Stability of localized standing waves in fluid-filled tubes

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October 20, 2011

1 Introduction

2 Basic equations and linearization

We model the tube as incompressible, isotropic, hyperelastic, cylindrical membrane. The tube has a constant undeformed radius R and a constant undeformed thickness H. The tube is assumed to be infinitely long, and end conditions are imposed at infinity. We use cylindrical coordinates, and undeformed configuration is given by coordinates R, Θ, Z.

We assume that the axisymmetry remains throughout the entire deformation; the deformed configuration is expressed using cylindrical polar coordinates r, θ, z, where $r = r(Z,t)$, $\theta = \theta(Z,t)$, $z = z(Z,t)$, and t denotes time.

The principal directions of the deformation correspond to the lines of latitude, the meridian and the normal to the deformed surface, and the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = (r'^2 + z'^2)^{\frac{1}{2}}, \quad \lambda_3 = \frac{h}{H}, \tag{2.1}$$

where the indices 1, 2, 3 are used for the circumferential, axial and radial directions respectively, a prime represents differentiation with respect to Z, and h denotes the deformed thickness.

The principal Cauchy stresses σ_1, σ_2, σ_3 in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i W_i - p, \qquad i = 1, 2, 3 \quad \text{(no summation)}, \tag{2.2}$$

where $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function, $W_i = \partial W / \partial \lambda_i$, and p is the pressure associated with the constraint of incompressibility. Utilizing the incompressibility constraint $\lambda_1 \lambda_2 \lambda_3 = 1$ and the membrane assumption of no stress through the thickness direction $\sigma_3 = 0$, we find

$$\sigma_i = \lambda_i \hat{W}_i, \qquad i = 1, 2 \tag{2.3}$$

where $\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1})$ and $\hat{W}_1 = \partial\hat{W}/\partial\lambda_1$ etc (Haughton and Ogden 1979).

As an example we give here frequently used three strain-energy functions, the Varga, Ogden and Gent materials, given respectively by,

$$W = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{2.4}$$

$$W = \mu \sum_{r=1}^{3} \mu_r(\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3)/\alpha_r, \tag{2.5}$$

$$W = -\frac{1}{2}\mu J_m \ln\left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_m}\right), \tag{2.6}$$

where μ is the shear modulus for infinitesimal deformations, $J_m > 0$ is a material constant representing the maximum stretch of the material and $\alpha_1 = 1.3$, $\alpha_2 = 5.0$, $\alpha_3 = -2.0$, $\mu_1 = 1.491$, $\mu_2 = 0.003$, $\mu_3 = -0.023$. The Ogden and Gent materials were proposed in Ogden , 1972 and Gent, 1996 respectively, and are popularly used to model rubber.

The equations of motion can be derived from the exact field equations of general non-linear shell theory, e. g. Budiansky, 1968, but Epstein and Johnson, 2001, gave a very readable self-contained derivation. We quote their results and rewrite them in the form:

$$\left[R\sigma_2 \frac{z'}{\lambda_2^2}\right]' - Prr' = \rho R\ddot{z}, \quad \left[R\sigma_2 \frac{r'}{\lambda_2^2}\right]' - \frac{\sigma_1}{\lambda_1} + Prz' = \rho R\ddot{r}, \tag{2.7}$$

where ρ is the density of the material,

$$z = z_\infty Z + u(Z,t), \quad r = r_\infty R + w(Z,t), \quad u, w \to 0, \text{ as } Z \to \infty,$$

and P is the internal pressure divided by the original wall thickness.

As far as the fluid-solid interaction, we follow Demiray, 1996 in adopting a simple model, whereby the conservation of mass and momentum are enforced under the assumption that the the fluid is ideal having constant density and the velocity profile is constant throughout the tube cross section. Denoting by v_f the fluid speed and ρ_f the fluid density divided by the original wall thickness, from the mass conservation law

$$0 = \frac{d}{dt}\iiint\limits_{V(t)} H\rho_f \, dx\, dy\, dz = H\rho_f\pi \frac{d}{dt}\int\limits_{z_1(t)}^{z_2(t)} r^2\, dz = H\rho_f\pi \int\limits_{z_1(t)}^{z_2(t)} \left(\frac{\partial r^2}{\partial t} + \frac{\partial r^2 v_f}{\partial z}\right) dz,$$

we obtain the continuity equation for smooth motions (Epstein and Johnston, 2001)

$$\frac{\partial r}{\partial t} + v_f \frac{\partial r}{\partial z} + \frac{r}{2}\frac{\partial v_f}{\partial z} = 0. \tag{2.8}$$

The Euler equation evidently reads

$$\rho_f \left(\frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial z} \right) + \frac{\partial P}{\partial z} = 0. \tag{2.9}$$

The connection between the actual longitudinal coordinate z and the reference coordinate Z is given through the motion of the wall of the tube, i. e.

$$z = z_\infty Z + u(Z, t).$$

For any dependant variable Ψ

$$\Psi' = \frac{\partial \Psi}{\partial z} z', \quad \dot{\Psi} = \frac{\partial \Psi}{\partial z} \dot{z} + \frac{\partial \Psi}{\partial t}.$$

Therefore, in the Lagrangian coordinates Z and t the equations (2.8), (2.9) read, correspondingly

$$\dot{r} z' - r' \dot{z} + v_f r' + \frac{1}{2} r v_f' = 0, \quad \rho_f \left[\dot{v}_f z' - v_f' \dot{z} + v_f v_f' \right] + P' = 0. \tag{2.10}$$

We may put radius R and shear modulus μ to unity by using the transformations

$$z \to Rz, \quad r \to Rr, \quad u \to Ru, \quad w \to Rw, \quad v_f \to R v_f, \quad P \to P\mu/R,$$
$$\sigma_{1,2} \to \sigma_{1,2}\mu, \quad \rho, \rho_f = \hat{\rho}\mu/R^2, \hat{\rho}_f \mu/R^3.$$

Evaluating the second equation in (2.7) at infinity we find the relation for the pressure at infinity

$$P_\infty = \frac{\hat{W}_1(r_\infty, z_\infty)}{r_\infty z_\infty}. \tag{2.11}$$

As discussed in Fu *et al.* (2009), two integrals of the equilibrium equations exist, given by,

$$\hat{W} - \lambda_2 \hat{W}_2 = C_1 = \hat{W}^{(\infty)} - z_\infty \hat{W}_2^{(\infty)}, \tag{2.12}$$

$$\frac{\hat{W}_2 z'}{\lambda_2} - \frac{1}{2} P_\infty \lambda_1^2 = C_2 = \hat{W}_1^{(\infty)} - \frac{1}{2} P_\infty r_\infty^2, \tag{2.13}$$

where a superscript ∞ represents evaluation at $\lambda_2 = z_\infty$, $\lambda_1 = r_\infty$; (2.12) was first derived by Pipkin, 1968.

For an infinite tube with open ends we remote axial stretch z_∞ represents a prestrain of the material which is prescribed by the load applied at the end of the tube and is therefore treated as constant. For a tube with closed ends and no axial loading, we require that the force balance in the Z direction is zero, and hence $C_2 = 0$, giving the following relation from (2.13) and (2.11),

$$r_\infty \hat{W}_1(r_\infty, z_\infty) = 2 z_\infty \hat{W}_2(r_\infty, z_\infty), \tag{2.14}$$

which may be used to determine z_∞ for any given r_∞. Therefore we take r_∞ as the controlling parameter of the deformation, with z_∞ either determined by (2.14) or prescribed.

For Varga and Gent materials the condition (2.14) takes the form, respectively

$$1 + r_\infty^2 z_\infty - 2 r_\infty z_\infty^2 = 0, \qquad 1 + r_\infty^4 z_\infty^2 - 2 r_\infty^2 z_\infty^4 = 0$$

which may be solved explicitly for z_∞ (see Fig. 1 in Pearce and Fu, 2010).

3 Stability of weakly nonlinear solution

Following the analysis of Fu *et al.* 2008, when the fluid is at rest (e. g. $v_f = 0$, $P = P_\infty$) and there are no dependance on t, we may expand r' for values r close to r_∞ as

$$(r')^2 = (w')^2 = \omega(r_\infty)w^2 + \gamma(r_\infty)w^3 + O(w^4), \tag{3.1}$$

where

$$\omega(r_\infty) = \frac{r_\infty(\hat{W}_1^{(\infty)} - z_\infty \hat{W}_{12}^{(\infty)})^2 + z_\infty^2 \hat{W}_{22}^{(\infty)}(\hat{W}_1^{(\infty)} - r_\infty \hat{W}_{11}^{(\infty)})}{r_\infty z_\infty \hat{W}_2^{(\infty)} \hat{W}_{22}^{(\infty)}}.$$

The expression for $\gamma(r_\infty)$ is too long and so is not writen here. In Fu *et al.*, 2008, it is observed that the bifurcation condition is given by $\omega(r_\infty) = 0$ at $r_\infty = r_{cr}$.

On differentiating (3.1) with respect to Z, expanding it around r_{cr} and denoting $\epsilon = r_\infty - r_{cr}$, we obtain

$$w'' = \omega'(r_{cr})\epsilon w + \frac{3}{2}\gamma(r_\infty)w^2 + O(w^3)$$

or equivalently

$$\frac{d^2V}{d\xi^2} = V - V^2, \tag{3.2}$$

where

$$w = -\frac{2\epsilon\omega'(r_{cr})}{3\gamma(r_{cr})}V(\xi), \quad \xi = \sqrt{\epsilon\omega'(r_{cr})}Z. \tag{3.3}$$

Equation (3.2) has an exact solitary wave-type solution given by

$$V = V_0 \equiv \frac{3}{2}\mathrm{sech}^2\Big(\frac{\xi}{2}\Big), \tag{3.4}$$

which will be refered to as the weakly nonlinear solution.

We further assume that our dependent variables depend on slow time variable τ, defined by

$$\tau = \epsilon t,$$

and spacial variable ξ given by (3.3). Looking for the perturbation solution of the form

$$r = r_{cr} + \epsilon + \epsilon\{w_1(\xi,\tau) + \epsilon w_2(\xi,\tau) + \dots\}, \tag{3.5}$$

$$z = (z'_{cr} + \epsilon z'_p + \dots) + \sqrt{\epsilon}\{u_1(\xi,\tau) + \epsilon u_2(\xi,\tau) + \dots\}, \tag{3.6}$$

$$v_f = \epsilon^{3/2} v_{f1} + \dots, \tag{3.7}$$

$$P = P_\infty + \epsilon^2 p_1(\xi,t) + \dots, \tag{3.8}$$

where z_p' is a constant. For P_∞ given by (2.11) with $r_\infty = r_{cr} + \epsilon$, $z_\infty = z_{cr}' + \epsilon z_p' + \dots$ we have the Taylor expansion

$$P_\infty = \frac{\hat{W}_1(r_{cr}, z_{cr}')}{r_{cr} z_{cr}'} + \epsilon P_1 + \dots . \tag{3.9}$$

On substituting (3.5)-(3.9) into equations of motion (2.7), (2.10) and equating the powers of ϵ, we obtain

$$L \begin{bmatrix} w_1 \\ \sqrt{\omega'(r_{cr})} u_{1\xi} \end{bmatrix} = 0, \quad L = \begin{bmatrix} -\hat{W}_1/z_{cr}' + \hat{W}_{12} & \hat{W}_{22} \\ z_{cr}'(\hat{W}_1 - r_{cr}\hat{W}_{11}) & r_{cr}(\hat{W}_1 - z_{cr}'\hat{W}_{12}) \end{bmatrix}, \tag{3.10}$$

where $\hat{W}_1$, $\hat{W}_{11}$, $\hat{W}_{12}$, $\hat{W}_{22}$ are all evaluated at $r = r_{cr}$, and $u_{1\xi}$ denotes $\partial u_1/\partial \xi$. It is easy to see that $\det L = 0$, thus the matrix equation (3.10) has a non-trivial solution for w_1 and $u_{1\xi}$.

Proceeding to the next order, we find

$$L \begin{bmatrix} w_{2\xi\xi} \\ \sqrt{\omega(r_{cr})} u_{2\xi\xi\xi} \end{bmatrix} = \mathbf{b}, \quad p_{1\xi\xi} = \frac{2\hat{\rho}_f z_{cr}^2}{\omega'(r_{cr}) r_{cr}} w_{1\tau\tau}, \tag{3.11}$$

where after substitution of the second equality in (3.11) the vector $\mathbf{b}$ only contains w_1 and its derivatives. Forming the dot product of (3.11) with the null eigenvector of the adjoint of L, we then obtain the evolution equation in the form

$$\frac{\partial^2 V}{\partial \xi^2} - c_{1n} \frac{\partial^2 V}{\partial \tau^2} = c_2 \frac{\partial^4 V}{\partial \xi^4} + c_3 \frac{\partial^2 V^2}{\partial \xi^2}, \tag{3.12}$$

where c_{1n}, c_2, c_3 are known constants, and V is given by

$$w_1 = -\frac{2\omega'(r_{cr})}{3\gamma(r_{cr})} V(\xi, \tau).$$

If V is independent of τ, (3.12) must reduce to (3.2), therefore $c_2 = c_3 = 1$. The constant c_{1n} can be easily determined from the following linear analysis. Linearize (3.12) and then look for the solution of the form

$$V = \mathrm{e}^{\mathrm{i}K(\xi - v\tau)} = \exp\left(\mathrm{i}K\sqrt{\epsilon\omega'(r_{cr})} \big(Z - \sqrt{\frac{\epsilon}{\omega'(r_{cr})}} vt\big) \right), \tag{3.13}$$

where

$$v^2 = \frac{1 + K^2}{c_{1n}}.$$

From (3.13) it is seen that the actual wave number $\hat{k}$ and phase speed $\hat{c}$ are given by

$$\hat{k} = K\sqrt{\epsilon\omega'(r_{cr})}, \quad \hat{c} = v\sqrt{\frac{\epsilon}{\omega'(r_{cr})}}.$$

It follows then that

$$\hat{c}^2 = \frac{\epsilon}{c_{1n}\omega'(r_{cr})} + \frac{\hat{k}^2}{c_{1n}\omega'^2(r_{cr})} = \frac{r_\infty - r_{cr}}{c_{1n}\omega'(r_{cr})} + \frac{\hat{k}^2}{c_{1n}\omega'^2(r_{cr})}. \tag{3.14}$$

From equation (2.8) of Fu and Il'ichev, 2009, where $v_{f\infty}$ is put to zero, with the help of (3.14), we obtain

$$c_{1n} = c_1 + \frac{2\gamma_1\hat{\rho}_f r_{cr}^2 z_{cr}'^3}{\omega'^2(r_{cr})} = \frac{2\hat{\rho}_f r_{cr}^2 z_{cr}'^5}{\omega'^2(r_{cr})}\hat{W}_{22}, \tag{3.15}$$

where c_1 is given by (5.14) in Pearce and Fu, 2010. The prestressed tube material is strongly elliptic, i. e.

$$\gamma_1 = z_{cr}'^2\hat{W}_{22} > 0,$$

therefore from (3.15) it is seen that $c_{1n} > c_1$. We then normalize the time τ in our case as $\tau = c_{1n}T$, and in the pressure controlled case ($P = P_\infty$, $v_f \equiv 0$), treated by Pearce and Fu, 2010, as $\tau = c_1 T$. In both cases we get the normalized equation for weakly nonlinear waves

$$\frac{\partial^2 V}{\partial \xi^2} - \frac{\partial^2 V}{\partial T^2} = \frac{\partial^4 V}{\partial \xi^4} + \frac{\partial^2 V^2}{\partial \xi^2}. \tag{3.16}$$

We than look for eigenfunction $B(\xi)$ of linearized equation (3.16)

$$V = V_0(\xi) + B(\xi)\mathrm{e}^{\sigma T},$$

where $V_0(\xi)$ is given by (3.4). It can be verified that there exist unstable eigenvalue $\sigma = \sigma_0$. Coming back to old time τ we find that

$$\sigma_0 = c_{1n}\sigma_n = c_1\sigma_o,$$

or

$$\sigma_n = \frac{c_1}{c_{1n}}\sigma_o, \tag{3.17}$$

where σ_o is the unstable eigenvalue for pressure controlled case found in Pearce and Fu, 2010, and σ_n is the unstable eigenvalue in our case of the presence of fluid. From (3.17) it is seen, that $\sigma_n < \sigma_o$, i. e. that the presence of the fluid stabilizes the weakly nonlinear aneurythm (3.4).

4 Eigenvalue problem for fully nonlinear solution

We consider axisymmetric pertubations and write

$$r(Z) = \overline{r}(Z) + \delta w(Z)\mathrm{e}^{\eta t}, \quad z(Z,t) = \overline{z}(Z) + \delta u(Z)\mathrm{e}^{\eta t}, \quad P(Z,t) = \overline{P} + \delta P(Z)\mathrm{e}^{\eta t}, \tag{4.1}$$

where overlined quantities refer to standing solitary-wave solution of (2.7), (2.10) and linearizing in terms δu and δw, we find

$$\left[\frac{\overline{W}_2\delta u' + \frac{\overline{z}'}{\overline{\lambda}_2^2}\left(\overline{\lambda}_2\overline{W}_{22} - \overline{W}_2\right)\left(\overline{r}'\delta w' + \overline{z}'\delta u'\right) + \delta w\overline{z}'\overline{W}_{12}}{\overline{\lambda}_2}\right]' - \overline{P}(\overline{r}\delta w' + \delta w\overline{r}') - \overline{r}\overline{r}'\delta P = \hat{\rho}\eta^2\delta u, \tag{4.2}$$

$$\left[\frac{\overline{W}_2\delta w' + \frac{\overline{r}'}{\overline{\lambda}_2^2}\left(\overline{\lambda}_2\overline{W}_{22} - \overline{W}_2\right)\left(\overline{r}'\delta w' + \overline{z}'\delta u'\right) + \delta w\overline{r}'\overline{W}_{12}}{\overline{\lambda}_2}\right]' -$$
$$-\frac{\overline{W}_{12}}{\overline{\lambda}_2}(\overline{r}'\delta w' + \overline{z}'\delta u') - \delta w\overline{W}_{11} + \overline{P}(\overline{r}\delta u' + \delta w\overline{z}') + \overline{r}\overline{z}'\delta P = \hat{\rho}\eta^2\delta w, \tag{4.3}$$

$$2\hat{\rho}_f\eta^2\overline{z}'^2\delta w - 2\hat{\rho}_f\eta^2\overline{z}'\overline{r}'\delta u - \overline{r}\delta P'' + \left(\overline{r}\frac{\overline{z}''}{\overline{z}'} - 2\overline{r}'\right)\delta P' = 0. \tag{4.4}$$

It can be seen that (4.2)-(4.4) is a system of three coupled linear non-homogeneous second order differential equations, and the dependence on η is entirely through η^2. We denote $\alpha = \hat{\rho}\eta^2$. The system (4.2)-(4.4) can be written in the matrix form

$$\mathbf{y}' = \mathcal{M}\mathbf{y}, \tag{4.5}$$

where $\mathbf{y} = (\delta u, \delta u', \delta w, \delta w', \delta P, \delta P')^T$, $\mathcal{M} = \{m_{ij}\}$, $i, j = 1, ..., 6$, the expressions for m_{ij} were calculated, but is not written here for brevity. Similarly

$$\mathbf{x}' = -\mathbf{x}\mathcal{M}, \tag{4.6}$$

for $\mathbf{x}$ a six-dimensional row vector is the adjoint system. Since each $\mathbf{y}(\eta, Z)$ and $\mathbf{x}(\eta, Z)$ satisfies (4.5) and (4.6), respectively,

$$\frac{d}{dZ}\mathbf{x}(\eta, Z)\cdot\mathbf{y}(\eta, Z) = 0; \tag{4.7}$$

thus $\mathbf{x}(\eta, Z)\cdot\mathbf{y}(\eta, Z)$ is independent of Z.

From the conditions governing the decay of the underlying state as $Z \to \pm\infty$ we require $\overline{r}(Z) \to r_\infty$, $\overline{z}'(Z) \to z_\infty$, and hence the matrix $\mathcal{M}_\infty$ now takes the form

$$\mathcal{M}_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\alpha}{\hat{W}_{22}^{(\infty)}} & 0 & 0 & \frac{\hat{W}_1^{(\infty)} - z_\infty\hat{W}_{12}^{(\infty)}}{z_\infty\hat{W}_{22}^{(\infty)}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-\hat{W}_1^{(\infty)} + z_\infty\hat{W}_{12}^{(\infty)}}{\hat{W}_2^{(\infty)}} & \frac{-z_\infty\hat{W}_1^{(\infty)} + r_\infty z_\infty(\alpha + \hat{W}_{11}^{(\infty)})}{r_\infty\hat{W}_2^{(\infty)}} & 0 & -\frac{r_\infty z_\infty^2}{\hat{W}_2^{(\infty)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{2b\alpha z_\infty^2}{r_\infty} & 0 & 0 & 0 \end{pmatrix}, \tag{4.8}$$

Figure 1: Dependence of ν on a for the tube with closed ends and the Gent material ($\alpha = -a$, $a > 0$); (a) $r_\infty = 1.54$; (b) $r_\infty = 1.3$; $b = 5$

where $b = \hat{\rho_f}/\hat{\rho}$[1].

The equation (4.5) asymptotes to the constant coefficient problem with exponential solutions $\exp(kZ)$, for values k related to the parameter α by the equation

$$\det\big(\mathcal{M}_\infty - kI\big) = 0,$$

where I is the identity matrix, or

$$\begin{aligned} &k^6 - \frac{1}{r_\infty z_\infty \hat{W}_2^{(\infty)}\hat{W}_{22}^{(\infty)}}\big[z_\infty^2 \hat{W}_{22}^{(\infty)}\hat{W}_1^{(\infty)} + r_\infty\big(\hat{W}_1^{(\infty)2} - 2z_\infty \hat{W}_{12}^{(\infty)}\hat{W}_1^{(\infty)} - \\ &-z_\infty(\alpha\hat{W}_2^{(\infty)} + \alpha z_\infty \hat{W}_{22}^{(\infty)} - z_\infty \hat{W}_{12}^{(\infty)2} + z_\infty \hat{W}_{11}^{(\infty)}\hat{W}_{22}^{(\infty)})\big)\big]k^4 + \\ &+\frac{1}{r_\infty \hat{W}_2^{(\infty)}\hat{W}_{22}^{(\infty)}}\alpha z_\infty\big(r_\infty\alpha - \hat{W}_1^{(\infty)} + r_\infty(\hat{W}_{11}^{(\infty)} + 2b\hat{W}_{22}^{(\infty)} z_\infty^3)\big)k^2 - \frac{2}{\hat{W}_2^{(\infty)}\hat{W}_{22}^{(\infty)}} z_\infty^4 b\alpha^2 = 0. \end{aligned} \tag{4.9}$$

It can be easily seen that the six eigenvalues of $\mathcal{M}_\infty$ are given by $\pm k_1$, $\pm k_2$, $\pm k_3$. Moreover,
(i) (4.9) has pure imaginary roots if and only if η is pure imaginary (α is real and negative). For if η is imaginary ($\alpha = -a$, $a > 0$), then $\nu_{1,2} = \phi_{1,2}(a)$ ($\nu = k^2$) are negative, and ν_3 is positive. Conversely, the two functions $\phi_{1,2}$ are strictly monotonic in a, $\phi_{1,2}(0) = 0$, and $|\phi_{1,2}(a)| \to \infty$ as $|a| \to \infty$. Figure 1 illustrates the dependance of ν on a for the two different values of r_∞ for the Gent material;
(ii) when $\eta \neq 0$ is real, (4.9) has three roots in each of the positive and negative complex half planes.

Thus, from (i) and (ii) it follows that for η not on the imaginary axis, (4.9) has three roots in the right complex half-plane and three in the left complex half-plane. For η in the right half-plane, denote the three roots in the left half-plane with increasing real part by $k_1(\eta)$, $k_2(\eta)$, and $k_3(\eta)$.

It is necessary to know when $\mathcal{M}_\infty$ has multiple eigenvalues. Equation (4.9) has multiple roots when the resultant of this equation and its derivative is zero. We have $\alpha = 0$ always corresponding to the quadruple eigenvalue $k = 0$.

[1]If we denote the density of the fluid by ρ_{fl} ($\rho_{fl} = H\rho_f$), then $b = \frac{\rho_{fl}R}{\rho H}$.

The bulging aneurythm solutions, for example, exist inside some range $r_0 \le r_\infty \le r_{cr}$ (see Pearce, Fu, 2010). For these solutions and Gent material in the right half plane lie four η for which the equation (4.9) has multiple roots. For $r_0 < r_a < r_\infty < r_{cr}$ we have (except $\eta = 0$) $\eta = \eta_1$ with the lowest real part is real, then a couple of complex conjugate $\eta = \eta_{2,3}$, and $\eta = \eta_4$ with the greatest real part is real again. For $\eta = \eta_1$ and $\eta = \eta_4$ the equality $k_2 = k_3$ holds. The root k_1 with the lowest real part is real as well. For $\eta = \eta_{2,3}$ the equality $k_1 = k_2$ holds, and all eigenvalues are complex. If η is on the real axis, and $\eta > \eta_4$ and $0 < \eta < \eta_1$, k_1, k_2, k_3 are all real. When $\eta_1 < \eta < \eta_4$, $k_{2,3}$ are complex conjugate and k_1 is real. For $r_a \approx 1.48761$, the real η_4 equals to the real part of $\eta_{2,3}$, and for $r_0 < r_\infty < r_a$ we have η_1 and η_2 real, and $\eta_{3,4}$ are complex conjugate. For $\eta = \eta_{3,4}$ the equality $k_1 = k_2$ holds, and all eigenvalues k are complex. For the real $\eta = \eta_1$ and $\eta = \eta_2$, $k_2 = k_3$ and all eigenvalues are real. In the case $r_\infty < r_a$ for any real $\eta > \eta_2$ and $\eta < \eta_1$ the eigenvalues k_1, k_2 and k_3 are real. For $\eta_1 < \eta < \eta_2$, k_1 is real and $k_{2,3}$ are complex conjugate.

We construct solutions $\mathbf{y}_i$ of (4.5) and $\mathbf{x}$ of (4.6) for $i = 1, 2, 3$ such that

$$\begin{aligned} &\lim_{Z\to\infty} \mathrm{e}^{-k_i(\eta)Z}\mathbf{y}_i(\eta, Z) = \mathbf{r}_i(\eta), \\ &\lim_{Z\to-\infty} \mathrm{e}^{k_i(\eta)Z}\mathbf{x}_i(\eta, Z) = \mathbf{l}_i(\eta), \end{aligned} \tag{4.10}$$

where $\mathbf{r}_i$ are appropriately normalized (see Alexander and Sachs, 1995) column-right eigenvectors of $\mathcal{M}_\infty$, and $\mathbf{l}_i$ row-left eigenvectors of this matrix. A solitary wave solution is linearly unstable if there exist η_0, $\operatorname{Re}\eta_0 > 0$ and solutions to (4.5), (4.6) and (4.10) with $\eta = \eta_0$, exponentially decaying at both infinities.

5 Exterior systems and Evans function

Consider the vector fields $\mathbf{y}^\wedge(\eta, Z)$, indexed with the components

$$y^\wedge_{i\wedge j\wedge k} = y^i_1(y^j_2 y^k_3 - y^j_3 y^k_2) - y^i_2(y^j_1 y^k_3 - y^j_3 y^k_1) + y^i_3(y^j_1 y^k_2 - y^j_2 y^k_1), \tag{5.1}$$

for $i < j < k$, $\quad i, j, k = 1, ..., 6$. That is, $y^\wedge_{i\wedge j\wedge k}$ is 3×3 determinant of the ith, jth and kth rows of the matrix

$$\begin{pmatrix} y^1_1 & y^1_2 & y^1_3 \\ y^2_1 & y^2_2 & y^2_3 \\ \vdots & \vdots & \vdots \\ y^6_1 & y^6_2 & y^6_3 \end{pmatrix}.$$

The same notation is used for the left solutions of the system (4.6).

We number the triples $i \wedge j \wedge k$ by the following way:

$$1 \to 1\wedge 2\wedge 3, \quad 2 \to 1\wedge 2\wedge 4, \quad 3 \to 1\wedge 2\wedge 5, \quad 4 \to 1\wedge 2\wedge 6, \quad 5 \to 1\wedge 3\wedge 4,$$

$$6 \to 1\wedge 3\wedge 5, \quad 7 \to 1\wedge 3\wedge 6, \quad 8 \to 1\wedge 4\wedge 5, \quad 9 \to 1\wedge 4\wedge 6, \quad 10 \to 1\wedge 5\wedge 6,$$

$$11 \to 2\wedge 3\wedge 4, \quad 12 \to 2\wedge 3\wedge 5, \quad 13 \to 2\wedge 3\wedge 6, \quad 14 \to 2\wedge 4\wedge 5, \quad 15 \to 2\wedge 4\wedge 6,$$

$$16 \to 2\wedge 5\wedge 6, \quad 17 \to 3\wedge 4\wedge 5, \quad 18 \to 3\wedge 4\wedge 6, \quad 19 \to 3\wedge 5\wedge 6, \quad 20 \to 4\wedge 5\wedge 6.$$

Note that as a straightforward matter of linear algebra,

$$\det \begin{pmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \mathbf{x}_1 \cdot \mathbf{y}_2 & \mathbf{x}_1\mathbf{y}_3 \\ \mathbf{x}_2 \cdot \mathbf{y}_1 & \mathbf{x}_2 \cdot \mathbf{y}_2 & \mathbf{x}_2\mathbf{y}_3 \\ \mathbf{x}_3 \cdot \mathbf{y}_1 & \mathbf{x}_3 \cdot \mathbf{y}_2 & \mathbf{x}_3\mathbf{y}_3 \end{pmatrix} = \mathbf{x}^\wedge \cdot \mathbf{y}^\wedge. \tag{5.2}$$

The vector satisfies a linear system

$$\mathbf{y}^{\wedge'} = \mathcal{M}^\wedge \mathbf{y}^\wedge, \tag{5.3}$$

and similarly

$$\mathbf{x}^{\wedge'} = -\mathbf{x}^\wedge \mathcal{M}^\wedge. \tag{5.4}$$

These systems are called exterior systems. If

$$\mathcal{M}^\wedge = \{m^\wedge_{i\wedge j\wedge k,\, i'\wedge j'\wedge k'}\},$$

$i < j < k$, $i' < j' < k'$, by differentiating (5.1),

if $i \neq i' \neq j' \neq k'$, $j \neq i' \neq j' \neq k'$, $k \neq i' \neq j' \neq k'$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = 0$;

if $i = i'$ (or $j = i'$, $j = j'$, $k = i'$, $i = j'$, $k = j'$, $i = k'$, $j = k'$, $k = k'$), and the other two indices without prime are not equal to those with prime, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = 0$;

if $i = i'$, $j = j'$, $k \neq k'$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = m_{k,k'}$;

if $i = i'$, $j = k'$, $(k \neq j')$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = -m_{k,j'}$;

if $i = j'$, $j = k'$, $(k \neq i')$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = m_{k,i'}$;

if $j = i'$, $k = j'$, $(i \neq k')$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = m_{i,k'}$;

if $j = j'$, $k = k'$, $i \neq i'$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = m_{i,i'}$;

if $j = i'$, $k = k'$, $(i \neq j')$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = -m_{i,j'}$;

if $i = i'$, $k = j'$, $(j \neq k')$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = -m_{j,k'}$;

if $i = i'$, $k = k'$, $j \neq j'$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = m_{j,j'}$;

if $i = j'$, $k = k'$, $(j \neq i')$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = -m_{j,i'}$;

if $i = i'$, $j = j'$, $k = k'$, then $m^\wedge_{i\wedge j\wedge k, i'\wedge j'\wedge k'} = m_{ii} + m_{jj} + m_{kk}$.

It can be easily seen from (5.1), that the matrix $\mathcal{M}^\wedge_\infty$ has eigenvalues

$$k_i(\eta) + k_j(\eta) + k_l(\eta), \quad 1 \leq i < j < l \leq 6.$$

For η in the right complex half-plane, the matrix $\mathcal{M}_\infty$ has three eigenvalues in the left half-plane. Thus the asymptotic matrix $\mathcal{M}^\wedge_\infty$ has simple left-most eigenvalue $k^\wedge(\eta) =$

$k_1(\eta)+k_2(\eta)+k_3(\eta)$ for η in the right half-plane. There are solutions $\mathbf{y}^\wedge$, $\mathbf{x}^\wedge$ of (5.3), (5.4) such that

$$\lim_{Z\to\infty} \mathrm{e}^{-k^\wedge(\eta)Z}\mathbf{y}^\wedge(\eta,Z)=\mathbf{r}^\wedge(\eta), \tag{5.5}$$

$$\lim_{Z\to-\infty} \mathrm{e}^{k^\wedge(\eta)Z}\mathbf{x}^\wedge(\eta,Z)=\mathbf{l}^\wedge(\eta), \tag{5.6}$$

where $\mathbf{r}^\wedge(\eta)$ and $\mathbf{l}^\wedge(\eta)$ are the eigenvectors associated to $k^\wedge(\eta)$. Moreover, because the eigenvalue is simple, the constructions of Alexander and Sachs, 1995 are valid and analytic in the entire half-plane.

We define Evans function by

$$D(\eta)=\mathbf{x}(\eta)^\wedge\cdot\mathbf{y}(\eta)^\wedge \tag{5.7}$$

The so-defined Evans function is analytic in the entire complex right half-plane of η and it is real for real η. The vectors $\mathbf{y}(\eta)$ and $\mathbf{x}(\eta)$ are the solutions of (5.3), (5.4) and (5.5), (5.6) . By the exterior analogue of (4.7), the Evans function is independent of Z. Note that by (5.2), for small $|\eta|$

$$D(\eta)=\det\begin{pmatrix}\mathbf{x}_1\cdot\mathbf{y}_1 & \mathbf{x}_1\cdot\mathbf{y}_2 & \mathbf{x}_1\mathbf{y}_3\\ \mathbf{x}_2\cdot\mathbf{y}_1 & \mathbf{x}_2\cdot\mathbf{y}_2 & \mathbf{x}_2\mathbf{y}_3\\ \mathbf{x}_3\cdot\mathbf{y}_1 & \mathbf{x}_3\cdot\mathbf{y}_2 & \mathbf{x}_3\mathbf{y}_3\end{pmatrix},$$

and

$$\mathbf{l}^\wedge\cdot\mathbf{r}^\wedge=\det\begin{pmatrix}\mathbf{l}_1\cdot\mathbf{r}_1 & \mathbf{l}_1\cdot\mathbf{r}_2 & \mathbf{l}_1\mathbf{r}_3\\ \mathbf{l}_2\cdot\mathbf{r}_1 & \mathbf{l}_2\cdot\mathbf{r}_2 & \mathbf{l}_2\mathbf{r}_3\\ \mathbf{l}_3\cdot\mathbf{r}_1 & \mathbf{l}_3\cdot\mathbf{r}_2 & \mathbf{l}_3\mathbf{r}_3\end{pmatrix}.$$