TRANSITION TO INSTABILITY OF THE INTERFACE IN GEOTHERMAL SYSTEMS

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Abstract – High-temperature geothermal reservoir in porous media is under consideration, consisting of two high-permeability layers, which are separated by a low-permeability stratum. The thermodynamic conditions are assumed to imply that the upper and lower high-permeability layers are filled in by water and by vapour, respectively. In these circumstances the low-permeability stratum possesses the phase transition interface, separating domains occupied by water and vapour. The stable stationary regimes of vertical phase flow between water and vapour layers in the low-permeability stratum may exist. Stability of such regimes where the heavier fluid is located over the lighter one is supported by a heat transfer, caused by a temperature gradient in the Earth's interior. We give the classification of the possible types of transition to instability of the vertical flows in such a system under the condition of smallness of the advective heat transfer in comparison with the conductive one. It is found that in the non-degenerate case there exist three different scenarios of the onset of instability of the stationary vertical phase transition flows. Two of them are accompanied by the bifurcations of the destabilizing vertical flow, leading to appearance of horizontally non-homogeneous regimes with non-constant shape of the interface. The bifurcations correspond to the simple resonance and 1 : 1-resonance, which typically arise in reversible systems.

Key words: Geothermal system, phase transition interface, threshold of instability, bifurcation

1. Introduction

In the framework of classical fluid dynamics the equilibrium of heavy fluid over the lighter one is always unstable. The instability in question is known as the Rayleigh-Taylor instability (see e. g. Chandrasekhar, 1967). Yet, when supplementary physical mechanisms have to be taken into consideration in a hydrodynamic model, such an equilibrium is possible. Geothermal reservoir gives an example of the natural system where the thermodynamic states are realized, supporting the stable existence of water (the heavier fluid) over vapour (the lighter fluid). Full-scale investigations of natural geothermal systems showed, that in a great number of reservoirs the situation takes place, when a water layer of a considerable thickness is located over a layer of superheated vapour (White *et al.*, 1971; Grant, 1983). The existence of such a configuration is explained from the thermodynamic point of view by a considerable temperature gradient typically taking place in geothermal system. In this case the thermodynamic conditions at large depths correspond to vapour, and at smaller depths – to water.

Schubert and Straus (1980) treated the example of the geothermal reservoir of the homogeneous vertical extend embedded in the rigid rock. In such a reservoir the basic state describes the motionless equilibrium

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of the water layer over the vapour one. The stability of such a configuration for some range of values of the permeability of the geothermal reservoir was shown by means of numerical analysis of the dispersion relation.

Tsypkin and Il'ichev (2004) analysed the more general case of a geothermal system in a sense that the phase motion and phase transition in a base state are permitted. A high-temperature geothermal reservoir was considered, consisting of two high-permeability layers, which are separated by a low-permeability stratum. It was assumed that the thermodynamic conditions imply that the upper and lower high-permeability layers are filled in by water and by vapour, respectively. Then in the low-permeability layer there exists the phase transition interface, separating domains occupied by water and vapour. The geometry of the reservoir is presented in Fig.1. The reservoir itself represents the fragment of the general reservoir, based on geologocal structure consisting of blocks of lower permeability embedded in the base rock [2]. In dependence on values of pressure in the high-permeability layers, either the regime of vapourization, when the water moves downwards, or the regime of condensation, corresponding to the vapour motion upwards, take place. In the rest state phase transitions are absent, and the pressure distribution in the high-permeability layers coincides with the hydrostatic one.

Our principal restriction is that we assume that the conductive heat transfer in the system considerably exceeds the advective one. This restriction, nevertheless, permits to treat a lot of natural geothermal systems. This statement is based on the following estimates. For the water domain, combining Darcy's law with the heat conservation equation, we obtain the dimensionless parameter that specifies the ratio of advective and conductive terms (Tsypkin and Il'ichev, 2004):

$$\frac{\rho_w C_w}{\mu_w \lambda_1} k(\delta P - \rho_w g l), \tag{1.1}$$

where l is the characteristic length scale and δP is the dynamical pressure variation. For other notations see table at the beginning of sec. 2 of the paper. Permeability and pressure may vary strongly, at the same time the other physical parameters in (1.1) vary slightly. Therefore, after substitution of the characteristic values of parameters in (1.1) the condition of smallness of advective transfer in the water domain can be written in the following form

$$k \mid \delta P - \rho_w g l \mid \ll 10^{-10}\text{N}, \tag{1.2}$$

Analogously, one has for the vapour domain

$$k \mid \delta P - \rho_v g l \mid \ll 3 \cdot 10^{-9}\text{N}. \tag{1.3}$$

The criteria (1.2), (1.3) occur to be valid for a lot of flows in natural geothermal systems.

Calculations showed that there exist the regimes of motion with phase transitions, when the water layer over the vapour layer in geothermal system occurs to be stable also for permeability values which exceed the value got in Schubert and Straus (1980) by the order of magnitude. The new threshold value of the permeability explains the fact of stability of a lot of geothermal systems, observed in nature.

The character of the arising secondary flows depends on the type of instability and, therefore, it seems to be important to investigate in detail the possible types of transition to instability. In this paper we present the results of the analytic investigation of stability of a flow in the geothermal system treated by Tsypkin and Il'ichev (2004). The criteria of the transition to instability satisfying the principle of exchange of stabilities (when the time exponent of the least stable normal mode is zero at the margin of stability) are obtained in explicit form. It is established that the transition to instability of the interface under the variation of physical parameters takes place by means of one of the following four mechanisms:

- comes spontaneously for all wave numbers of perturbations (degenerate case);
- unstable wave number arises at infinity;
- a threshold of instability is determined by double-zero wave number;
- a threshold of instability is determined by a pair of multiple non-zero wave numbers.

In the two last cases the transition to instability is accompanied by bifurcations of the simple resonance and 1 : 1-resonance, respectively. These bifurcations lead to the branching of base regimes, describing horizontally homogeneous vertical phase flows, and appearance of secondary regimes depending on the horizontal coordinate.

The paper organized as follows. In sec. 2 we give formulation, describe base regimes and present the dispersion relation. In sec. 3 the types of transition to instability in the degenerate case (when the phase transition interface of the base regime is located exactly in the middle of the low-permeability stratum) and in the general case of location of the interface are described. In section 4 we discuss the results and give our conclusions.

2. Formulation, basic states and dispersion relation

Notations

Latin symbols

a	[m^2 s^{-1}]	thermal diffusivity
C	[J K^{-1} kg^{-1}]	specific heat at constant pressure
g	[m s^{-2}]	acceleration of gravity
h	[m]	location parameter of the interface
k	[m^2]	permeability
L	[m]	thickness of the low permeable stratum
m	[1]	porosity
P	[Pa]	pressure
q	[J kg^{-1}]	specific heat of phase transition
t	[s]	time
T	[K]	temperature
V	[m s^{-1}]	speed of the phase transition interface
x	[m]	vertical coordinate

Greek symbols

η	[m]	perturbation of the interface
κ	[m^{-1}]	wave number
λ	[W m^{-1} K]	thermal conductivity
μ	[Pa s]	viscosity
ρ	[kg m^{-3}]	density
σ	[s^{-1}]	spectral parameter

Subscripts

1	water domain
2	vapour domain
n	normal
s	skeleton of porous medium
v	vapour
w	water
0	boundary value at $x = 0$
$+$	right ahead of the interface in the water saturated domain
$-$	right behind the interface in the vapour domain
$*$	at the phase transition front

Superscript

0	boundary value at $x = L$

We consider high-temperature geothermal reservoir, consisting of two high-permeability horizontal layers, separated by a low-permeability stratum having sufficiently large horizontal extension. The low-permeability stratum is covered by a strip $y \in (-\infty, \infty) \times [0, L] \ni x$ with the vertical x-axis directed downwards. The upper

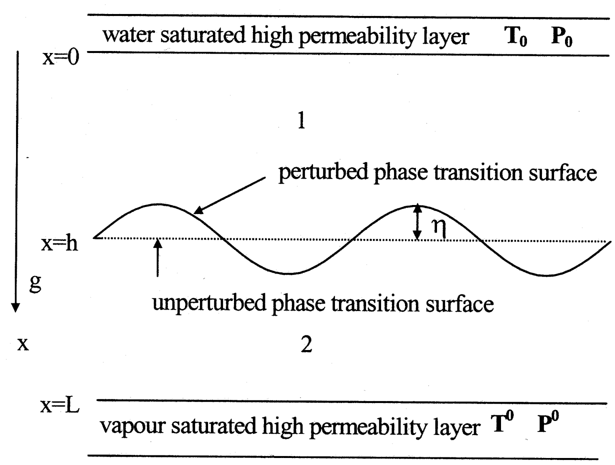

FIGURE 1. Geometry of the problem; 1 – water domain in the low-permeability stratum, 2 – vapour domain in the low-permeability stratum.

and lower high-permeability layers are occupied by water and vapour, respectively. Inside the low-permeability stratum there exists the phase transition interface, separating domains filled in by different phases (Fig.1).

Processes of heat and mass transfer in the framework of equilibrium thermodynamics can be described by mass and energy conservation laws, the Darcy law for water and vapour with allowance for gravity, the equations of state and thermodynamic relations (O'Sullivan, 1985). The water and vapour are assumed to be incompressible. Then the basic system of equations for the two domains of the low-permeability stratum separated by the interface read

$$\begin{aligned}
&\text{div } v_j = 0, \qquad v_j = -\frac{k}{\mu_j}(\text{ grad } P - \rho_j g \mathbf{e}_x),\\
&(\rho C)_{1,2}\frac{\partial T}{\partial t} + \rho_j C_j v_j \text{ grad } T = \text{ div } (\lambda_{1,2} \text{ grad } T),\\
&\lambda_{1,2} = m\lambda_j + (1-m)\lambda_s,\\
&(\rho C)_{1,2} = m\rho_j C_j + (1-m)\rho_s C_s, \quad j = v, w.
\end{aligned} \tag{2.1}$$

The conservation of mass and energy across the interface are formulated as the conditions of the thermodynamical equilibrium jump of the water saturation function (Fitzgerald and Woods, 1994; Tsypkin, 1997). These relations have the form

$$\begin{aligned}
&m(1-\frac{\rho_v}{\rho_w})V_n = \frac{k}{\mu_v}\frac{\rho_v}{\rho_w}(\text{ grad } P)_{n+} - \frac{k}{\mu_w}(\text{ grad } P)_{n-} + \\
&\qquad\qquad + \frac{k}{\mu_w}\rho_w g(1 - \frac{\mu_w}{\mu_v}\frac{\rho_v^2}{\rho_w^2}),\\
&mq\rho_w V_n = \lambda_-(\text{ grad } T)_{n-} - \lambda_+(\text{ grad } T)_{n+} - \frac{kq\rho_w}{\mu_w}((\text{ grad } P)_{n-} - \rho_w g),\\
&T_+ = T_- = T_*, \qquad P_+ = P_- = P_*, \quad \ln\frac{P_*}{P_a} = A + \frac{B}{T_*},\\
&A = 12.512, \quad B = -4611.73, \quad P_a = 10^5\text{Pa}.
\end{aligned} \tag{2.2}$$

The second term in the left-hand side of the heat transfer equation in (2.1) describes the heat advective transfer while the right-hand side describes the heat conductive transfer. We consider the flows, where advective transfer can be neglected in comparison with conductive transfer. Assuming that estimates (1.2), (1.3) are valid, we ignore the nonlinear advective heat transfer in the energy equation and obtain usual linear equation of heat transfer for the both domains.

Boundary conditions for the pressure and temperature in high-permeability layers are given by

$$x = 0: \ P = P_0, \quad T = T_0; \qquad x = L: \ P = P^0, \quad T = T^0. \tag{2.3}$$

If the pressures and temperatures in the high-permeability layers don't vary, then the stationary regime of flow is realized, which is characterized by linear distribution of temperature and pressure in domains, saturated by water and vapour, correspondingly. Substituting these distributions into the system of boundary conditions (2.2), we get the unknown location of the interface $x = h$, and also the pressure P_* and temperature T_* on this surface (Tsypkin and Il'ichev, 2004).

For investigation of normal stability of a stationary flow with the phase transition interface we linearize the basic equations. The perturbations of the pressure and temperature satisfy

$$\begin{aligned} &\Delta P = 0, \qquad \frac{\partial T}{\partial t} = a_{1,2}\Delta T, \qquad 0 < x < h \cup h < x < L, \\ &a_{1,2} = \frac{\lambda_{1,2}}{(\rho C)_{1,2}}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \end{aligned} \tag{2.4}$$

For the sake of simplicity we assume that the specific heat and heat conductivity depend only on the corresponding parameters of the skeleton: $a = a_1 = a_2$.

The boundary conditions for perturbations read

$$\begin{aligned} &P = 0, \ \ T = 0, \qquad \text{at } \ x = 0, \ \ L, \\ &P_- = P_+ + \frac{P_0}{L}\Gamma_1\eta, \quad \Gamma_1 = \frac{1}{H} + \frac{P_1 H - P_f}{H(1-H)}, \quad \text{at } \ x = h, \\ &T_- = T_+ + \frac{T_0}{L}\Gamma_2\eta, \quad \Gamma_2 = \frac{1}{H} + \frac{T_1 H - T_f}{H(1-H)}, \quad \text{at } \ x = h, \\ &P_- = \left(\frac{\partial f(T)}{\partial T}\right)_{T=T_*} T_- + \left[\left(\frac{\partial f(T)}{\partial T}\right)_{T=T_*}\left(\frac{\partial T}{\partial x}\right)_- - \left(\frac{\partial P}{\partial x}\right)_-\right]\eta = \\ &= -\frac{P_0 B}{T_0^2}\Gamma T_- - \frac{P_0}{L}\Gamma_0\eta, \quad \Gamma = \frac{P_f}{T_f^2}, \quad \Gamma_0 = \frac{B}{T_0}\Gamma\frac{T_f - 1}{H} + \frac{P_f - 1}{H}, \ \text{ at } x = h, \\ &m(1-R)\frac{\partial \eta}{\partial t} = \frac{k}{\mu_v}R\left(\frac{\partial P}{\partial x}\right)_+ - \frac{k}{\mu_w}\left(\frac{\partial P}{\partial x}\right)_-, \quad R = \frac{\rho_v}{\rho_w}, \ \text{ at } \ x = h, \\ &mq\rho_w\frac{\partial \eta}{\partial t} = \lambda_-\left(\frac{\partial T}{\partial x}\right)_- - \lambda_+\left(\frac{\partial T}{\partial x}\right)_+ - \frac{kq\rho_w}{\mu_w}\left(\frac{\partial P}{\partial x}\right)_-, \ \text{ at } \ x = h, \end{aligned} \tag{2.5}$$

where $x = h + \eta(t, y)$ is the interface equation, $T_f = T_*/T_0$, $T_1 = T^0/T_0$, $P_f = P_*/P_0$, $P_1 = P^0/P_0$, $H = h/L$.

Assuming, that the unknown functions can be represented as

$$\{P(x, y, t), T(x, y, t), \eta(y, t)\} = \{\hat{P}(x), \hat{T}(x), \hat{\eta}\} \exp(\hat{\sigma} t + i\hat{\kappa} y),$$

we get the dispersion equation from the condition of existence of the non-trivial solution:

$$\begin{aligned}
F(\sigma,\kappa)\equiv&\alpha\ \coth[\alpha(1-H)]\Big[\frac{\omega_w}{a}\Gamma_3\kappa\ \coth(\kappa H)-(1-R)\sigma+\\
&+\frac{\omega_v}{a}\Gamma_4 R\kappa\ \coth[\kappa(1-H)]\Big]+\alpha\ \coth(\alpha H)\Big[\frac{\omega_v}{a}\Gamma_5 R\kappa\ \coth[\kappa(1-H)]+\\
&+\frac{\omega_w}{a}\Gamma_0\kappa\ \coth(\kappa H)-(1-R)\sigma\Big]+\frac{\omega_v B}{T_0}\frac{mq\rho_w R}{\lambda T_0}\Gamma\kappa\Big[\sigma\ \coth[\kappa(1-H)]+\\
&+\sigma\frac{\mu_v}{\mu_w}\coth(\kappa H)+\Gamma_1\kappa\frac{\omega_w}{a}\coth(\kappa H)\coth[\kappa(1-H)]\Big]=0.
\end{aligned}\tag{2.6}$$

Here

$$\begin{aligned}
&\Gamma_3=\frac{B}{T_0}\Gamma\frac{T_1-T_f}{1-H}+\frac{P_f-1}{H},\qquad \Gamma_4=\frac{B}{T_0}\Gamma\frac{T_1-T_f}{1-H}+\frac{P_1-P_f}{1-H},\\
&\Gamma_5=\frac{B}{T_0}\Gamma\frac{T_f-1}{H}+\frac{P_1-P_f}{1-H},\quad \alpha=\sqrt{\kappa^2+\sigma},\quad \kappa=\hat{\kappa}/L,\\
&\sigma=a\hat{\sigma}/L^2,\quad \omega_w=\frac{P_0 k}{m\mu_w},\quad \omega_v=\frac{P_0 k}{m\mu_v}.
\end{aligned}$$

From (2.6) one gets the asymptotic $\sigma=\sigma_0|\kappa|$, $\kappa\to\pm\infty$,

$$\begin{aligned}
\sigma_0\quad=\quad&\left[\frac{\omega_v}{a}R(\Gamma_4+\Gamma_5)+\frac{\omega_w}{a}(\Gamma_0+\Gamma_3)+\frac{\omega_w\omega_v}{a}\frac{B}{T_0}\frac{mq\rho_w R}{\lambda T_0}\Gamma\Gamma_1\right]\times\\
&\left[2(1-R)-\frac{\omega_v}{a}\frac{B}{T_0}\frac{mq\rho_w R}{\lambda T_0}\left(1+\frac{\mu_v}{\mu_w}\right)\Gamma\right]^{-1}.
\end{aligned}\tag{2.7}$$

3. **Transition to instability**

In this section the possible types of the exchange of stabilities of the system are considered. In the degenerate case of the location of phase transition interface $x=h=L/2$ the onset of instability takes place simultaneously for all normal modes. In the general case $h\neq L/2$ there are three different mechanisms of the exchange of stabilities.

Putting $\sigma=0$ in the dispersion equation (2.6) and dividing it by the non-vanishing function $\kappa^2\coth^2(1-H)\kappa$, one gets the equation

$$\Gamma_0\omega_w Z^2(\kappa)+\left(\omega_w\Gamma_3+\omega_v\Gamma_5 R+\frac{\omega_v\omega_w Bmq\rho_w R}{\lambda T_0^2}\Gamma\Gamma_1\right)Z(\kappa)+\omega_v\Gamma_4 R=0,\tag{3.1}$$

where

$$Z(\kappa)=\frac{\coth H\kappa}{\coth(1-H)\kappa}.$$

The quadratic equation (3.1) has a solution

$$Z(\kappa)=Z_{1,2}=d\pm\sqrt{d^2-b},\tag{3.2}$$

where

$$d=-\frac{1}{2\Gamma_0}\left(\Gamma_3+\frac{\omega_v}{\omega_w}\Gamma_5 R+\frac{\omega_v Bmq\rho_w R}{\lambda T_0^2}\Gamma\Gamma_1\right),\quad b=\frac{\omega_v}{\omega_w}\frac{\Gamma_4}{\Gamma_0}R.$$

In the left hand side of the equality (3.2) stands the even and monotonous for $\kappa>0$ function $Z(\kappa)$, $\min[(1-H)/H,1]\leq Z(\kappa)\leq\max[(1-H)/H,1]$. The values of $Z(\kappa)$ satisfying (3.2) are real if the expression under the square root sign is positive. Instability takes place if the equation (3.2) has finite simple real roots (their number is even because of the symmetry of the function $Z(\kappa)$), and the threshold of instability is attained when the real roots of this equation arise. In the absence of the real roots of the equation (3.2), their origin under variation of the physical parameters may occur in the following cases:

- $D = d^2 - b > 0$, $d < \min[(1 - H)/H, 1]$ and $Z_2 < Z_1 = \min[(1 - H)/H, 1]$;
- $D = d^2 - b > 0$, $d > \max[(1 - H)/H, 1]$ and $Z_1 > Z_2 = \max[(1 - H)/H, 1]$;
- $D = 0$, $\min[(1 - H)H, 1] < d < \max[(1 - H)H, 1]$.

3.1. *Spontaneous transition to instability in the degenerate case*

First, let us construct the example of the regime when the transition interface is located exactly in the middle of low-permeability stratum. Typically, such a regime corresponds to the equilibrium base state without phase motion (filtration speed equals zero). Let, for example, the pressure in the upper water saturated layer be $P_0 = 10^6$ Pa. Then, one can uniquely find the pressure on the interface. The difference between this pressure and the pressure at any boundary equals the hydrostatic pressure. For $L = 40$ m, for example, $P_* = 1.17418 \cdot 10^6$ Pa. Adding the hydrostatic pressure of the vapour in the vapour saturated domain to this value, we obtain the pressure at the lower boundary at $x = L$: $P^0 = 1.17512 \cdot 10^6$ Pa. These values of pressure correspond to the dynamical rest state. In order to satisfy the thermodynamic conditions it is necessary to choose the temperature distribution with boundary values lying, respectively, in the water and vapour domains in the Clapeyron plane. Consequently, the temperature at the upper boundary at $x = 0$ must be smaller than the temperature of water boiling: $T_0 < T_f(P_0) = 451.7134$ K, and at the lower boundary at $x = L$ the temperature has to be larger than the vapourization temperature: $T^0 > T_f(P^0) = 458.97$ K.

The value of the vapourization temperature T_* on the interface has to belong to the Clapeyron curve at $P_* = 1.17418 \cdot 10^6$ Pa and, therefore, this temperature can be uniquely determined and it equals $T_* = 458.93$ K. Arbitrariness exists only in a choice of the temperature gradient. For example, putting $T_0 = 450$ K and using the absence of phase transitions (continuity of derivatives of temperature across the interface) and also the known value of the temperature at the interface, we get the following value of the temperature at the lower boundary $x = L$: $T^0 = 467.86$ K. Let us note, that this solution is realized at all values of the permeability coefficient.

The dynamic equilibrium or rest state solution takes place at the prescribed boundary values of the pressure and when the temperature coincides with the phase transition temperature on the interface. Variation of the pressure or temperature distribution causes the onset of phase transition and phase motion for the stationary solution. It means that the rest state solutions represent isolated solutions in the sense that a small variation of the boundary values leads to a solution with phase motion.

Now, consider the transition to instability of the regime characterized by the phase transition interface location at the point $x = h = L/2$ on the vertical. The set of solutions of (2.1)-(2.3), containing such regimes is not empty as it is demonstrated above. Assume that for some values of the physical parameters the regime in question is stable. Evidently, the necessary condition of stability ($\sigma < 0$ for all κ) occurs to be the negativity of the value σ_0 in (2.7). This is equivalent to the negativity of the numerator

$$\sigma_1 = \omega_v R(\Gamma_4 + \Gamma_5) + \omega_w(\Gamma_0 + \Gamma_3) + \omega_w\omega_v \frac{B}{T_0}\frac{mq\rho_w R}{\lambda T_0}\Gamma\Gamma_1 \tag{3.3}$$

of σ_0, because the denominator of this quantity is always positive. The system comes to the margin of stability when under variation of the parameters the value $\kappa_0 > 0$ arises, such that $\sigma(\kappa_0) = 0$. Substituting $\sigma = 0$ in the dispersion relation (2.6) and taking into account the condition $h = L/2$ ($H = 1/2$) one gets the equality $\sigma_1 = 0$ at all κ. This implies that for $\sigma_1 > 0$ the quantity σ (if real) occurs to be positive exactly for all κ and completely (for all κ) unstable regime is bounded away from the completely stable one by the hypersurface $\sigma_1 = 0$ in the space of parameters. Thus, the equality of σ_1 to zero represents a criterion of the exchange of stabilities for the degenerate stationary regimes, characterized by a certain location of the interface $x = h = L/2$. The instability in this case takes place spontaneously in such a way, that all wavenumbers become unstable at once, when the

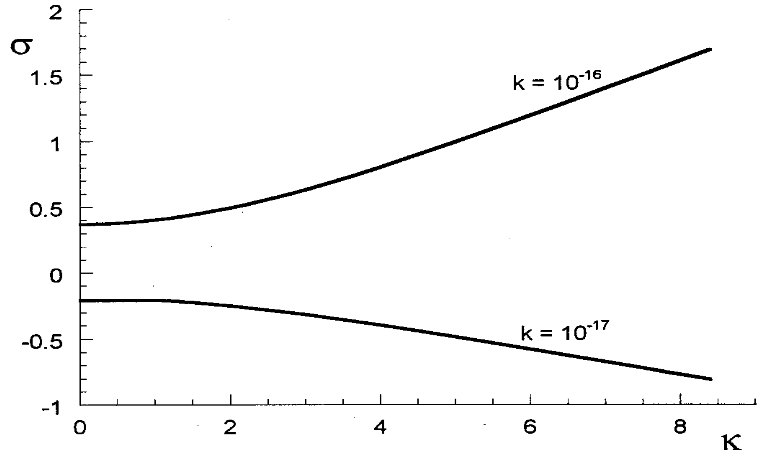

FIGURE 2. The spontaneous transition to instability of the base regime with $h = L/2$.

system crosses the threshold of instability. The example a transition to instability for the regime, described at the beginning of this subsection as the permeability varies is illustrated in Fig. 2. In this example, the instability takes place when $k > k_0$, $k_0 \approx 8.9\,10^{-17}$ m^2 at fixed values of the pressure and temperature on the boundaries of the low-permeability stratum. According to the criterion derived, the quantity σ_1 changes its sign when k crosses k_0.

3.2. *Origin of unstable wave numbers at infinity*

Let $H < 1/2$, i. e. $\min[(1-H)/H, 1] = 1$. In this case the transition to instability for $d < 1$ takes place at $Z_1 = 1$. The latter equality is attained at $\kappa = \pm\infty$. Therefore, the infinite wavenumber corresponds to the onset of instability in this case. When further variation of parameters leads to the development of instability, the interval of the unstable wave numbers, affiliated with growing normal modes, widens from infinity.

In this setting, as follows from (3.2), the condition of the onset of instability $Z_1 = 1$ is equivalent to the condition $\sigma_1 = 0$, where σ_1 is given by (3.3). The domain of the parameters given by $\sigma_1 < 0$ corresponds to a stable regime and the domain given by $\sigma_1 > 0$ corresponds to an unstable regime. If $H > 1/2$ ($\max[(1-H)/H, 1] = 1$), the onset of instability for $d > 1$ takes place analogously but at $Z_2 = 1$, and the ends of the interval of wave numbers, affiliated with decaying normal modes, come from infinity. The dispersion curves $\sigma = \sigma(\kappa)$ of the most unstable modes of the regimes where the prescribed type of transition to instability is realized are pictured in Fig. 3a,b. The transition to instability in this case is illustrated in Fig. 5a,c.

3.3. *Onset of instability at $\kappa = 0$*

For regimes with $H > 1/2$ ($\min[(1-H)/H, 1] = (1-H)/H$) and $d < (1-H)/H$ the transition to instability takes place at $Z_1 = (1-H)/H$ or, as follows from (3.2), for

$$\sigma_2 = \sigma_1 - \frac{1-2H}{H}\left(\frac{\omega_v H}{1-H}\Gamma_4 R - \omega_w \Gamma_0\right) = 0. \qquad (3.4)$$

This equality is attained at the zero value of the wavenumber κ. If instability is developed, the end points of the interval of the wavenumbers, affiliated with growing normal modes, diverge from the origin in the (κ, σ)-plane. The domain of the parameters $\sigma_2 < 0$ corresponds to a stable regime and the domain $\sigma_2 > 0$ corresponds to an unstable one. The stable and unstable dispersion curves $\sigma = \sigma(\kappa)$ of the most unstable branch of the dispersion relation for the regime where this type of transition to instability is realized are pictured in Fig. 4. The onset

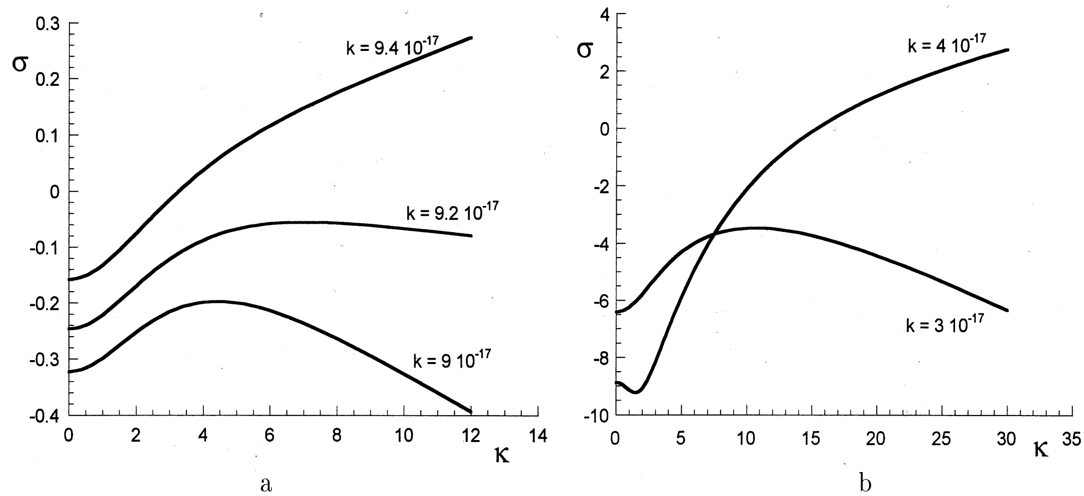

FIGURE 3. The dispersion curves $\sigma = \sigma(\kappa)$ of the most unstable branch of the dispersion relation, corresponding to regimes with $T_0 = 450$ K, $P_0 = 10^6$ Pa, $P^0 = 1.1 \cdot 10^6$ Pa; $L = 40$ m, $T^0 = 470$ K ($H < 1/2$) (a); $L = 100$ m, $T^0 = 480$ K ($H > 1/2$) (b). In the case a the transition to instability is illustrated in Fig. 5a; in the case b – in Fig. 5c.

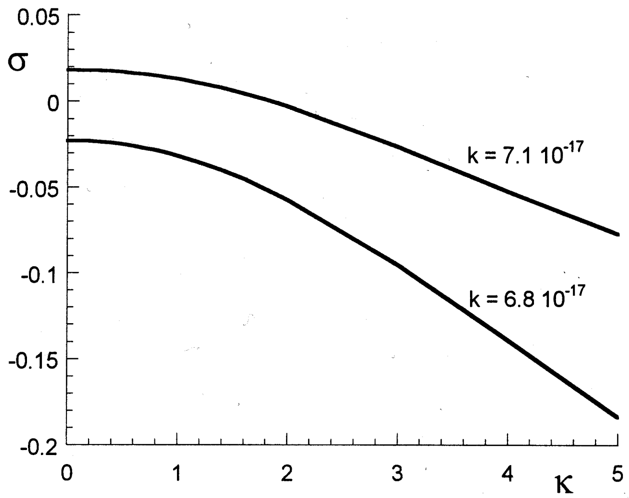

FIGURE 4. Dispersion curves $\sigma = \sigma(\kappa)$ of the most unstable branch of the dispersion relation for the regimes at $L = 40$ m, $T_0 = 450$ K, $T^0 = 467.46$ K, $P_0 = 10^6$ Pa, $P^0 = 1.19 \cdot 10^6$ Pa. Corresponds to Fig. 5d.

of instability for regimes with $H < 1/2$ ($\max[(1-H)/H, 1] = (1-H)/H$) and $d > (1-H)/H$ takes place completely analogously but at $Z_2 = (1-H)/H$. The last equality is equivalent to (3.4). The transition to instability for these cases is illustrated in Fig 5b, 5d

3.4. *Onset of instability at $\kappa \neq 0$*

Now, let us consider the case $D = 0$, $\min[(1-H)/H, 1] < d < \max[(1-H)/H, 1]$ of the onset of instability. The equality $D = 0$ is attained at the two non-zero points κ, symmetric about zero. Each of these points gives

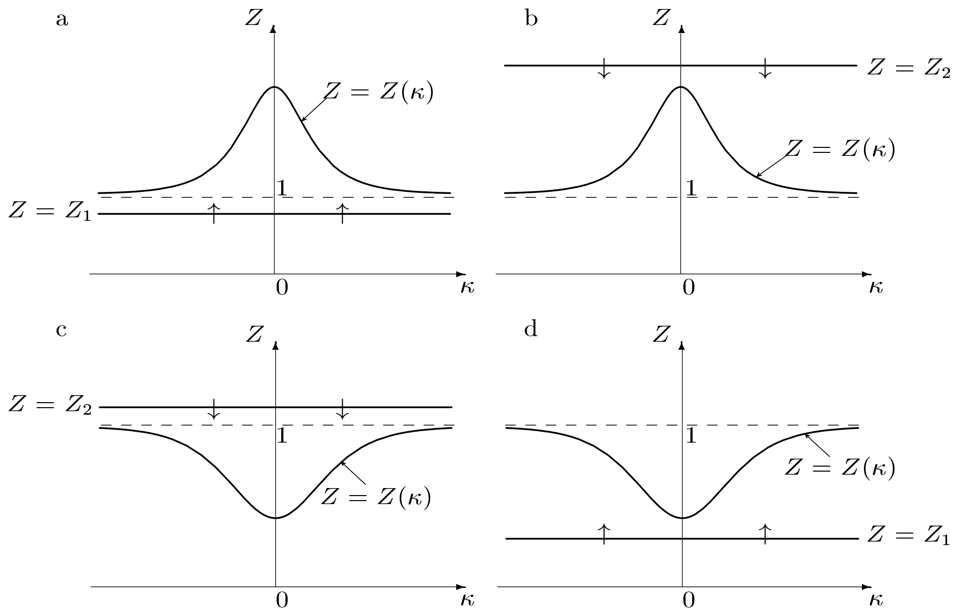

FIGURE 5. Transition to instability in cases $H < 1/2$ (a,b) and $H > 1/2$ (c,d). The threshold of instability at $\kappa = \pm\infty$ is attained, when either the line $Z = Z_1$ moving upward (a), or the line $Z = Z_2$ moving downward (c), reach the position of the asymptote of the graph $Z = Z(\kappa)$. The threshold of instability at $\kappa = 0$ is attained, when either the line $Z = Z_2$, moving downward (b), or the line $Z = Z_1$ moving upward (d) touch the curve $Z = Z(\kappa)$ at the point $\kappa = 0$.

the double root of the equation (3.2). Crossing the hypersurface in the space of parameters, determined by this equality, the system loses its stability: at $\sigma_1 < 0$ the domain of parameters $D < 0$ corresponds to a stable regime, and the domain $D > 0$ corresponds to a non-stable one. For $D = \mu < 0$, the equation (3.2) has no real roots. At $\mu = 0$ the transition to instability takes place and the roots of (3.2) come to the real axis from the complex plane. In this case the contact of the branch of the dispersion relation with the axis $\sigma = 0$ occurs in a pair of non-zero wave numbers. If variation of parameters leads to the positivity of the value of D ($\mu > 0$), two symmetric about the origin real segments of wave numbers arise, bounded by the roots of (3.2). These wavenumbers are affiliated with unstable normal modes. The stable and unstable dispersion curves $\sigma = \sigma(\kappa)$ of the most unstable branch of the dispersion relation for the regime where this type of transition to instability is realized are pictured in Fig. 6. The transition to instability itself is illustrated in fig. 7a,b.

4. Conclusion and Discussion

In this work we establish the following mechanisms of transition to instability of the interface between the domains occupied by water and vapour in high-temperature geothermal reservoirs.

1. Transition to instability realized in such a way, that all normal modes occur to be growing. In this setting the margin of stability $\sigma = 0$ of the system is attained at all κ and all normal modes become growing simultaneously (see Fig. 2). This mechanism of transition to instability is realized in the degenerate case $H = 1/2$ of phase transition interface location;

2. $D > 0$; transition to instability takes place through the infinite wave number, i. e. the neutrally stable normal modes, corresponding to a threshold of instability appear to be modes, affiliated to the wavenumber $\kappa = \pm\infty$ (see Fig 3 and Fig. 5 a,c). Such a mechanism of transition to instability is realized at $\sigma_1 = 0$ either if $d < 1$ for $H < 1/2$, or if $d > 1$ for $H > 1/2$ (see (3.3), (3.2)). The system is stable at $\sigma_1 < 0$;

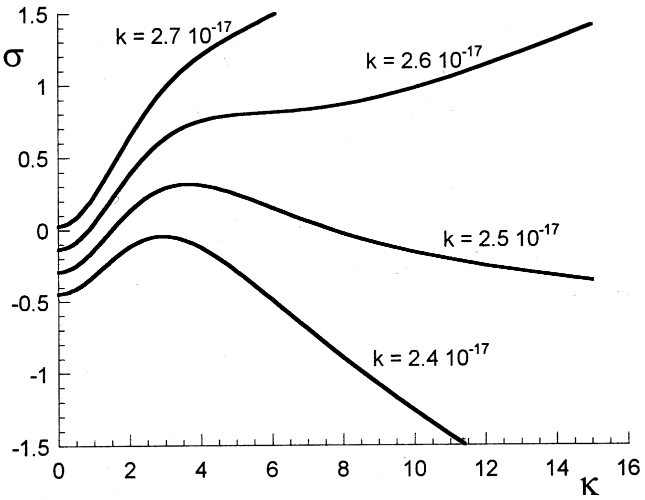

FIGURE 6. Dispersion curves $\sigma = \sigma(\kappa)$ of the most unstable branch of the dispersion relation for the regimes at $L = 400$ m, $T_0 = 450$ K, $T^0 = 538.96$ K, $P_0 = 10^6$ Pa, $P^0 = 3.62 \cdot 10^6$ Pa ($h = 300$ m). Corresponds to Fig. 7b.

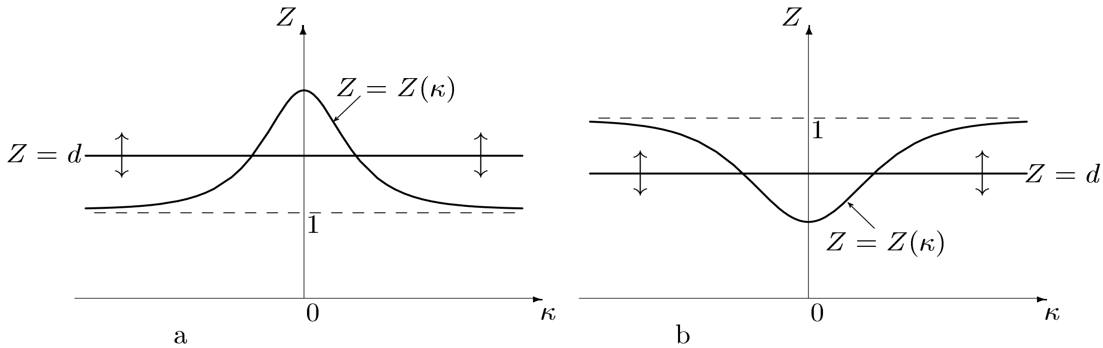

FIGURE 7. Transition to instability for D in zero neighbourhood; $H < 1/2$ (a) and $H > 1/2$ (b). For $D = 0$ the line $Z = d$ arises, which is subjected to disintegration into two lines moving in the opposite directions when the instability is in progress.

3. $D > 0$; the threshold of instability is attained for $\sigma_2 = 0$ (see (3.4)) at $\kappa = 0$ (see Fig. 4 and Fig. 5 b,d) . It takes place either for $H > 1/2$, if $d < (1 - H)/H$, or for $H < 1/2$, if $d > (1 - H)/H$. The system is stable if $\sigma_2 < 0$;

4. the threshold of instability is attained for $D = 0$ and $\min[(1 - H)/H, 1] < d < \max[(1 - H)/H, 1]$ at the pair of double non-zero values of the wavenumber κ (see Fig. 6 and Fig. 7). At $\sigma_1 < 0$ the domain of parameters $D < 0$ corresponds to a stable regime, and the domain $D > 0$ corresponds to a non-stable one.

In this paper we considered only non-oscillatory instabilities, when time exponents σ, affiliated with growing normal modes, remain real after the exchange of stabilities takes place. We are not still able to prove that there is no exchange of stabilities other than the non-oscillatory one, though all the numerous examples we ever treated give a strong support to this conclusion.

The system of equations (2.4) for time independent solutions may be written in the form of an infinite-dimensional dynamical system, where the role of "time" is played by the unbounded horizontal variable y:

$$\dot{\mathbf{w}} = \mathcal{A}\mathbf{w},$$

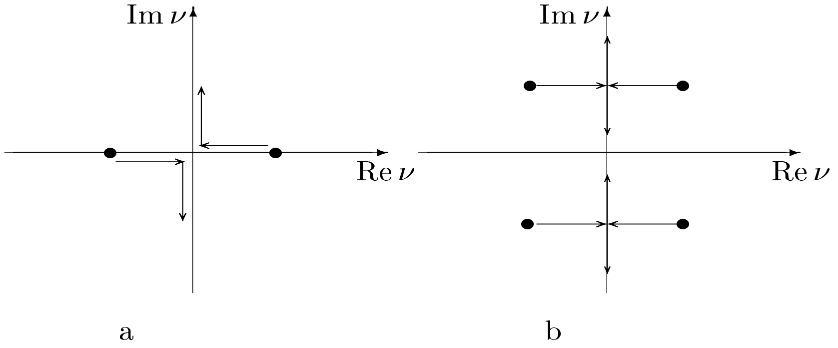

FIGURE 8. Dynamics of eigenvalues ν arriving at the imaginary axis as the parameter μ varies from negative to positive values for the cases of the transition to instability 3 (a) 4 (b).

where dot denotes differentiation with respect to the variable y, and

$$\mathbf{w} = \{\mathbf{v}_1, \mathbf{v}_2\}^T, \quad \mathbf{v}_i = \{T_i, \dot{T}_i, P_i, \dot{P}_i\}, \quad i = 1, 2, \quad \mathcal{A} = \mathrm{diag}(\mathcal{B}, \mathcal{B}), \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}.$$

Line vector-functions $\mathbf{v}_1$ and $\mathbf{v}_2$ are defined for $x \in [0, h]$ and for $x \in [h, L]$, respectively. We include the relations got from the linearized boundary conditions (2.5) into the domain of definition of the linear operator $\mathcal{A}$:

$$\begin{aligned} &T_1 = 0, \quad P_1 = 0, \quad \text{at } x = 0; \\ &T_2 = 0, \quad P_2 = 0, \quad \text{at } x = L; \\ &\mathcal{C}\mathbf{u}|_{x=h} = 0, \end{aligned}$$

where

$$\mathcal{C} = \begin{pmatrix} 1 & -1 & -\frac{T_0\Gamma_2}{P_0\Gamma_1} & \frac{T_0\Gamma_2}{P_0\Gamma_1} \\ 0 & -\frac{P_0 B\Gamma_1\Gamma}{T_0^2(\Gamma_1+\Gamma_0)} & \frac{\Gamma_0}{\Gamma_1+\Gamma_2} & -1 \\ 0 & 0 & \frac{R}{\mu_v}\frac{\partial}{\partial x} & -\frac{1}{\mu_w}\frac{\partial}{\partial x} \\ -\lambda_+\frac{\partial}{\partial x} & \lambda_-\frac{\partial}{\partial x} & 0 & -\frac{kq\rho_w}{\mu_w}\frac{\partial}{\partial x} \end{pmatrix}, \quad \mathbf{u} = \{T_1, T_2, P_1, P_2\}^T.$$

Bifurcation takes place, when eigenvalues ν of the operator $\mathcal{A}$, come to the imaginary axis. The eigenvalue problem $\mathcal{A}\mathbf{f} = \nu\mathbf{f}$ yields the dispersion equation (2.6) with $\sigma = 0$ up to the correspondence $\kappa = \mathrm{i}\,\nu$. It occurs then, that in the cases of the transition to instability, when the onset of instability is attained if the dispersion equation (2.6) has either double zero root, or the pair of non-zero double roots at $\sigma = 0$, the eigenvalues ν arrive at the imaginary axis, so the loss of stability is accompanied by a bifurcation in these cases. Bifurcation leads to a formation of horizontally non-homogeneous secondary regimes, bifurcating from the vertical flow, loosing its stability.

Because of the reversibility of the equations (2.1) and boundary conditions (2.2) for $\partial/\partial t = 0$, the eigenvalues ν arrive at the imaginary axis in pairs. The dynamics of the eigenvalues when μ crosses the origin in the direction of the positive real semi-axis for the cases 3 and 4 of transition to instability above is shown in fig. 8a,b. The bifurcations, corresponding to such a dynamics, are well known as bifurcations of the simple resonance and 1 : 1-resonance, respectively (see e. g. Iooss and Adelmeyer, 1992).

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