3.3}$$

$$f_{28}^{(2)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz + \frac{1}{xyz} + xz + \frac{1}{yz} \tag{3.4}$$

$$f_{28}^{(3)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + \frac{1}{xy} + xz + \frac{1}{yz} \tag{3.5}$$

Then $\Phi_{f_{28}^{(1)}}(t) = \Phi_{f_{28}^{(2)}}(t) = \Phi_{f_{28}^{(3)}}(t)$ and up to Givental's constant are equal to $I_{Y_28}(t)$.

Example 3.6 (example of degree 30)**.** Consider the differential operator

$$\begin{aligned} L_{30} = D^3 - tD(D+1)(2D+1) - t^2(D+1)(43(D+1)^2+5)- \\ - 78t^3(2D+3)(D+2)(D+1) - 216t^4(D+3)(D+2)(D+1) \end{aligned} \tag{3.7}$$

The solution of this equation in coordinate q is given by

$$\eta(q)\eta(q^3)\eta(q^5)\eta(q^{15})$$

. In coordinate t after the shift of first term to zero it is

$$1 + 6t^2 + 24t^3 + 162t^4 + 1080t^5 + 7620t^6 + 55440t^7 + \dots$$

Consider Laurent polynomials

$$f_{30}^{(1)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \tag{3.8}$$

$$f_{30}^{(2)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + xz + \frac{1}{yz} + xyz \tag{3.9}$$

$$f_{30}^{(3)} = x + y + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x}{y} + \frac{y}{x} + xy + xz + xyz \tag{3.10}$$

Then $\Phi_{f_{30}^{(1)}}(t) = \Phi_{f_{30}^{(2)}}(t) = \Phi_{f_{30}^{(3)}}(t)$ and up to Givental's constant are equal to $I_{Y_{30}}(t)$.

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