Solitary waves in fluid-filled elastic tubes: existence, persistence, and the role of axial displacement

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Abstract

We re-examine the problem of solitary wave propagation in a fluid-filled elastic membrane tube using a much simplified procedure. It is shown that there always exist two families of solitary waves with speeds close to those given by the linear dispersion relation, whether the fluid is initially stationary or not, and that it is not asymptotically consistent to neglect the axial displacement even in a long-wave approximation. It is also shown that the solitary wave solution obtained by neglecting higher-order terms persist for the full system of equations in the sense that the full system has solutions of the solitary-wave type and each exact solution is uniformly approximated by the corresponding leading-order solution.

1 Introduction

Nonlinear wave propagation in arteries is a subject that has been much studied over the past three decades. As a very good approximation, the arterial blood flow can be modeled as an incompressible viscous or inviscid fluid flowing in a distensible elastic membrane tube. The linearized governing equations admit dispersive-wave solutions. Thus, when small but finite amplitude traveling waves are considered in the long wavelength limit, we expect to see the famous KdV equation, or a modified KdV equation if viscous effects are taken into account, to emerge as the evolution equation for the wave amplitude. There now exists a good number of papers devoted to the derivation of the KdV or the modified KdV equation for arterial blood flows; see, for instance, Johnson (1970), Hashizume (1985), Cowley (1987, 1988), Yomosa (1987), Demiray (1996), Eraby *et al.* (1992), Demiray (1997), Demiray and Dost (1998), Antar and Demiray (1999) and the references therein. Various approximations have been adopted, some are ad hoc and some can be justified as being asymptotically self-consistent. In particular, we note that Hashizume (1985) and Yomosa (1987) approximated both the governing equation and the constitutive relation for the membrane, Demiray (1996) assumed that the axial displacement in the membrane can be neglected, whereas Demiray (1997) assumed that the axial displacement was so small that the governing equations can be linearized in terms of it. Most of these studies assume that the fluid is inviscid, and can be approximated by a one-dimensional model where the radial velocity and dependence on the radial variable can be neglected.

The present study is motivated by the results of Epstein and Johnston (2001, hereafter referred to as EJ) and those of Fu *et al* (2008, hereafter referred to as FPL), the latter authors examined the problem of localized bulging/necking in an inflated membrane tube with a view to model aneurysm formation. Whereas previous studies invariably used a multiple-scale perturbation procedure to derive the evolution equation, EJ noted that the problem has a variational formulation and, as a result, found two integrals (conservation laws) for the governing equations. They showed that solitary waves of finite amplitude can be determined exactly by a simple numerical procedure. Surprisingly, they did not specialize their analysis to the case when the speed in each case is close to the corresponding linear wave speed, thus failing to make a correct connection with the results obtained from a weakly nonlinear analysis. In fact, they concluded incorrectly, or at least give the reader the impression, that a solitary wave cannot propagate in a fluid-filled elastic membrane tube if the fluid is stationary prior to wave propagation. One of the aims of the present study is to show that when EJ's formulation is specialized to the case when the speed is close to a linear wave speed, it does yield a KdV equation, hence guaranteeing the existence of a solitary wave. To this end, we use the formulation of FPL which can be viewed as an improved variation of EJ's original formulation. By deriving the KdV equation explicitly, first for the case when axial displacement is fully taken into account and then for the case when it is neglected, we shall show that it is asymptotically inconsistent to neglect the axial displacement, even in the long-wave approximation. This has previously been pointed out by EJ, but their comparison was not made for the long-wave case and the reader was still left to wonder whether the axial displacement might be neglected in the long-wave limit. Our above-mentioned result serves to resolve this uncertainty.

The rest of this paper is organized into five sections as follows. In the next section, we quote the governing equations and the associated integrals, and rewrite them in a more simplified form. The linear dispersion relation is noted down for later reference. We then apply, in Section 3, the procedure of FPL to derive a single amplitude equation of the form

$$\left(\frac{dw}{d\bar{Z}}\right)^2 = \omega(c,\lambda)w^2 + \gamma(c,\lambda)w^3 + O(w^4) \tag{1.1}$$

for the radial displacement w, where $\bar{Z} = Z - ct$, Z is the axial coordinate, c is the wave speed, λ is a measure of the pre-stress, and $\omega(c,\lambda)$ and $\gamma(c,\lambda)$ have a explicit expression for a general strain-energy function. For each fixed λ, $\omega(c,\lambda) = 0$ gives the dispersion relation; whereas when $c = 0$ and the fluid is stationary, $\omega(0,\lambda) = 0$ gives the bifurcation condition for the onset of localized bulging/necking, which connects with the analysis of FPL. In a small neighborhood of a linear speed, c_1 say, the full amplitude equation can then be approximated by

$$\left(\frac{dw}{d\bar{Z}}\right)^2 = (c - c_1)\frac{\partial}{\partial c}\omega(c_1, \lambda)w^2 + \gamma(c_1, \lambda)w^3, \tag{1.2}$$

the derivative of which is then the KdV equation specialized to a traveling wave.

In Section 4, we compare (1.1) with its counterpart when the axial displacement is neglected. In the final section, we establish the persistence by proving that the solitary wave solution given by (1.2) is indeed a uniformly valid approximation of the full solution governed by (1.1).

2 Governing equations and the dispersion relation

We consider propagation of nonlinear traveling waves in an infinite fluid-filled membrane tube of averaged radius R and thickness H in its unstressed configuration, assuming that the tube is made of an incompressible hyperelastic material and always maintains its axisymmetry. We assume that prior to wave propagation, the tube is already subjected to a finite deformation with principal stretches $\lambda_{1\infty}$ and $\lambda_{2\infty}$ in the axial and circumferential directions, respectively, and the fluid has constant speed $\hat{v}_{f\infty}$ and exerts a constant pressure Hp_∞ on the tube wall (where H is inserted to simplify the notation later). We use Z to measure distance in the axial direction in the unstressed configuration, and $w(Z,t)$ and $u(Z,t)$ to denote the incremental displacement in the axial and radial directions, respectively. Thus, the position vector of a material particle in the tube in the current configuration has the form

$$\boldsymbol{r} = r(Z)\boldsymbol{e}_r + z(Z)\boldsymbol{e}_z, \quad \text{with} \quad r = \lambda_{2\infty}R + w, \qquad z = \lambda_{1\infty}Z + u, \tag{2.1}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ are unit vectors in the radial and axial directions, respectively.

Since the deformation is axially symmetric, the principal directions of stretch coincide with the meridians (1-direction), the lines of latitude (2-direction), and the normal to the deformed surface. Thus, the principal stretches are given by

$$\lambda_1 = \sqrt{z'^2 + r'^2}, \qquad \lambda_2 = \frac{r}{R}, \qquad \lambda_3 = 1/(\lambda_1\lambda_2), \tag{2.2}$$

where a prime denotes differentiation with respect to Z. The principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$ in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i \hat{W}_i - \hat{p}, \qquad i = 1, 2, 3 \text{ (no summation)}, \tag{2.3}$$

where $\hat{W} = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function, $\hat{W}_i = \partial\hat{W}/\partial\lambda_i$, and $\hat{p}$ is the pressure associated with the constraint of incompressibility; see, for instance, Ogden (1997). Utilizing the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ and the membrane assumption $\sigma_3 = 0$ we find

$$\sigma_i = \lambda_i W_i, \qquad i = 1, 2 \text{ (no summation)}, \tag{2.4}$$

where $W(\lambda_1,\lambda_2)=\hat{W}(\lambda_1,\lambda_2,\lambda_1^{-1}\lambda_2^{-1})$ and $W_1=\partial W/\partial\lambda_1$ etc. Without loss of generality, we from now on take $R=1$.

The governing equations for u and w can be obtained from exact field equations of general nonlinear shell theory, see, for instance, Budiansky (1968), but EJ gave a very readable self-contained derivation. We quote their results and rewrite them in the form

$$\left[\sigma_1\frac{z'}{\lambda_1^2}\right]'-p\lambda_2 r'=\rho\ddot{u}, \tag{2.5}$$

$$\left[\sigma_1\frac{r'}{\lambda_1^2}\right]'-\frac{\sigma_2}{r}+p\lambda_2 z'=\rho\ddot{w}, \tag{2.6}$$

where p is the fluid pressure divided by the wall thickness H and a superimposed dot denote differentiation with respect time t. With the fluid assumed to be inviscid and the fluid flow to be one-dimensional, EJ showed that for traveling waves with speed $\hat{c}$ the fluid equations can be integrated exactly, leading to the following simple expression for the scaled pressure:

$$p=p_\infty+v_f(1-\lambda_{2\infty}^4/\lambda_2^4),\quad v_f=\frac{\rho_f}{2H}(\lambda_{1\infty}\hat{c}-\hat{v}_{f\infty})^2, \tag{2.7}$$

where ρ_f is the density of the fluid. In view of the fact that the cross-section area is $A=\pi\lambda_2^2$, this relation is simply a quadratic pressure-area law.

Linearizing the governing equations (2.5) and (2.6) in the neighborhood of

$$\lambda_1=\lambda_{1\infty},\quad \lambda_2=\lambda_{2\infty},\quad u=w=0,$$

and then looking for a traveling wave solution with wave number $\hat{k}$ and speed $\hat{c}$, we obtain the linear dispersion relation

$$\begin{aligned}\left(k^2m^2+2m\right)c^4-4mv_{f\infty}c^3-\left(m\alpha_0k^2+m\gamma_1k^2-2mv_{f\infty}^2-m\beta_0+m\beta_1+2\gamma_1\right)c^2\\ +4v_{f\infty}\gamma_1c-(\alpha_1-\beta_0)^2-2v_{f\infty}^2\gamma_1+k^2\alpha_0\gamma_1-\beta_0\gamma_1+\beta_1\gamma_1=0,\end{aligned} \tag{2.8}$$

where

$$c=\frac{\lambda_{1\infty}}{c_0}\hat{c},\quad c_0=\sqrt{\frac{\mu h}{\rho_f r_0}},\quad m=\frac{\rho h}{\rho_f r_0},\quad v_{f\infty}=\frac{\hat{v}_{f\infty}}{c_0},\quad k=\frac{\lambda_{2\infty}}{\lambda_{1\infty}}\hat{k}, \tag{2.9}$$

$$\begin{gathered}\mu\alpha_0=\lambda_{1\infty}W_1^{(\infty)},\quad \mu\alpha_1=\lambda_{1\infty}\lambda_{2\infty}W_{12}^{(\infty)},\quad \mu\beta_0=\lambda_{2\infty}W_2^{(\infty)},\\ \mu\beta_1=\lambda_{2\infty}^2W_{22}^{(\infty)},\quad \mu\gamma_1=\lambda_{1\infty}^2W_{11}^{(\infty)},\quad \mu\alpha_2=\frac{1}{2}\lambda_{1\infty}\lambda_{2\infty}^2W_{122}^{(\infty)},\\ \mu\beta_2=\frac{1}{2}\lambda_{2\infty}^3W_{222}^{(\infty)},\quad \mu\gamma_2=\frac{1}{2}\lambda_{1\infty}^2\lambda_{2\infty}W_{112}^{(\infty)},\quad \mu\gamma_3=\frac{1}{2}\lambda_{1\infty}^3W_{111}^{(\infty)}.\end{gathered} \tag{2.10}$$

In the above expressions, $r_0(=\lambda_{2\infty}R=\lambda_{2\infty})$ and h are respectively the radius and thickness of the tube prior to wave propagation, μ is a typical material modulus, and the superscripts (∞) on the W's signify evaluation at $\lambda_1=\lambda_{1\infty}$, $\lambda_2=\lambda_{2\infty}$. In defining the constants in (2.9) and (2.10) we have followed the scheme of Demiray and Dost (1998).

When the fluid is stationary prior to wave propagation, corresponding to $v_{f\infty} = 0$, the dispersion relation (2.8) reduces to

$$\left(k^2m^2 + 2m\right)c^4 - \left(m\alpha_0 k^2 + m\gamma_1 k^2 - m\beta_0 + m\beta_1 + 2\gamma_1\right)c^2$$

$$-(\alpha_1 - \beta_0)^2 + k^2\alpha_0\gamma_1 - \beta_0\gamma_1 + \beta_1\gamma_1 = 0. \tag{2.11}$$

For $k \ll 1$, we look for a series solution of the form

$$c = g(1 - \sigma k^2) + O(k^4), \tag{2.12}$$

where g and σ are constants. On substituting (2.12) into (2.11) and equating the coefficients of k^0 and k^2 to zero, we find that g satisfies

$$2mg^4 + (m\beta_0 - m\beta_1 - 2\gamma_1)\,g^2 + (\beta_1 - \beta_0)\gamma_1 - (\alpha_1 - \beta_0)^2 = 0, \tag{2.13}$$

and σ is given by

$$\sigma = \frac{m^2g^4 - (m\alpha_0 + m\gamma_1)g^2 + \alpha_0\gamma_1}{2g^2\left(4mg^2 + m\beta_0 - m\beta_1 - 2\gamma_1\right)}. \tag{2.14}$$

Demiray and Dost (1998) derived a dispersion relation by treating the fluid flow as being two-dimensional and assuming that the flow is stationary prior to wave propagation. As expected, our leading order result (2.13) agrees with their result, but their expression for σ takes the slightly different form

$$\sigma = \frac{(m^2 + m/4)g^4 - (m\alpha_0 + \gamma_1/4 + m\gamma_1)g^2 + \alpha_0\gamma_1}{2g^2\left(4mg^2 + m\beta_0 - m\beta_1 - 2\gamma_1\right)}. \tag{2.15}$$

The extra terms in (2.15) are from dependence of fluid flow on the radial variable and can be seen to be negligible in the limit $m \gg 1$.

3 Solitary-wave solutions

For a traveling-wave solution in which the dependence on Z and t is through $Z - ct$, EJ showed that (2.5) and (2.6) have two integrals (conservation laws). After some manipulation, we find that their original integrals can be rewritten in the simpler form

$$W(\lambda_1, \lambda_2) - \lambda_1 W_1 + \frac{1}{2}\rho c^2 \lambda_1^2 = C_1, \tag{3.1}$$

$$(W_1/\lambda_1 - \rho c^2)z' - \frac{1}{2}\lambda_2^2\left\{p_\infty + v_f(1 + \lambda_{2\infty}^4/\lambda_2^4)\right\} = C_2, \tag{3.2}$$

where the two constants C_1 and C_2 each take the value of the corresponding left hand side evaluated at $\pm\infty$. Here and hereafter, dependent variables are all functions of $Z - ct$, a prime now denotes differentiation with respect to $Z - ct$, and we shall use Z to denote $Z - ct$ to avoid introducing extra notation.

The above conservation laws with $c = 0$ are well-known in the finite elasticity community: the conservation law (3.2) can be obtained by straightforward integration of (2.5); the other conservation law (3.1) was first noted by Pipkin (1968).

We observe that the two equations (3.1) and (3.2) may be rewritten as

$$f(\lambda_1, \lambda_2) = 0, \quad z' = g(\lambda_1, \lambda_2), \tag{3.3}$$

where

$$f = W(\lambda_1, \lambda_2) - \lambda_1 W_1 + \frac{1}{2}\rho c^2 \lambda_1^2 - C_1,$$

$$g = (W_1/\lambda_1 - \rho c^2)^{-1} \left\{ \frac{1}{2}\lambda_2^2 \left\{ p_\infty + v_f(1 + \lambda_{2\infty}^4/\lambda_2^4) \right\} + C_2 \right\}.$$

Equations (3.3) are of the same form as those studied by FPL. We may thus follow these authors' methodology and derive the amplitude equations as follows. Firstly, we write

$$\lambda_2 = \lambda_{2\infty} + w, \tag{3.4}$$

and assume $|w|$ to be small. Equation $(3.3)_1$ then defines λ_1 as a function of w implicitly. This function can be expanded as

$$\lambda_1 = 1 + d_1 w + \frac{1}{2} d_2 w^2 + O(w^3), \tag{3.5}$$

where the coefficients d_1, d_2 etc can be obtained by substituting (3.4) and (3.5) into the left hand side of $(3.3)_1$, expanding in terms of w, and then equating the coefficients of w^0, w, w^2, etc to zero. This can easily be carried out using a symbolic manipulation package such as *Mathematica*. Next, we substitute (3.4) and (3.5) into $(3.3)_2$ and expand in terms of w to obtain

$$z' = 1 + g_1 w + \frac{1}{2} g_2 w^2 + O(w^3), \tag{3.6}$$

where g_1 and g_2, are constants with known expressions. Finally, on substituting (3.4)–(3.6) into $(2.2)_1$ and again expanding in terms of w, we arrive at the amplitude equation

$$(w')^2 = \omega(c, \lambda) w^2 + \gamma(c, \lambda) w^3 + O(w^4), \tag{3.7}$$

where

$$\omega(c, \lambda) = \frac{\left(-(\alpha_1 - \beta_0)^2 + \beta_0 \left(mc^2 - \gamma_1\right) + \left(2\left(c - v_{\mathrm{f}\infty}\right)^2 - \beta_1\right)\left(mc^2 - \gamma_1\right)\right)\lambda_{1\infty}^2}{\left(mc^2 - \alpha_0\right)\left(mc^2 - \gamma_1\right)\lambda_{2\infty}^2},$$

$$\frac{3\left(mc^2 - \alpha_0\right)^2 \left(\gamma_1 - mc^2\right)^3 \lambda_{2\infty}^3}{\lambda_{1\infty}^2} \cdot \gamma(c, \lambda) = 6m^4 v_{\mathrm{f}\infty}^2 c^8 + 2m^4 \beta_2 c^8 + 3m^3 \beta_0^2 c^6$$
$$+6m^3 \tilde{v}_{f\infty}\beta_0 c^6 - 6m^3 \alpha_2 \beta_0 c^6 - 18m^3 \tilde{v}_{f\infty}\gamma_1 c^6 - 6m^3 \beta_2 \gamma_1 c^6 + 18m^2 \tilde{v}_{f\infty}\gamma_1^2 c^4 + 6m^2 \beta_2 \gamma_1^2 c^4$$
$$-3m^2 \beta_0^2 \gamma_1 c^4 - 6m^2 \tilde{v}_{f\infty}\beta_0 \gamma_1 c^4 + 12m^2 \alpha_2 \beta_0 \gamma_1 c^4 - 3m^2 \beta_0 \beta_1 \gamma_1 c^4 + 6m^2 \beta_0^2 \gamma_2 c^4 - 6m \tilde{v}_{f\infty}\gamma_1^3 c^2$$
$$-2m\beta_2 \gamma_1^3 c^2 - 3m\beta_0^2 \gamma_1^2 c^2 - 6m\tilde{v}_{f\infty}^2 \beta_0 \gamma_1^2 c^2 - 6m\alpha_2 \beta_0 \gamma_1^2 c^2 + 6m\beta_0 \beta_1 \gamma_1^2 c^2 - 3m\beta_0^3 \gamma_1 c^2 - 6m\beta_0^2 \gamma_1 \gamma_2 c^2$$

$$
\begin{gathered}
-2m\beta_0^3\gamma_3c^2 + m\alpha_1^3\left(3mc^2 - 3\gamma_1 + 2\gamma_3\right)c^2 + 3\beta_0^2\gamma_1^3 + 6\tilde{v}_{f\infty}\beta_0\gamma_1^3 - 3\beta_0\beta_1\gamma_1^3 + 3\beta_0^3\gamma_1^2 \\
-3\alpha_1\left(-2m\alpha_2\left(\gamma_1 - mc^2\right)^2c^2 + m\left(4\tilde{v}_{f\infty} - \beta_1\right)\left(\gamma_1 - mc^2\right)^2c^2\right. \\
\left.+2m\beta_0\left(mc^2 - \gamma_1\right)\left(mc^2 - \gamma_1 + 2\gamma_2\right)c^2 - \beta_0^2\left(m^2c^4 + m\gamma_1c^2 + 2m\gamma_3c^2 - 2\gamma_1^2\right)\right) \\
+\alpha_1^2\left(6m\left(mc^2 - \gamma_1\right)\gamma_2c^2 + 3\beta_0\left(\gamma_1mc^2 - 2m\left(mc^2 + \gamma_3\right)c^2 + \gamma_1^2\right)\right) \\
+\alpha_0\left(\left(-3mc^2 + 3\gamma_1 - 2\gamma_3\right)\alpha_1^3 + \left(6\left(\gamma_1 - mc^2\right)\gamma_2 + \beta_0\left(9mc^2 - 9\gamma_1 + 6\gamma_3\right)\right)\alpha_1^2\right. \\
+3\left(\left(-3mc^2 + 3\gamma_1 - 2\gamma_3\right)\beta_0^2 + 2\left(mc^2 - \gamma_1\right)\left(mc^2 - \gamma_1 + 2\gamma_2\right)\beta_0 - 2\alpha_2\left(\gamma_1 - mc^2\right)^2\right. \\
\left.+\left(4\tilde{v}_{f\infty} - \beta_1\right)\left(\gamma_1 - mc^2\right)^2\right)\alpha_1 - 2\left(3\tilde{v}_{f\infty} + \beta_2\right)\left(mc^2 - \gamma_1\right)^3 \\
\left.-3\beta_0\left(4\tilde{v}_{f\infty} - 2\alpha_2 - \beta_1\right)\left(\gamma_1 - mc^2\right)^2 - 6\beta_0^2\left(mc^2 - \gamma_1\right)\left(mc^2 - \gamma_1 + \gamma_2\right) + \beta_0^3\left(3mc^2 - 3\gamma_1 + 2\gamma_3\right)\right).
\end{gathered}
$$

In the last expression, $\tilde{v}_{f\infty} = (v_{f\infty} - c)^2$.

In the limit $c \to 0, v_f \to 0$, (3.7) reduces to the amplitude equation given by FPL. For each fixed choice of $\lambda_{1\infty}$ and $\lambda_{2\infty}$, the traveling wave problem can be viewed as a bifurcation problem with the speed c acting as the bifurcation parameter. The bifurcation condition is given by

$$\omega(c, \lambda_{1\infty}, \lambda_{2\infty}) = 0, \tag{3.8}$$

which, as expected, is equivalent to the dispersion relation (2.11) for $k = 0$.

Denote by c_1 a solution of (3.8), write $\epsilon = |c - c_1|$, and assume ϵ to be small. Equation (3.7) may be replaced by

$$(w')^2 = \omega_c(c_1, \lambda_{1\infty}, \lambda_{2\infty})(c - c_1)w^2 + \gamma(c_1, \lambda_{1\infty}, \lambda_{2\infty})w^3 + O(\epsilon^2 w^2, w^4), \tag{3.9}$$

where ω_c denotes $\partial\omega/\partial c$. Requiring the first three terms in (3.9) to have the same order, we deduce that w must necessarily be of order ϵ and its variation takes place on a lengthscale of order $\epsilon^{-1/2}$. Thus, in terms of the new variables ξ and a_0 defined by

$$w = -\frac{2\epsilon|\omega_c|}{3\gamma}a_0(\xi), \qquad \xi = \sqrt{\epsilon\,|\omega_c|}Z,$$

equation (3.9) takes the form

$$(a_0')^2 = \mathrm{sgn}[(c - c_1)\omega_c]\, a_0^2 - \frac{2}{3}a_0^3 + O(\epsilon), \tag{3.10}$$

where we have suppressed the dependence of ω_c and γ on $c_1, \lambda_{1\infty}$ and $\lambda_{2\infty}$. It can easily be shown that provided $(c - c_1)\omega_c > 0$, this equation, with the $O(\epsilon)$ terms neglected, has a solitary-wave solution given by $a_0 = b_0$, where

$$b_0 = \frac{3}{2}[\cosh\frac{1}{2}\xi]^{-2}. \tag{3.11}$$

We observe that when $\gamma < 0$ the solitary wave is a wave of elevation, whereas when $\gamma > 0$ the solitary wave is a wave of depression. The sign of ω_c determines whether the solitary wave is supersonic (i.e. $c > c_1$) or subsonic (i.e. $c < c_1$).

We would have liked to compare the differentiated form of our equation (3.9) with Demiray and Dost's (1998) equation (46), but it seems that their series expansion (19) is not self-consistent: $\gamma_1(\partial w/\partial z)^2 + (\gamma_1/2)(\partial u/\partial z)^2$ should be added to the first expression and $\alpha_1(\partial w/\partial z)^2 + (\alpha_1/2)(\partial u/\partial z)^2$ should be added to the second expression in their equation (19). We thus decide not to make any comparisons.

4 The role of axial displacement

A very simple model that has been adopted in some previous studies is one in which u and w'^2 are viewed to be negligible in a small-amplitude and long-wave approximation; see, for instance, Demiray (1996) and Epstein and Johnston (1999). In this case, we have

$$\lambda_1 \equiv \lambda_{1\infty}, \qquad \lambda_2 = \lambda_{2\infty} + w. \tag{4.1}$$

The first term in (2.6) is approximated by

$$\left[\sigma_1 \frac{r'}{\lambda_1^2}\right]' = \frac{1}{\lambda_{1\infty}^2}[\sigma_1 w']' = \frac{1}{\lambda_{1\infty}^2}\left(\sigma_1 w'' + \frac{\partial \sigma_1}{\partial w} w'^2\right) \approx \frac{\sigma_1}{\lambda_{1\infty}^2} w''.$$

Equation (2.6) is then approximated by

$$\left(\frac{\sigma_1}{\lambda_{1\infty}^2} - \rho\hat{c}^2\right) w'' - \frac{\sigma_2}{\lambda_2} + p\lambda_2\lambda_{1\infty} = 0. \tag{4.2}$$

Linearizing this equation in terms of w and taking $\hat{v}_\infty = 0$, we obtain

$$(\alpha_0 - mc^2)w'' + \lambda_{1\infty}^2\lambda_{2\infty}^{-2}(2c^2 + \beta_0 - \beta_1)w = 0, \tag{4.3}$$

where the various constants are given by (2.9) and (2.10). Thus, taking w to be proportional to $\mathrm{e}^{\mathrm{i}\hat{k}(Z-\hat{c}t)}$, we obtain the dispersion relation

$$k^2(mc^2 - \alpha_0) + 2c^2 + \beta_0 - \beta_1 = 0, \tag{4.4}$$

where k is given by $(2.9)_5$. The only way to justify this dispersion relation is to take the limit $m \gg 1, k \ll 1$ with $mk^2 = O(1)$ in (2.11). In this limit, (2.11) can be replaced by

$$\left(k^2m^2 + 2m\right)c^4 - \left(m\alpha_0k^2 + m\gamma_1k^2 - m\beta_0 + m\beta_1 + 2\gamma_1\right)c^2 = 0$$

or equivalently by

$$k^2(mc^2 - \alpha_0) + 2c^2 + \gamma_1k^2 - \beta_0 + \beta_1 = 0, \tag{4.5}$$

which agrees with (4.4) if the term $\gamma_1 k^2$, which is small, is dropped from the last equation (then the term $k^2\alpha_0$ should also be dropped from both equations for consistencey). However, when $mk^2 = O(1)$, the correction term σk^2 in (2.12) becomes $O(1)$ and the expansion (2.12) breaks down. Thus, we conclude that it is not asymptotically consistent to ignore the axial displacement.

The nonlinear equation in this approximation is given by

$$w'' = \frac{\lambda_{1\infty}^2 \left(2c^2 + \beta_0 - \beta_1\right)}{\lambda_{2\infty}^2 \left(mc^2 - \alpha_0\right)} w$$

$$- \frac{\lambda_{1\infty}^2 \left(m\left(3c^2 + \beta_2\right)c^2 + \alpha_1\left(-2c^2 - \beta_0 + \beta_1\right) - \alpha_0\left(3c^2 + \beta_2\right)\right)}{\lambda_{2\infty}^3 \left(mc^2 - \alpha_0\right)^2} w^2 + O(w^3). \qquad (4.6)$$

Denoting the linear wave speed $\sqrt{(\beta_1 - \beta_0)/2}$ by $\hat{c}_0$, expanding the right hand side of (4.6) in a small neighborhood of $c = \hat{c}_0$, and then neglecting terms of order $(c - \hat{c}_0)^2 w$ or w^3, we obtain

$$w'' = -\frac{4\sqrt{2(\beta_1 - \beta_0)}\lambda_{1\infty}^2}{\lambda_{2\infty}^2 \left(2\alpha_0 + m\left(\beta_0 - \beta_1\right)\right)}\left(c - \hat{c}_0\right) w - \frac{\lambda_{2\infty}^2 \left(3\beta_0 - 3\beta_1 - 2\beta_2\right)}{\lambda_{2\infty}^3 \left(2\alpha_0 + m\left(\beta_0 - \beta_1\right)\right)} w^2. \qquad (4.7)$$

This reproduces Demiray's (1996) equation (34) which was derived using a multiple-scale expansion.

5 Persistence of the solitary wave solutions

From now on we assume that $(c - c_1)\omega_c > 0$. On differentiating (3.10) with respect to ξ, we obtain

$$\mathcal{M}(a_0) \equiv a_0'' - a_0 + a_0^2 + p(\varepsilon, a_0) = 0, \qquad (5.1)$$

where $p(\varepsilon, a_0) = O(\varepsilon)$. Equation (5.1) is obviously reversible, that is, it is invariant under the action of the group

$$a_0'' \to a_0'', \quad a_0' \to -a_0', \quad a_0 \to a_0.$$

This reversibility means that we may restrict ourselves to the case when a_0 is an even function. We have the following result:

Lemma (Iooss & Kirchgässner 1992): For a small enough ε_0 and $\varepsilon \in (0, \varepsilon_0]$ there exists a family of solitary wave solutions $\mathbf{a}$ satisfying equation (5.1). Moreover,

$$|\mathbf{a} - \mathbf{b}| \leq C\varepsilon \exp(-\lambda|\xi|),$$

where C, λ are constants, and $C > 0, 0 < \lambda < 1$.
Proof. We first define the Banach functional spaces

$$C_{\lambda j}^e = \left\{ f_0 \in C^j(\mathbb{R}) : \sup_\xi |\exp(\lambda|\xi|) f_0^{(m)} < \infty, \quad j = 0, 1, 2; \quad m \leq j; \quad f_0(\xi) = f_0(-\xi) \right\}.$$

It is evident that $b_0 \in C_{1,2}^e$.

Let $\mathcal{M} : C^e_{\lambda,2} \to C^e_{\lambda,0}$, where $\lambda < 1$. From the implicit function theorem the existence and uniqueness of $a_0 \in C^e_{\lambda,2}$, $\varepsilon \in (0, \varepsilon_0]$ and satisfying the equation (5.1) will follow, if the operator

$$\mathcal{L} = \frac{\partial \mathcal{M}}{\partial a_0}(\varepsilon, b_0)\big|_{\varepsilon=0} \colon C^e_{\lambda,2} \to C^e_{\lambda,0}$$

has a bounded inverse. The proof of the first part of the assertion of the lemma is, therefore, reduced to the demonstration of this fact.

It can be easily seen that

$$\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - 1 + 2b_0,$$

and the inverse exists if and only if the equation

$$\mathcal{L} f = g \tag{5.2}$$

has a unique solution for any $g \in C^e_{\lambda,0}$. The solvability of (5.2) follows from the following.

The homogeneous equation $\mathcal{L} v = 0$ has no solutions in $C^e_{\lambda,2}$. In fact, the solution $v_1 = \partial_\xi b_0$ is an odd one, and the linearly independent solution $v_2 = c_1 v_1 + c_2 v_1 \int w_1^{-2}(\xi)\mathrm{d}\xi$ is an even function (if the constants c_1 and c_2 are chosen so that $v_2(0) = 1$ and $v_2'(0) = 0$), but the increasing one. It follows, that the solution of (5.2) is unique if exists.

The required solution of the equation (5.2) is given by the formula

$$v = v_2 \int_y^\infty v_1 g \,\mathrm{d}y + v_1 \int_0^y v_2 g \,\mathrm{d}y. \tag{5.3}$$

Moreover, it follows from (5.3) that $||v||_{C^e_{\lambda,2}} \leq C||g||_{C^e_{\lambda,0}}$, where $C > 0$ is a constant.

The first part of the assertion of the lemma is, therefore, proved. We next write the solution of the equation (5.1) in the form $a_0 = b_0 + \hat{b}_0$. The function $\hat{b}_0$ obeys the equation

$$\mathcal{L}\hat{b}_0 = D\big(\varepsilon, \hat{b}_0, \hat{b}_0', \xi\big),$$

where $D(\cdot) = O(\varepsilon, \hat{b}_0^2, \hat{b}_0'^2)$. Consequently

$$||\hat{b}_0||_{C^e_{\lambda,2}} \leq ||\mathcal{L}^{-1}|| \cdot ||D\big(\varepsilon, \hat{b}_0, \hat{b}_0', \xi\big)||_{C^e_{\lambda,0}},$$

from where the assertion of the lemma concerning the norm follows immediately.

We have proved that the family of solitary wave solutions a_0 exists and is unique for small enough amplitudes. Returning to the unscaled variables, we conclude that the fluid-filled membrane tube supports a unique family of solitary wave solutions bifurcating from the trivial solution at each linear wave speed.

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