[1] A. Il'ichev, G.Tsypkin Catastrophic transition to instability

Abstract

Instability of a water layer located over an air-vapor layer in a horizontally infinite two dimensional domains of a porous medium is considered. A new mechanism of transition to instability of vertical flows developed in such a system is treated when the most unstable normal mode is affiliated with the zero wave number. Secondary structures bifurcating from the vertical base flow in a neighborhood of the threshold of instability obey the Kolmogorov-Petrovsky-Piscounov (KPP) diffusion-type equation. For the transition in question the KPP equation represents the analogue of the Ginzburg-Landau equation for the transition when the most unstable mode has a nonzero wave number. It is shown that in some neighborhood of the critical parameters there exist two different plane phase transition interfaces coinciding at the threshold of instability and ceasing to exist when the threshold is overcome. One of these interfaces is unstable, whereas the other is stable. It is shown nevertheless, that even the stable interface is destroyed by some perturbations of the unstable one due to nonlinear interplay of disturbances.

Front (interface) of evaporation, Rayleigh-Taylor instability, modulation equation

Catastrophic transition to instability of evaporation front in a porous medium

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1 Introduction

The model under consideration in this paper describes, for example, the convective and filtration processes in mines, tunnels and other constructions, having contact with natural massifs. The functioning of such engineering systems is accompanied by heat and mass exchange between the construction and surrounding rock [1]. Artificial ventilation makes it possible to keep the micro-climate, necessary for exploitation. Ventilation is accompanied by evaporation from a ceiling of the construction while the ground water moves downwards under the action of gravity or pressure in the water horizon. Water can enter the underground construction either in liquid or vapor states. If the surrounding rock has relatively low permeability it is natural to assume that the underground water moving towards the ceiling of the construction evaporates in a porous space and diffuses into the underground construction as a vapor. In this case a region of the rock exists which is saturated with a mixture of vapor and air and located under the water saturated domain. The interface of evaporation, separating these two domains can be either stable or unstable. Considering the stability of the interface we notice that the process of water evaporation is slow and as a consequence the influence of the heat absorbtion during evaporation is negligible. To be specific we consider the isothermal problem when the temperature of the surrounding rocks T is equal to that of the ventilated air. This makes it possible to eliminate from consideration the temperature field and to reduce the problem to the purely hydrodynamic one.

It is well known that for immiscible fluids the configuration with heavier fluid overlying the lighter one is always subjected to the Rayleigh-Taylor instability even in a porous medium having an arbitrary small permeability [6]. The interface separating immiscible fluids has to deform in a way to prevent the both fluids from mixing while the phase transition interface deforms in a way

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to keep its temperature and pressure values on the Clapeyron curve of phase equilibrium. This difference in physical properties of the interfaces in question explains the possibility of existence of a stable phase transition interface even in a case when the heavier fluid overlies the lighter one in the porous medium. For the first time the stability of such a configuration was considered in [6] where an example of a geothermal system is treated where existence of two domains saturated by motionless water and vapor is supposed.

In the present paper the criterion of stability of a system in the frame of the physical problem under consideration is obtained, and also the possible types of the transition to instability were enumerated. It was discovered that one of these types is a transition when the most unstable mode is affiliated with a zero wave number. Such a transition has a number of specific features, in particular, evolution of a narrow band of weakly unstable modes is described by the nonlinear KPP equation of a diffusion type [4].

This equation was derived for description of evolution of weakly nonlinear and weakly unstable narrow band of modes for the transition type in question in general case under the validity of the principle of exchange of stabilities [3]. As far as we know the KPP equation was not treated before as an amplitude modulation equation in hydrodynamic stability.

In this paper the KPP equation is derived in a case of the loss of stability of the plane phase transition interface for the problem in question. A number of the properties of the transition being described by this equation are discussed.

The paper organized as follows. In sec. 2 we formulate the problem. In sec. 3 we get the solutions, describing the base flow with phase transition to be subjected to the stability analysis. In sec. 4 we present the linear stability analysis of the base flows. In sec. 5 we derive the modulation KPP equation in a neighborhood of the threshold of long-wave instability. Section 6 is devoted to discussion of features of the weakly nonlinear stage of development of phase transition interface perturbations. In sec. 7 we make the conclusion and discussion. Appendix is devoted to the derivation of the KPP equation in the general case of long-wave instability satisfying the principle of exchange of stabilities.

2 Formulation

Let the high permeability water horizon with the water pressure P_0 bounded from below by the plane $z = 0$, be located over the ceiling $z = L$ (the z-axis is directed downwards). The rock in a layer $0 < z < L$ has a low permeability and at the surface $z = L$ it is in contact with air of humidity ν_a which is smaller then the humidity of saturation, i. e. the partial pressure in the air is smaller than the pressure of saturation of the vapor in the air at a given value of temperature T. In this case the low permeability porous media $0 < z < L$ contains the water layer $0 < z < h$ and the layer $h < z < L$, saturated by a mixture of the air and water vapor (Fig. 1) and adjacent to the space of the underground construction $z > L$.

We assume that there exists the plane interface where the evaporation occurs

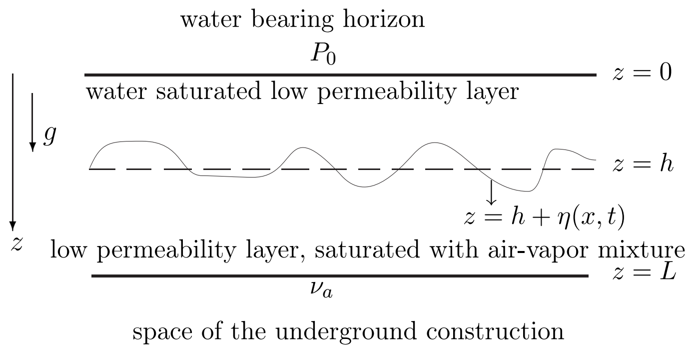

Figure 1: Schematic of the system considered; see text for definitions.

between the water saturated domain and the domain, containing homogeneous mixture of the air and vapor. Then in the domain filled in by the liquid phase the inflow of the water from the high permeability horizon takes place towards the interface of evaporation. The vapor arising at the interface diffuses through the air-vapor domain in direction of free surface $z = L$ (the ceiling of the underground construction) being in contact with air. The vapor diffusion occurs in the case when the partial pressure of the vapor in the neighborhood of the interface of evaporation is greater than the partial pressure at the free surface $z = L$.

Assuming the fluids to be incompressible we get the continuity equation and Darcy's law as the governing equations in the water saturated domain

$$\text{div } v_w = 0, \qquad v_w = -\frac{k}{\mu_w} \text{ grad } (P - \rho_w g z). \tag{2.1}$$

The governing equations in the domain saturated by the air-vapor mixture represents the equation of vapor diffusion and the Clapeyron equation for gases:

$$\frac{\partial \rho_v}{\partial t} = \text{div } D\text{grad } \rho_v \quad P_v = \rho_v R_v T, \quad P_a = \rho_a R_a T. \tag{2.2}$$

Here v is the filtration velocity, m is porosity, k the permeability, μ the viscosity, P the pressure, g the gravity, ρ the density, T the temperature, D the diffusion coefficient. Typical values are (see e.g. [5]) $D = 2.4 \times 10^{-5}$ m^{2}s^{-1}, $P_a = 10^5$ Pa, $R_a = 287$ J kg^{-1}K^{-1}, $R_v = 461$ J kg^{-1}K^{-1}.

The subscripts v, w, a correspond to the vapor, water and air, respectively. Instead of the equation for the vapor density it is convenient to use the analogous equation for the humidity function $\nu = \rho_v/(\rho_a + \rho_v)$. This equation follows from (2.2) under the condition of smallness of the partial pressure of the vapor in comparison with the atmospheric pressure [7]:

$$\frac{\partial \nu}{\partial t} = D \, \Delta \nu. \tag{2.3}$$

The system of equations (2.1) in the water domain is reduced to the Laplace equation

$$\Delta\ P = 0. \tag{2.4}$$

For determination of conditions at the interface we assume that the full pressure in the domain of the air-vapor mixture coincides with the pressure in a free space of the underground construction and equals to the atmospheric one. It means that there is no gas filtration in the air-vapor domain. On the phase transition interface $z = h + \eta(x, t)$ the difference between the pressure in the water domain and that one in the gas domain depends on the physical properties of rocks and it equals to the capillary pressure. The boundary condition at the interface for a pressure jump reads

$$P_- = P_+ + P_c \equiv P_a + P_c, \tag{2.5}$$

where signs "−" and "+" stand for the water and air-vapor domain, respectively. Here the capillary pressure P_c is negative, when the rock is wettable and positive for the non-wettable rock. The boundary condition for the humidity at the interface follows from the definition of the humidity function and Clapeyron's equations for the air and vapor:

$$\nu = \nu_* = \frac{R_a}{R_v}\frac{P_{v*}}{P_a}. \tag{2.6}$$

The dependance of the partial pressure on the temperature can be presented in the form [9]

$$P_v = F(T) \qquad \text{or} \qquad \nu_* = \frac{R_a}{R_v}\frac{F(T)}{P_a},$$

$$F(z) = 10^5 \exp\left[-7226.6\left(\frac{1}{z} - \frac{1}{373.16}\right) + 8.2\ln\frac{373.16}{z} - 0.0057(373.16 - z)\right].$$

From the assumption that the processes under consideration are isothermal it follows that the humidity on the interface being a function of the temperature is a constant in the framework of our model. The water mass conservation law at the interface has the form

$$\left(1 - \frac{\rho_v}{\rho_w}\right) V_n = -\frac{k}{m\mu_w}\left[\text{ grad } (P - \rho_w g z)\right]_{n-} + D\frac{\rho_a}{\rho_w}\left(\text{ grad } \nu\right)_{n+}, \tag{2.7}$$

where the subscript n denotes normal components, and V_n is a normal velocity of the interface.

The boundary condition at the upper boundary $z = 0$ and at the lower boundary, coinciding with the ceiling are written as

$$z = 0: \quad P = P_0; \quad z = L: \quad \nu = \nu_a. \tag{2.8}$$

3 Base flow

If the parameters of the water layer and of the circulated air are constant, then the surface of evaporation occupies some equilibrium position $z = h$, depending on parameters, initial and boundary values. This position has to be determined from the solving the stationary problem. The solution corresponding to the stationary vertical base flow is determined from the equations

$$\frac{\mathrm{d}^2 P(z)}{\mathrm{d}z^2} = 0, \qquad \frac{\mathrm{d}^2 \nu(z)}{\mathrm{d}z^2} = 0.$$

The mass conservation law at the interface $z = h$ in the stationary case has the form

$$\frac{k}{m\mu_w} \left[\left(\frac{\mathrm{d}P}{\mathrm{d}z} \right)_- - \rho_w g \right] = D \frac{\rho_a}{\rho_w} \left(\frac{\mathrm{d}\nu}{\mathrm{d}z} \right)_+ . \tag{3.1}$$

The solutions in the water domain $0 < z < h$ and vapor domain $h < z < L$ have the form, respectively

$$P_{st} = P_0 + \frac{P_a + P_c - P_0}{h}\, z, \qquad \nu_{st} = \frac{\nu_a - \nu_*}{L - h}\, z + \frac{L\nu_* - h\nu_a}{L - h}. \tag{3.2}$$

Substituting the solutions (3.2) into the conditions at the interface (3.1) one gets the quadratic equation for the determination of h. This equation in the dimensionless form reads

$$\left(1 + \frac{P_c}{P_a} - \frac{P_0}{P_a} \right) \frac{1}{H} - \frac{\rho_w g L}{P_a} = \frac{D}{\omega} \frac{\rho_a}{\rho_w} \frac{\nu_a - \nu_*}{1 - H}, \qquad \omega = \frac{k P_a}{m\mu_w}, \qquad H = \frac{h}{L}. \tag{3.3}$$

The solution of (3.3) is given by

$$H_{l,s} = -\frac{1}{2} (\beta - \alpha - 1) \pm \frac{1}{2} \sqrt{(\beta - \alpha - 1)^2 - 4\alpha}, \quad H_l \geq H_s,$$

$$\alpha = \frac{P_c + P_a - P_0}{\rho_w g L}, \qquad \beta = \frac{D}{\omega} \frac{\rho_a}{\rho_w} (\nu_* - \nu_a) \frac{P_a}{\rho_w g L}. \tag{3.4}$$

The parameter α measures the deviation of the aquifer pressure P_0 from hydrostatic, while the parameter β measures the importance of the downflow driven by evaporation relative to a purely pressure driven downflow.

Next let us consider some general features of the roots of the quadratic equation (3.3). From the solution (3.4) it follows that for the neutral $P_c = 0$ or wettable ($P_c < 0$) porous medium when $P_0 - P_a - P_c > 0$, one root is positive and the other one is negative. The physical meaning has only the positive root, corresponding to the sign plus at the radical in (3.4). For the non-wettable medium when $P_c > 0$, the transition of the second root through zero to the positive real axis is possible. Such a transition corresponds, for example, to the pressure P_0 fall in the water horizon, when the difference $P_0 - P_a - P_c$ decreases. For $P_0 - P_a - P_c = 0$ the second root equals zero. For the further decreasing of P_0, when $P_0 - P_a - P_c < 0$ the both roots of the quadratic equation become positive and when some critical value P_0 is achieved the confluence of the roots takes place and the stationary solution ceases to exist for P_0 below the critical value. For the critical value of P_0, as it follows from (3.4), the quadratic equation has a double root

$$H = \sqrt{\alpha}. \tag{3.5}$$

Let us write the quadratic equation (3.3) in the form

$$-\beta = \alpha \frac{1-H}{H} - (1-H).$$

Using (3.5), we present the right hand side of this equality as

$$\frac{(1-H)^2}{H^2} \left[\alpha \frac{H}{1-H} - \frac{H^2}{1-H} \right] \equiv$$
$$\equiv -\frac{(1-H)^2}{H^2} \alpha \left(-\frac{H}{1-H} + \frac{1}{1-H} \right) \equiv -\frac{(1-H)^2}{H^2} \alpha.$$

The resulting condition of the coincidence of roots now reads

$$\beta = \left(1 - \sqrt{\alpha}\right)^2. \tag{3.6}$$

The solution considered, describing the stationary process of water evaporation at the interface corresponds to the configuration when the heavier fluid (water) overlies the lighter one (air-vapor mixture). In the case of the existence of the unique solution for the location of the phase transition interface it is evident, that the solution in question gives the inflow of moisture into the underground construction as a result of the diffusion of the vapor through the rock. If there exist two solutions, i. e. two locations of the interface satisfy the equation (3.3), then the analysis of the both solutions is required and one of them can (and it does!) occur unstable. The region $\mathcal{D}$ of parameter space (α, β) where the both solutions for H exist and remain positive is given by

$$\mathcal{D} = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \, |, \; 0 < \alpha < 1 \cap \sqrt{\alpha} + \sqrt{\beta} < 1 \right\},$$

and they merge at the boundary

$$\partial\mathcal{D} = \left\{ (\alpha, \beta) \, |, \; 0 < \alpha < 1 \cap \sqrt{\alpha} + \sqrt{\beta} = 1 \right\}.$$

4 Linear stability

Destabilization of the base regime is possible only in the case of non-wettable porous medium ($\alpha > 0$), consequently we consider this case only. The equations

(2.3) and (2.4) are linear therefore the linearization about the solution (3.2) of boundary conditions only has to be done. Let P', ν', η be the perturbations of the base regime pressure, humidity and the interface position respectively. The linearization of water conservation law reads

$$\frac{\partial \eta}{\partial t} = -\frac{k}{m\mu_w}\left(\frac{\partial P'}{\partial z}\right)_- + D\frac{\rho_a}{\rho_w}\left(\frac{\partial \nu'}{\partial z}\right)_+ . \tag{4.1}$$

The conditions for pressure jump (2.5) and the condition of the constancy of the humidity (2.6) at the interface give

$$P' + \frac{dP_{st}}{dz}\eta = 0, \qquad \nu' + \frac{d\nu_{st}}{dz}\eta = 0, \qquad z(t,y) = h. \tag{4.2}$$

We look for the solutions of (4.1), (4.2) in the form $\{P', \nu', \eta\} = \{\hat{P}(z), \hat{\nu}(z), \hat{\eta}\}\exp(\sigma t + i\kappa y)$. The expressions for the amplitudes read

$$\hat{P}(z) = \hat{P}_-\frac{\sinh \kappa z}{\sinh \kappa h}, \tag{4.3}$$

$$\hat{\nu}(z) = \hat{\nu}_+\frac{\sinh a(L-z)}{\sinh a(L-z)}, \tag{4.4}$$

where $a^2 = \sigma/D + \kappa^2$.

It follows from (4.3), (4.4) that

$$\left(\frac{d\hat{P}(z)}{dz}\right)_{z=h} = \kappa\hat{P}_-\coth\,\kappa h, \qquad \left(\frac{d\hat{\nu}(z)}{dz}\right)_{z=h} = -a\hat{\nu}_+\coth\,a(L-h). \tag{4.5}$$

Substituting (4.5) into (4.1), (4.2) we get that the nontrivial solution $\hat{\eta}$, $\hat{P}_-$, $\hat{\nu}_+$ exists if the following dispersion relation holds

$$\gamma\,\Sigma - \alpha\,\frac{K\coth\,KH}{H} + \beta\,\sqrt{\Sigma + K^2}\,\frac{\coth\,(1-H)\sqrt{\Sigma + K^2}}{1-H} = 0, \tag{4.6}$$

where $K = \kappa L$, $\Sigma = \sigma L^2/D$, $\gamma = \frac{D}{\omega}\frac{P_a}{\rho_w g L}$.

We consider the mechanism of the loss of stability satisfying the principle of exchange of stabilities when Σ equals zero at the margin of stability. The equation of the marginal set is got by putting $\Sigma = 0$ in (4.6). We have

$$f(K) \equiv \frac{\coth K(1-H)}{\coth KH} = \frac{\alpha}{\beta}\,\frac{1-H}{H} \equiv \Delta. \tag{4.7}$$

It follows from (4.6) that for $K \to \infty$, $\Sigma \to \Sigma_0 K$, where

$$\Sigma_0 = \frac{\alpha}{\gamma}\,\frac{1}{H} - \frac{\beta}{\gamma}\,\frac{1}{1-H}.$$

From (4.7) and the fact that the inequality $\Sigma_0 > 0$ evidently implies instability of the base regime in question, it follows that there are two scenarios of transition to instability in the case of the non-wettable porous media.

It is easy to establish that if the values of the left hand side of the equation (4.7) for some K is less than the value of the right hand side then the base solution is unstable i. e. $\Sigma > 0$ for these K. Assume that

$$\frac{\coth K(1-H)}{\coth KH} < \Delta, \tag{4.8}$$

and the solution is stable, i. e. $\Sigma < 0$ for any K. If $\Sigma < 0$ the inequality follows

$$\frac{\sqrt{\Sigma + K^2}\coth\sqrt{\Sigma + K^2}(1-H)}{K\coth KH} < \frac{\coth K(1-H)}{\coth KH}. \tag{4.9}$$

Simultaneously, from the dispersion relation (4.6) it follows that if $\Sigma < 0$, the inequality holds

$$-\Delta\ K\coth KH + \sqrt{\Sigma + K^2}\coth\sqrt{\Sigma + K^2}(1-H) > 0,$$

or

$$\frac{\sqrt{\Sigma + K^2}\coth\sqrt{\Sigma + K^2}(1-H)}{K\coth KH} > \Delta$$

that contradicts the inequalities (4.8) and (4.9). Therefore, if (4.8) is valid, $\Sigma > 0$ must hold.

For the sake of clarity we use the geometrical interpretation of the equation (4.7). The left hand side of the equation represents a function that depends on K and tends to 1 for $K \to \infty$. This function has either minimum for $H < 1/2$ or maximum for $H > 1/2$ at $K = 0$. The right hand side of the equation (4.7) does not depend on K and represents the line parallel to the K-axis.

The first scenario of the transition is realized for $H > 1/2$. Fig. 2a illustrates the stability of the base stationary solution. Decrease of the aquifer pressure P_0 implies the increase of Δ and the line $\zeta = \Delta$ becomes asymptotic to the graph of the function $\zeta = f(K) = \coth K(1-H)\coth^{-1} KH$ in the (K,ζ)-plane, when $\Sigma = 0$. The further decrease of P_0 implies that Δ becomes greater than unity and the line $\zeta = \Delta$ intersects the graph of $f(K)$ in two points symmetric about zero. The increment $\Sigma > 0$ in two semi-infinite intervals symmetric about zero (Fig. 2b). The corresponding transition to instability is in the (K,Σ)-plane is presented in Fig. 2c.

For $H < 1/2$ one has the different scenario of the emergence of instability. Fig. 3a illustrates the stable case when the marginal set (4.7) is empty i. e. there is no intersection between the curve $\zeta = f(K)$ and the line $\zeta = \Delta$. When the pressure P_0 decreases the straight line $\zeta = \Delta$ moves upward while the minimum of the curve $\zeta = f(K)$ moves downward. The marginal stability is reached at the moment when they have the point of contact at $K = 0$. The second root of the equation (3.3) is unstable (Fig. 3b), but when P_0 decreases the opposite motion

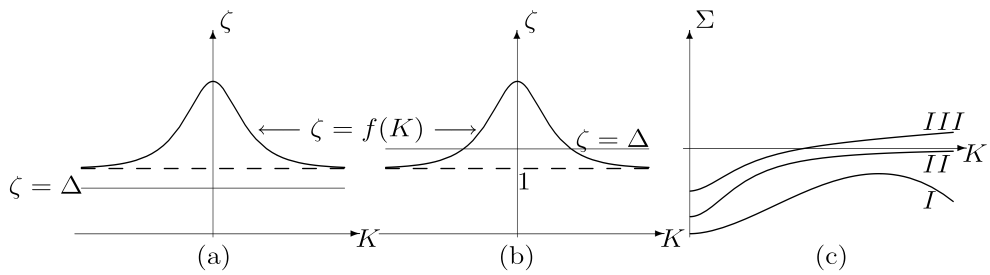

Figure 2: Case $H > 1/2$. Comparative locations of the line $\zeta = \Delta$ and the graph $\zeta = f(K)$ for the stable (a) and for the unstable (b) fronts. Sketches of dispersion curves $\Sigma = \Sigma(K)$ (c): for stable (I), marginal (II) and unstable (III) front.

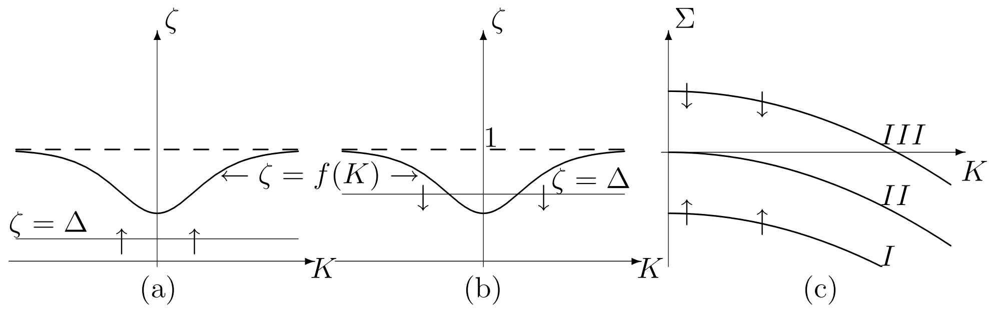

Figure 3: Case $H < 1/2$. Comparative locations of the line $\zeta = \Delta$ and the graph $\zeta = f(K)$ for the stable $H = H_l$ (a) and for the unstable $H = H_s$ (b) fronts. Sketches of dispersion curves $\Sigma = \Sigma(K)$ (c): for stable $H = H_l$ (I), marginal (II) and unstable $H = H_s$ (III) fronts. The vertical arrows denote the tendency to the marginal state on the way to instability.

takes place, i. e. the line moves downward while the curve moves upward. As a result the second root reaches the threshold of instability simultaneously with the first one and at they coincide at the margin of stability. The moment of the marginal stability (when the line $\zeta = \Delta$ touches the graph of the function $\zeta = f(K)$ in the (K, ζ)-plane at $K = 0$) in the (K, Σ)- plane is illustrated in Fig. 3c. Expanding the left hand side of (4.7) in the zero neighborhood and neglecting the higher order terms one gets the criterion of the transition at $K = 0$:

$$\frac{H^2}{(1-H)^2} = \frac{\alpha}{\beta}. \tag{4.10}$$

The equation (4.10) coincides with the condition (3.6) of the merger of the roots for the base solution and the double root correspond to the marginal set in the parameter space. When the quantity Δ further increases the real solutions of the quadratic equation (3.3) cease to exist. This means that there are no stationary base regimes with phase transition interface in the parameter domain in question.

Therefore the stability criterion (4.7) gives two possible routes to instability. The first one occurs at infinite wave number, corresponds to the condition $\Delta = 1$, and is possible for front positions $H > 1/2$; the second one occurs for zero wave number, corresponds to the criterion $f(0) = \Delta$, and is possible for front positions $H < 1/2$. The conditions of stability $\Delta < 1$, $f(0) > \Delta$ (see Figs. 2a,b and 3a,b) imply that the root H_s corresponding to a higher front position (recall that the z-axis is directed downwards) is unstable for $(\alpha, \beta) \in \mathcal{D}$. The root H_l, which corresponds to a lower interface position is linearly stable in the region $\mathcal{D}_0 = \mathcal{D}_1 \cup \mathcal{D}_2$, where

$$\mathcal{D}_1 = \{\alpha, \beta) \in \mathcal{D} \mid 0 < \alpha < 1/4\}, \quad \mathcal{D}_2 = \{(\alpha, \beta) \in \mathcal{D} \mid 1/4 < \alpha < 1/2 \cap 0 < \beta < 1/2-\alpha\}.$$

It is linearly unstable in the region $\mathcal{D}_3 = \mathcal{D} \setminus \mathcal{D}_0$ (Fig 4). The components of the boundary $\Gamma_0 = \Gamma_1 \cup \Gamma_2$ where H_l becomes unstable is given by

$$\Gamma_1 = \{(\alpha, \beta) \mid 1/4 < \alpha < 1/2,\ \alpha+\beta = 1/2\}, \quad \Gamma_2 = \{(\alpha, \beta) \mid 0 < \alpha < 1/4,\ \sqrt{\alpha}+\sqrt{\beta} = 1\}.$$

The root H_l becomes unstable through the $K = \infty$ transition on Γ_1, while it becomes unstable through $K = 0$ transition on the boundary Γ_2.

5 Derivation of the KPP equation

In this section we confine our discussion to the case of the long-wave instability in a neighborhood of the boundary at which $K = 0$ transition occurs, i e. for $(\alpha, \beta) \in \mathcal{D}_2$ close to the boundary Γ_2 . The equation is derived describing the secondary structures, bifurcating from the base state (3.2) in a small neighborhood of the instability threshold for the mentioned type of instability. Taking into account the long-wave nature of instability determine the small dimensionless parameters χ and ε:

$$\chi = L^2/\lambda^2, \quad \varepsilon = \eta_a/L,$$

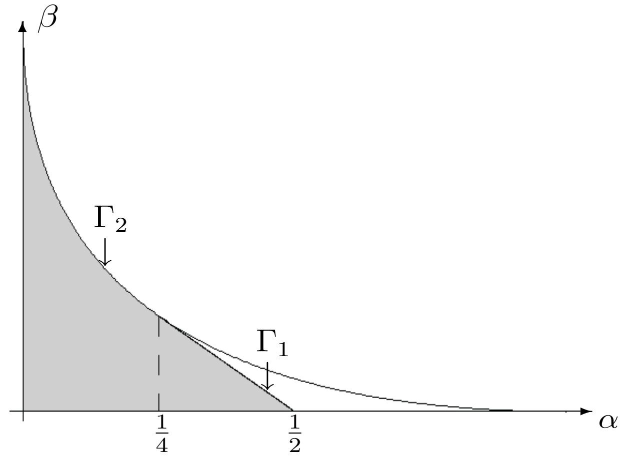

Figure 4: The region $\mathcal{D}$ of existence of the roots H_s, H_l for $\alpha > 0$. The root H_l is stable in the domain $\mathcal{D}_0$ coloured with grey, it becomes unstable on the boundary $\Gamma_0 = \Gamma_1 \cup \Gamma_2$: on Γ_1 through $K = \infty$ transition, on Γ_2 through $K = 0$ transition.

where λ and η_a are characteristic values of wave length and amplitude. Determine the dimensionless variables (keeping the former notations)

$$x \to \lambda x, \quad t \to \frac{L^2}{D} t, \quad \eta \to \eta_a \eta, \quad z \to Lz.$$

The equations (2.3), (2.4) are now rewritten in the following dimensionless form ($\partial_x = \partial/\partial x$, $\partial^2_{xx} = \partial^2/\partial x^2$, etc.).

$$\big(\chi \partial^2_{xx} + \partial^2_{zz}\big) P = 0, \quad \partial_t \nu = \big(\epsilon \partial^2_{xx} + \partial^2_{zz}\big)\nu. \tag{5.1}$$

Further put

$$P = P_{st} + P', \quad \nu = \nu^s_{st} + \nu',$$

where P', ν' are the pressure and humidity perturbations in the domains of low permeability rock saturated by water and air-vapor mixture, respectively.

Make the expansion of the pressure and humidity with respect to the vertical coordinate z in the both domains at the position $z = H$:

$$P = P_a + P_c + P' + \left(\frac{P_a + P_c - P_0}{H} + \partial_z P'\right)(z - H) + \frac{1}{2}\partial^2_{zz} P'(z-H)^2 + \frac{1}{6}\partial^3_{zzz} P'(z-H)^3 ...,$$
$$\nu = \nu_* + \nu' + \left(\frac{\nu_a - \nu_*}{1 - H} + \partial_z \nu'\right)(z - H) + \frac{1}{2}\partial^2_{zz}\nu'(z-H)^2 + \frac{1}{6}\partial^3_{zzz}\nu'(z-H)^3 ... \tag{5.2}$$

Next substitute the expressions (5.2) for the pressure into the boundary conditions (2.8) at $z = 0$, and the expressions for the humidity into the boundary conditions at the boundary $z = 1$. We have

$$\partial_z P' = \frac{P'}{H} + \frac{1}{2}\partial^2_{zz} P' H - \frac{1}{6}\partial^3_{zzz} P' H^2 + ...,$$
$$\partial_z \nu' = -\frac{\nu'}{1-H} - \frac{1}{2}\partial^2_{zz}\nu'(1-H) - \frac{1}{6}\partial^3_{zzz}\nu'(1-H)^2 ... \tag{5.3}$$

Next express the z-derivatives in terms of the x-derivatives from the equations (5.1):

$$\partial^2_{zz} P' = -\chi \partial^2_{xx} P', \quad \partial^3_{zzz} P' = -\chi \partial_{xx}\big(\partial_z P'\big), ...,$$
$$\partial^2_{zz}\nu' = \big(\partial_t - \chi\partial^2_{xx}\big)\nu', \quad \partial^3_{zzz}\nu' = \big(\partial_t - \chi\partial^2_{xx}\big)\partial_z \nu', ... \tag{5.4}$$

Substituting (5.4) into (5.3) and resolving the resulting equation with respect to χ, we get

$$\partial_z P' = \frac{P'}{H} - \frac{\chi H}{3}\partial^2_{xx} P' ..., \quad \partial_z \nu' = -\frac{\nu'}{1-H} - \frac{1-H}{3}\big(\partial_t - \chi \partial^2_{xx}\big)\nu' ... \tag{5.5}$$

From the boundary conditions (2.5), (2.6) using (5.5) one gets the asymptotic expressions for pressure and humidity on the perturbed interface

$$P' = -\frac{P_a + P_c - P_0}{H}\varepsilon\eta + \frac{P_a + P_c - P_0}{H^2}\varepsilon^2\eta^2 ...,$$
$$\nu' = -\frac{\nu_a - \nu_*}{1-H}\varepsilon\eta - \frac{\nu_1 - \nu_*}{(1-H)^2}\varepsilon^2\eta^2 ... \tag{5.6}$$

Substituting (5.5), (5.6) into the dimensionless conservation law (2.7) and putting $\chi = \varepsilon$ we get finally the equation (3.3) at ε^0, zero at ε, and the equation

$$c_0 \partial_\tau \eta = c_1 \eta + c_2 \partial_{xx}\eta + c_3 \eta^2, \quad \tau = \varepsilon t \tag{5.7}$$

at ε^2. Here

$$c_0 = \left(1 - \frac{\rho_a}{\rho_w}\frac{\nu_a - \nu_*}{3}\right) > 0, \quad c_1 = \varepsilon^{-1}\left(\frac{k}{D\mu_w m}\frac{P_a + P_c - P_0}{H^2} + \frac{\rho_a}{\rho_w}\frac{\nu_a - \nu_*}{(1-H)^2}\right),$$
$$c_2 = -\frac{1}{3}\left(\frac{k}{D\mu_w m}(P_a + P_c - P_0) + \frac{\rho_a}{\rho_w}(\nu_a - \nu_*)\right) > 0, \quad c_3 = -\left(\frac{k}{D\mu_w m}\frac{P_a + P_c - P_0}{H^3} + \frac{\rho_a}{\rho_w}\frac{\nu_* - \nu_a}{(1-H)^3}\right) <$$

The coefficient c_1 is of order 1 in a neighborhood of the marginal stability: the condition $\varepsilon = 0$ corresponds to the condition of marginal stability of the most unstable mode for the transition under consideration, coinciding with (4.10).

6 Evolution of disturbances

The equation (5.7) by use of the appropriate scaling may be reduced to the KPP form (the former notations are kept for the scaled variables)

$$\partial_\tau \eta = n\eta - \eta^2 + \partial^2_{xx}\eta \tag{6.1}$$

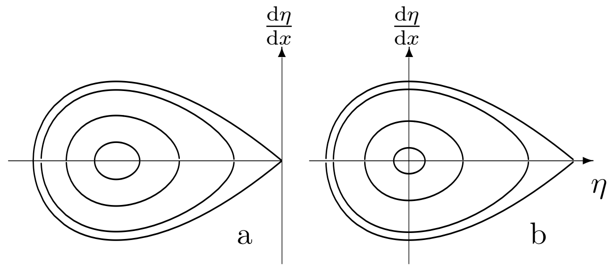

Figure 5: Phase portrait of the equation (6.1) when $\partial_\tau \eta = 0$; subcritical case (a) ($n = -1$); supercritical case (b) ($n = 1$).

where $n = \mathrm{sign}\,(c_1)$.

In Fig. 5 the phase portraits of the stationary equation

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} = -n\eta + \eta^2, \tag{6.2}$$

are presented. This equation describes the stationary regimes (non necessarily horizontally homogeneous) in some neighborhood of the instability threshold for $n = \mp 1$, respectively.

It can be seen from Fig. 5 that in a neighborhood of the threshold of instability there are two equilibrium points of the dynamical system which is equivalent to the equation (6.2). These points correspond to the base regimes, representing the horizontally homogeneous flows. In the subcritical domain ($n = -1$) the solution $\eta = 0$ is a stable one while the solution $\eta = -1$ is unstable. In the supercritical domain the solution $\eta = 0$ is unstable while the solution $\eta = 1$ is stable. Hence, the transition is accompanied by the shift of equilibrium states, describing the base regimes, to the constant value equal to the unity for the dimensionless equation (6.1). It is clear then, that for our problem the case $n = 1$ is reduced to the considering the unstable front $H = H_s$ as the base solution to be subjected to the stability analysis, i. e. perturbing it in the domain of parameters subcritical for the stable front H_l.

There is a homoclinic orbit in Fig. 5a representing a solitary wave solution of the KPP equation (6.1) for $n = -1$ (the analysis for the case $n = 1$ is reduced to this one by performing the shift $\eta \to \eta + 1$):

$$\eta = \eta_0(x) = -\frac{3}{2}\mathrm{sech}^2\frac{x}{2}. \tag{6.3}$$

This localized solution of the KPP equation is linearly unstable. In fact, linearization of (6.1) about the solitary-wave solution (6.3) gives the nonhomogeneous equation for perturbations $\delta\eta = \delta\eta(x,t)$:

$$\delta\eta_\tau + \delta\eta - \partial_{xx}\delta\eta + 2\eta_0(x)\delta\eta = 0. \tag{6.4}$$

Substituting $\delta\eta = w(x)\mathrm{e}^{-\omega\tau}$ in (6.4) we get the classical Sturm-Liouville spectral

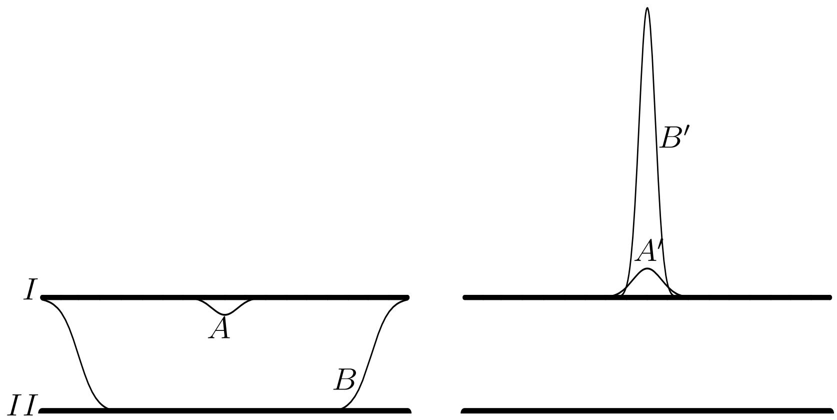

Figure 6: Schematic development of perturbations of the unstable interface (I) in a neighborhood of the threshold of instability. A is the initial perturbation, concentrated between the stable (II) and unstable interface, B the evolution of A after some time; A' is the initial perturbation satisfying the condition (6.5), B' its evolution after some time.

problem

$$\mathcal{L}w = \omega w, \quad \mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\eta_0 + 1.$$

It is well known that the lowest eigenvalue ω_s of the operator $\mathcal{L}$ is negative, therefore there exists the unstable eigenfunction $w_s(x)$ affiliated with ω_s and the solitary wave (6.3) is unstable.

It is easy to establish for the equation (6.1) that the linear exponential growth of the positive perturbation of the unstable base state is suppressed by the nonlinearity. As a result the perturbed dynamically unstable equilibrium state evolves to the stable equilibrium state [4]. If the unstable state is subjected to the non-positive perturbations satisfying

$$\int_{-\infty}^{\infty} \eta \, \mathrm{d}x < 0, \tag{6.5}$$

they grow unboundedly with time. This follows from the application of Gronwall's lemma to the inequality

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} \eta \, \mathrm{d}x < \int_{-\infty}^{\infty} \eta \, \mathrm{d}x$$

which in its turn follows from one time integration of the equation (6.1) for $n = 1$.

As the smaller root of (3.3) H_s is unstable and the larger one H_l is stable, one has the configuration of the location of interfaces pictured in Fig. 6. The positive perturbation of the unstable upper interface (we recall that the z-axis is directed downwards as in Fig. 1) evolves to the stable lower interface. The perturbation satisfying (6.5) grows to the domain overlying the unstable interface (Fig. 6). The behavior of these perturbations underlines either the stabilization of the system on the lower interface position or the tendency of penetration of the air-vapor into the domain occupied by water when the parameters vary near critical values below the threshold of instability of the front $H = H_l$.

7 Conclusion and Discussion

The subject of discussion of the present paper is a new type of transition to instability which is characterized by the first destabilization of the mode affiliated with the zero wave number. The transition in question is illustrated by the simple though physically realistic example of the flow in the porous medium with phase transition interface of evaporation where the heavier phase (water) overlies the lighter one (air-water vapor mixture). Our approach throughout was to seek the simplest rather than the most comprehensive description of the phenomena. Rigorously derived models for these processes can become very complex; in the spirit of our approach we make some assumptions which lead to a tractable model and which should capture the essential physics of the processes.

The type of the transition to instability under consideration was first discovered in the more complicated models of geothermal systems where flows with phase transition are permitted in the base state [2, 8]. Yet the complete analytical investigation of this model is not easy, and we formulate the simple physical model admitting the complete analysis in a closed analytical form.

We identify that the vanishing of the base regime to be subjected to a stability analysis coincides with its marginal stability. This implies the "catastrophic" reconstruction of system as a whole, base flow under consideration can move nowhere in the neighborhood of itself and therefore it has to change significantly, rapidly developing into some non-stationary regime. Consequently, the transition in question occurs under the action of the mechanism, different from the exponential growth of normal modes and corresponds to catastrophic scenario of the development of instability when the base regimes vanish.

We find out that in a neighborhood of the threshold of stability for the transition through $K = 0$ there exist exactly two base flows, corresponding to different locations of the evaporation front. One of these regimes is stable while the other one is unstable. At the marginal stability the coincidence of the both fronts takes place. The bifurcation diagram is given in Fig. 7. In this case the unfolding takes place which is typical for the subcritical bifurcations, consequently we find that one solution branch is stable, and the other one is subcritically unstable. Nevertheless, for the case under analysis it is impossible to continue the solution losing its stability (branch $H = H_l$) through the turning point $\mathcal{O}$.

We derive the nonlinear equation describing evolution of a narrow band of

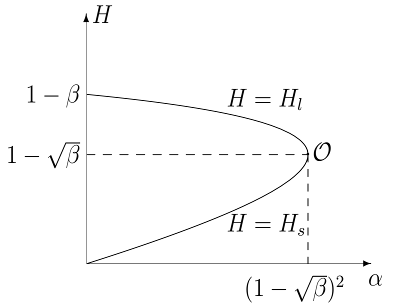

Figure 7: Bifurcation diagram H versus α at a fixed β in (3.4); $H = H_l$ is the stable branch, $H = H_s$ the unstable branch, $\mathcal{O}$ the turning point.

weakly unstable perturbations of a base state in a neighborhood of the stability threshold for the $K = 0$ transition. This equation represents the celebrated Kolmogorov-Petrovsky-Piscounov equation with simple expressions for its coefficients. The analysis of this equation allows one to predict the tendency of nonlinear development of perturbations. We find that the perturbations of the unstable evaporation front not necessarily evolve to the stable front and as a consequence the stable front can be destroyed by perturbations, having negative average, even below the threshold of stability.

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8 Appendix

In the Appendix we consider the homogeneous flow of the viscous fluid in the case of one spatial dimension, when the independent variables are x and t (a generalization of the analysis to the case of higher dimensions can be performed straightforwardly). Let the corresponding linearized hydrodynamic problem admit the normal mode solutions $\mathbf{A}\mathrm{e}^{(\mathrm{i}\kappa x+\sigma t)}$ with real wavenumbers κ and the dispersion relation $\mathcal{D}(\kappa, \sigma, R) = 0$, where R is a parameter of the problem and its variation leads to the loss of stability of the base flow in question. We assume that the problem is invariant under the reflectional symmetry $x \to -x$, and also that the principle of the exchange of stabilities is valid, i. e. $\operatorname{Im}\sigma = 0$ at the origin of instability and $\sigma = 0$ corresponds to the marginal mode.

It follows from the spatial reversibility that the expression $\mathbf{A}\mathrm{e}^{(-\mathrm{i}\kappa x+\sigma t)}$ is a solution of the linearized problem as well, i. e. $\mathcal{D}(\kappa, \sigma, R) = \mathcal{D}(-\kappa, \sigma, R)$ for any σ. We assume further, that the most unstable mode at a margin of instability is the mode affiliated with $\kappa(R_c) = 0$, where R_c is the critical value of the parameter. Then in the generic case the algebraic approximation of the dispersion relation for $|R-R_c| \ll 1$, describing a narrow band of weakly unstable

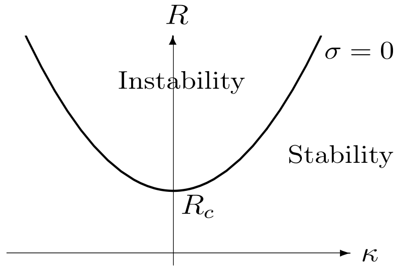

Figure 8: Critical curve $\sigma = 0$ in the (κ, R)-plane.

modes has the form

$$\sigma = a - b\kappa^2 + O(\kappa^4), \quad \kappa \to 0, \tag{A1}$$

where $a = r(R - R_c)$ $(r > 0)$, $0 < b \to b_c$ for $R \to R_c$. The marginal curve for the case in question is pictured schematically in Fig. 8. The width of a band of the unstable modes is of order $O(\varepsilon^{1/2})$, where $\varepsilon = |R - R_c|$ (Fig. 9).

For small enough ε the problem admits the approximate solutions in a form $\varepsilon A(X, T)$, where A is a real function, X and T are the slow coordinate and time, respectively. In the case of non-degenerate quadratic nonlinearity one has $X = \sqrt{\varepsilon}x$, $T = \varepsilon t$. The amplitude of the most unstable mode in correspondence with (A1) obeys the equation

$$\frac{\partial A}{\partial T} = \frac{a}{\varepsilon}A + b_c\frac{\partial^2 A}{\partial X^2} + d\,A^2, \tag{A2}$$

where $d \neq 0$ is a coefficient at the nonlinearity. The equation (A2) describes the evolution of a narrow band of weakly unstable long waves in a some neighborhood of a threshold of instability. By use of scaling transformations this equation is reduced to the form

$$\frac{\partial u}{\partial \tau} = nu - u^2 + \frac{\partial^2 u}{\partial \xi^2}, \quad n = \mp 1. \tag{A3}$$

In the supercritical case ($n = 1$) the equation (A3) represents the well known KPP equation [4]. In the subcritical case ($n = -1$) the equation is reduced to the supercritical equation (A3) by the transformation $u \to u - 1$.

In the subcritical domain ($n = -1$) the solution $u = 0$ is a stable one, while the solution $u = -1$ is unstable. In the supercritical domain the solution $u = 0$ is unstable, while the solution $u = 1$ is stable. Therefore the transition to instability is accompanied by the general shift of the equilibria solutions, describing the homogeneous flows. In this setting the exponential growth of the positive perturbation of the unstable equilibrium solution is suppressed by nonlinearity and as a result the perturbed dynamically unstable equilibrium solution tends to the stable equilibrium solution. If the unstable equilibrium solution is subjected to perturbations with negative average (6.5), then the perturbations grow unboundedly (see Fig. 6).

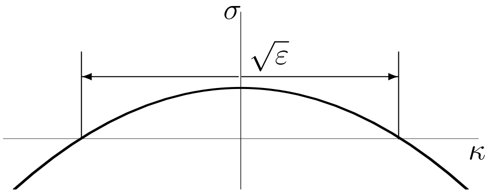

Figure 9: The form of the graph of $\sigma(\kappa, R)$ at a fixed R, $0 < \varepsilon = R - R_c \ll 1$.

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