Stability of an inflated hyperelastic membrane tube with localized wall thinning

A. T. Il'ichev[1,2] and Y.B. Fu[3,4]

[1]Steklov Mathematical Institute, Gubkina str. 8, 119991 Moscow, Russia

[2] Bauman Moscow Technical University, Baumanskaya str. 5, 105110 Moscow, Russia

[3]Department of Mathematics, Keele University, ST5 5BG, U.K.

[4]Department of Mechanics, Tianjin University, Tianjin 300072, China

Abstract

It is now well-known that when an infinitely long hyperelastic membrane tube free from any imperfections is inflated, a transcritical-type bifurcation may take place that corresponds to the sudden formation of a localized bulge. When the membrane tube is subjected to localized wall-thinning, the bifurcation curve would "unfold" into the turning-point type with the lower branch corresponding to uniform inflation in the absence of imperfections, and the upper branch to the larger amplitude bifurcated states. In this paper stability of bulged configurations corresponding to both branches is investigated with the use of the spectral method. It is shown that under pressure control and with respect to axi-symmetric perturbations, configurations corresponding to the lower branch are stable but those corresponding to the upper branch are unstable. To establish instability, we demonstrate the existence of an unstable eigenvalue (an eigenvalue with a positive real part). This is achieved using a construction of the Evans function that depends only on the spectral parameter. This function is analytic in the right half of the complex plane and has there zeroes coinciding with an unstable eigenvalue of the generalized spectral problem governing spectral stability/instability. We show that due to the fact that the skew-symmetric operator $\mathcal{J}$ involved in the Hamiltonian formulation of the basic equations is onto, the zeroes of the Evans function can only be located on the real axis of the complex plane. We also establish a connection between the spectral problem governing spectral (linear) stability and the one governing nonlinear (Lyapunov) stability.

1 Introduction

Our present study is closely related to studies of solitary waves in hyperelastic membrane tubes; for a review of the relevant literature we refer to [1]. On the one hand, a static localized bulge can be viewed as a solitary wave that has zero propagation speed, the zero speed being induced by the internal pressure in the membrane tube. On the other hand, solitary waves may play an important role in interrogating the health status of arteries (e .g. presence of an aneurysm) through signal processing [2]. The present study is also part of our recent research effort exploring the postulate that initiation of aneurysms is a bifurcation phenomenon [5]. This postulate is motivated by two main results. Firstly, assuming that the artery is axisymmetric and homogeneous in the sense that the initial value of its wall thickness H is a constant, a localized bulge can form when the internal pressure reaches a certain critical value [3]. Secondly, when imperfections such as localized wall weakening is introduced, the bifurcation pressure may fall to within the physiologically possible range [4].

Identifying an aneurysm with a static localized bulge implies that this configuration must be stable since otherwise it cannot be observed. It was found that in the homogeneous case although internal fluid inertia would reduce the growth rate of the single unstable mode significantly, it alone cannot stabilize the unstable mode completely [6]. The stabilization of the exponential growth of the aneurysm solution takes place in the presence of a non-zero mean flow, but in this case the standing bulging configuration may gain a non-zero speed, and we can speak only about orbital stability or stability in form[8]. This occurs due to translational symmetry of the problem as a whole.

When wall weakening is introduced, the problem in question is no longer invariant under translations, and in this case we may speak about the usual stability of the standing configuration. In this paper we investigate the stability of bulging configurations corresponding to both branches of the bifurcation curve. In the absence of any imperfections, the lower branch would correspond to uniform inflation whose stability has previously been studied by Shield [15], Haughton & Ogden [17], and Chen [16], and the upper branch would correspond to large amplitude bifurcated solutions whose instability has recently been established by Fu and Xie [18]. Our stability analysis is made by construction of the Evans function, depending only on the spectral parameter. The function is analytic in the right half of the complex plane and has there zeroes coinciding with unstable eigenvalues. We demonstrate that the zeroes of the Evans function can be located only on the real axis of the complex plane. Therefore, we need to establish behavior of this function only on the real axis which is technically possible, and based on this behavior make the conclusions not only about spectral instability of the bulging configurations under consideration, but also about its stability. In other words, absence of zeroes of the Evans function on the real axis implies linear stability of the aneurysm solution. Moreover, the correspondence of the spectrum of the related spectral problem in linear stability analysis to that one in the Lyapunov (nonlinear) stability analysis is established. The spectrum η of the linearized problem is related to the spectrum $-\alpha$ of the Hessian of the energy via the relation $\alpha = \rho\eta^2$, where ρ is the density of an elastic material of the tube. Therefore, with the Hessian being a self-adjoint operator, η can have only real and purely imaginary values, the last corresponding to the continuous spectrum. Linear instability is governed by the presence of a discrete spectrum.

The rest of the paper is divided into four sections as follows. After presenting the Hamiltonian form of the governing equations and some constitutive assumptions we discuss in Section 3 the construction of fully nonlinear bulging (aneurysm) solutions. We present the bifurcation diagram, reflecting the appearance of the standing bulging solutions, and also the third order dynamic system to be solved numerically to obtain the fully nonlinear bulging solutions. This is then followed by Section 4 where we discuss properties of the related spectral problems in linear and nonlinear stability approach and their correspondence. We construct the Evans function for both branches of the bifurcation diagram and examine its behavior on the real axis of the right half of the complex plane. According to the existence or absence of its zeroes conclusions about linear instability or stability of the aneurysm solutions in question are made. The paper is concluded in Section 5 with further discussions.

2 Formulation

We consider the inflation of a cylindrical membrane tube that is assumed to be incompressible, isotropic, and hyperelastic. In its undeformed configuration, the tube wall has thickness H that is not necessarily a constant, but the average of its outer and inner radii, hereafter referred to simply as the radius R, is a constant. The tube is assumed to be infinitely long, and end conditions are imposed at infinity. We use cylindrical coordinates, and undeformed and deformed configurations are described by coordinates (R, Θ, Z) and (r, θ, z), respectively.

We assume that the axisymmetry is maintained throughout the entire deformation, and so the deformation has the general form $r = r(Z, t)$, $\theta = \Theta$, $z = z(Z, t)$. The principal directions of the deformation correspond to the lines of latitude, the meridian and the normal to the deformed surface, and the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = (r'^2 + z'^2)^{\frac{1}{2}}, \quad \lambda_3 = \frac{h}{H}, \tag{2.1}$$

where a prime represents differentiation with respect to Z, and h denotes the deformed thickness.

The principal Cauchy stresses σ_1, σ_2, σ_3 in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i \hat{W}_i - p, \qquad i = 1, 2, 3 \quad \text{(no summation)}, \tag{2.2}$$

where $\hat{W} = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function, $\hat{W}_i = \partial \hat{W} / \partial \lambda_i$, and p is the pressure associated with the constraint of incompressibility. Utilizing the incompressibility constraint $\lambda_1 \lambda_2 \lambda_3 = 1$ and the membrane assumption of no stress through the thickness direction $\sigma_3 = 0$, we find

$$\sigma_i = \lambda_i W_i, \qquad i = 1, 2 \tag{2.3}$$

where $W(\lambda_1, \lambda_2) = \hat{W}(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1})$ and $W_1 = \partial W/\partial \lambda_1$ etc. [17].

In our numerical illustrations, we shall assume that the membrane material is described by the Ogden strain-energy function

$$\hat{W} = \sum_{r=1}^{3} \mu_r(\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3)/\alpha_r, \tag{2.4}$$

where

$$\alpha_1 = 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0, \quad \mu_1 = 1.491, \quad \mu_2 = 0.003, \quad \mu_3 = -0.023$$

are material constants given by Ogden [10], and the μ's have been scaled by the ground state shear modulus.

We consider the pressure controlled case when the inner pressure P is prescribed. The total energy of the configuration is $E = K + \Pi$, where K is the kinetic energy given by

$$K = \frac{1}{2}\int_{-L}^{L} \rho(\dot{r}^2 + \dot{z}^2) 2\pi R H \, dZ,$$

and Π is the potential energy which is the sum of the strain energy and the potential energy of pressure:

$$\Pi = \int_{-L}^{L} W(\lambda_1, \lambda_2) 2\pi R H \, dZ - P \int_{-L}^{L} \pi r^2 z' \, dZ.$$

In the above expressions the superimposed dot denotes differentiation with respect to time and L is the length of the tube in the undeformed configuration (which will eventually be taken to be infinite). The Hamiltonian, therefore, has the form

$$E(q_1, q_2, v_1, v_2) = \frac{1}{2}\int_{-\infty}^{\infty} \left\{ \rho R(v_1^2 + v_2^2) + 2R\big(W(\lambda_1, \lambda_2) - W^{(\infty)}\big)H - P\left[\frac{q_1^2}{H}\left(\frac{q_2}{\sqrt{H}}\right)' - r_\infty^2 z_\infty\right]\right\} dZ, \tag{2.5}$$

where $q_1 = \sqrt{H} r$, $q_2 = \sqrt{H} z$, $v_1 = \sqrt{H}\dot{r}$, $v_2 = \sqrt{H}\dot{z}$, $W^{(\infty)}$ is the value of the strain-energy function W at infinity where $\lambda_1 = r_\infty$, $\lambda_2 = z_\infty$. Here the constants are chosen such that the integral in (2.5) is convergent for $(r - r_\infty, z')$ exponentially decaying at infinity.

Employing R as the unit of length, we may put R in (2.5) to unity throughout this paper. If we denote $\mathbf{u} = \{q_1, q_2, v_1, v_2\}^\top$, the equations of motion in the pressure controlled case may be written formally as a Hamiltonian dynamical system

$$\frac{d\mathbf{u}}{dt} = \mathcal{J}\frac{\delta E}{\delta \mathbf{u}}, \quad \text{with} \quad \mathcal{J} = \frac{1}{\rho}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

and $\delta/\delta\mathbf{u}$ denotes variational derivative. It can easily be verified that equation (2.6) is equivalent to the more familiar form [12]

$$\rho\ddot{r} = \frac{1}{H}\left(\frac{HW_2 r'}{\lambda_2}\right)' + \frac{P}{H}rz' - W_1,$$
$$\rho\ddot{z} = \frac{1}{H}\left(\frac{HW_2 z'}{\lambda_2}\right)' - \frac{P}{H}rr'. \tag{2.7}$$

Taking the limit $Z \to \infty$ in the first equation (2.7), we obtain

$$P = \frac{H^{(\infty)}W_1^{(\infty)}}{r_\infty z_\infty}, \tag{2.8}$$

where superscript (∞) denotes evaluation at $Z = \infty$. We shall focus on the situation of an open-end membrane tube with fixed axial stretch z_∞; this models the state of arteries. The azimuthal stretch r_∞ can then be used as the control parameter in our bifurcation analysis, with the associated pressure calculated according to (2.8).

3 Weakly and fully nonlinear bulging solutions

The weakly nonlinear localized bulging solution has an amplitude of order ϵ, where ϵ is a small enough dimensionless quantity. As in [4] we assume that the variable thickness H has the form

$$H = H^{(\infty)}\big(1 + \epsilon^2 h(\xi)\big), \quad \xi = \epsilon Z, \quad h(\pm\infty) \to 0,$$

where $H^{(\infty)}$ is the constant wall thickness at infinity and the function $h(\xi)$ is to be prescribed.

It was shown in [4] that if $r - r_\infty = \epsilon y(\xi)$ for weakly nonlinear solutions (small ϵ), y must satisfy the differential equation

$$\frac{d^2 y}{d\xi^2} = \omega'_{cr} r_1 y + \frac{3}{2}\gamma_{cr} y^2 + \zeta h(\xi), \tag{3.1}$$

where r_1 is defined by $r_\infty = r_{cr} + \epsilon r_1$, $r_\infty = r_{cr}$ is the critical value at which a bulge will initiate without any imperfections), $\omega'_{cr} = d\omega(r_{cr})/dr_{cr}$, $\gamma_{cr} = \gamma(r_{cr})$, and explicit expressions for $\omega(r_\infty)$, $\gamma(r_\infty)$ and ζ in terms of the strain-energy function can be found in [3], [1] and [4], respectively. In [4] several classes of $h(\xi)$ are considered for which (3.1) has closed-form solutions. In particular, if $h(\xi)$ takes the form

$$h(\xi) = \frac{3}{2} d_1 y^2, \tag{3.2}$$

where d_1 is a constant, then (3.1) has an explicit localized solution given by

$$y = -\frac{\omega'_{cr} r_1}{\gamma_{cr} + \zeta d_1}\mathrm{sech}^2\Big(\frac{1}{2}\sqrt{\omega'_{cr} r_1}\,\xi\Big). \tag{3.3}$$

Denoting $h(0)$ by h_0, we have

$$h_0 = \frac{3}{2} d_1 y_1^2(0) = \frac{3d_1(\omega'_{cr} r_1)^2}{2(\gamma_{cr} + \zeta d_1)^2},$$

which can be solved to express d_1, and hence $h(\xi)$ and $y_1(\xi)$, in terms of h_0. We then obtain

$$r_0 - r_\infty = \varepsilon y_1(0) = -\frac{\omega'_{cr} \cdot (r_\infty - r_{cr})}{2\gamma_{cr}} \left[1 \pm \sqrt{1 - \frac{8\gamma_{cr} h_0 \zeta \varepsilon^2}{3\omega'^2_{cr}(r_\infty - r_{cr})^2}} \right]. \tag{3.4}$$

Plotted on the $(r_\infty, r_0 - r_\infty)$-plane, the above expression describes a parabola opening to the left when $\gamma_{cr} h_0 \zeta > 0$. The turning point (i.e. the nose of the parabola), beyond which no localized solutions can exist, corresponds to

$$r_\infty = r_{cr} + 2\sqrt{\frac{2}{3}} \cdot \frac{\varepsilon \sqrt{\gamma_{cr} \zeta h_0}}{\omega'_{cr}}. \tag{3.5}$$

For the Ogden material model and in the open end case with $z_\infty = 1$, we have

$$\zeta = 2.0328, \ r_{cr} = 1.6873, \ \omega'_{cr} = -3.2329, \ \gamma_{cr} = -1.3369.$$

Thus, the expression (3.5) is real only for the wall-thinning case (i.e. $h_0 < 0$), and its right hand side is less than r_{cr} by an amount that is proportional to the square root of the imperfection amplitude $H_\infty - H(0)$. This reflects the square root law for the imperfection sensitivity of this type of elastic localizations [4].

Although the particular choice of the wall thinning profile (3.2) leads to an exact solution that enables us to see explicitly how the bifurcation diagram unfolds from the perfect case, this profile is actually dependent on r_1 and hence on the value of r_∞. In our subsequent calculations, we shall consider the r_∞-independent profile

$$H(Z) = H_\infty(1 - 0.05\,\mathrm{sech}^4 Z). \tag{3.6}$$

The associated bifurcation diagram is obtained as follows. First, the equilibrium solutions can be obtained from (2.7) and may be written in the form (see, e.g., [4])

$$\begin{aligned} \lambda'_1 &= \lambda_2 \sin\phi, \\ \lambda'_2 &= \frac{W_1 - \lambda_2 W_{12}}{W_{22}} \sin\phi - \frac{H' W_2}{W_{22}}, \\ \phi' &= \frac{W_1}{W_2} - \frac{P\lambda_1\lambda_2}{H W_2}, \end{aligned} \tag{3.7}$$

where ϕ is the angle between the meridian and the Z-axis so that $r' = \lambda_2 \sin\phi$, $z' = \lambda_2 \cos\phi$. We also note that $(2.7)_2$ in the static case can be integrated once to give

$$\frac{H W_2 z'}{\lambda_2} - \frac{1}{2} P r^2 = C_1, \tag{3.8}$$

where the integration constant C_1 can be determined by evaluating the left hand side at ∞. The (symmetric) localized bulging solutions can then be determined by integrating the system (3.7) from $Z = 0$ towards ∞ subject to the initial conditions

$$\lambda_1(0) = r_0, \quad \lambda_2(0) = z_0', \quad \phi(0) = 0,$$

where r_0 is to be guessed in our shooting procedure, and the constant z_0' is related to r_0 by

$$f(r_0, z_0') \equiv H(0)W_2(r_0, z_0') - H^{(\infty)}W^{(\infty)} - \frac{1}{2}P(r_0^2 - r_\infty^2) = 0, \tag{3.9}$$

obtained from the integral (3.8). The solvability of (3.9) for z_0' is guaranteed by the fact that $\partial f/\partial z_0' = H(0)W_{22}(r_0, z_0') > 0$, whose satisfaction is verified numerically (this can also be made as a constitutive assumption, see, e.g., [17]). Thus, for each specified r_∞ and a guess for r_0, we solve (3.9) numerically to find the corresponding z_0'. We iterate on r_0 so that the decay condition [4]

$$r'(L) + \sqrt{\omega(r_\infty)}(r(L) - r_\infty) = 0 \tag{3.10}$$

is satisfied for a sufficiently large positive number L. In Figure 1, we have shown the dependence of $r(0) - r_\infty$ on r_∞ corresponding to the wall thickness profile (3.6). It is seen that it has a similar form to the one given by the analytical expression (3.5). We also observe that the upper branch very quickly approaches its counterpart in the absence of imperfections. This means that large amplitude bulged solutions do not feel the presence of the initial wall-thinning, the main effect of the latter is to reduce the bifurcation value of r_∞.

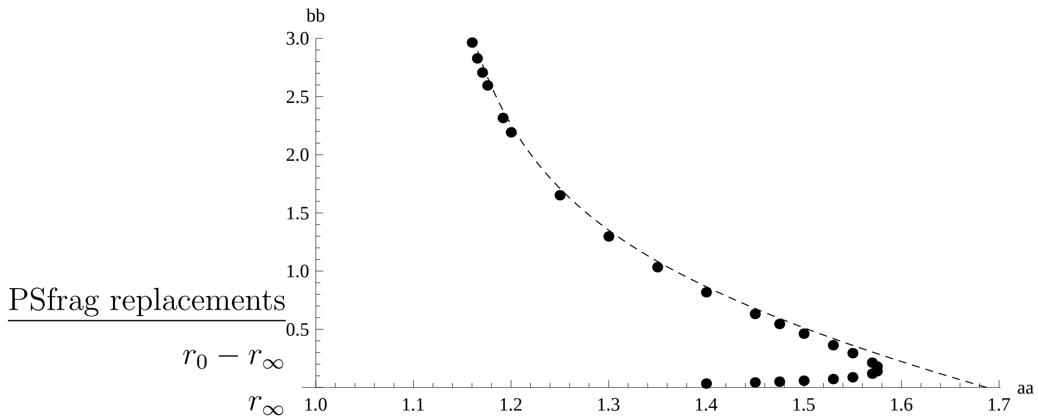

Figure 1: Dependence of $r(0) - r_\infty$ on r_∞ when the wall thickness is given by (3.6) (dark dots). The corresponding result when the wall thickness is uniform is given by the dashed line. The main effect of localized wall thinning is to reduce the bifurcation value of r_∞ from 1.687 to 1.576 (a 6.6% reduction); it has a negligible effect on large amplitude bulged solutions.

4 Stability analysis

First we consider linear stability of the bulging solutions $(\overline{q}_1(Z), \overline{q}_2(Z))$ determined in the previous section subject to axisymmetric perturbations. We thus write

$$q_1(Z,t) = \overline{q}_1(Z) + \Psi(Z)\,\mathrm{e}^{\eta t}, \quad q_2(Z,t) = \overline{q}_2(Z) + \Phi(Z)\,\mathrm{e}^{\eta t} \tag{4.1}$$

where the mode functions $\Psi(Z)$, $\Phi(Z)$ and the growth rate η are to be determined. On substituting expressions (4.1) into Eq. (2.7), and linearizing, we find

$$\mathcal{L}\mathbf{B} = \rho\eta^2\mathbf{B}, \quad \text{with} \quad \mathbf{B} = \{\Psi, \Phi\}^T, \tag{4.2}$$

where the differential operator $\mathcal{L}$ is not written out for brevity.

As the aneurysm solution is a stationary solution, we have

$$DE(\boldsymbol{\phi}) \equiv \frac{\delta}{\delta \mathbf{u}} E(\boldsymbol{\phi}) = 0$$

where $\boldsymbol{\phi} = \{\overline{q_1}, \overline{q_2}, 0, 0\}^T$ denotes the aneurysm solution. The Hessian of E evaluated at the aneurysm solution is

$$\mathcal{H} = D^2E(\boldsymbol{\phi}) = \begin{pmatrix} -\mathcal{L} & 0 \\ 0 & \rho I \end{pmatrix}, \tag{4.3}$$

where I is the 2×2 identity matrix. It is shown in [8] that the operator $\mathcal{H}$ constructed in this way is self-adjoint. It then follows that $\mathcal{L}$ must necessarily be a self-adjoint operator and it can have only real spectrum. Thus, the eigenvalues (discrete spectrum) $\rho\eta^2$ in (4.2) can only lie only on the real axis, that is, η can only be real or pure imaginary.

In the imperfect case, when H is not a constant, the problem no longer has translational invariance, and therefore zero in not an eigenvalue of $\mathcal{L}$. If $\mathcal{L}$ has only strictly negative spectrum, then it follows immediately that the vector $\{\overline{q}_1, \overline{q}_2, 0, 0\}$ is a minimum of the Hamiltonian and, hence the aneurysm solution is nonlinearly stable. Nevertheless, it evidently follows from (4.2), that the continuous spectrum of $\mathcal{L}$ which is determined by the system at $\pm\infty$, is the interval $(-\infty, 0)$. It is not separated from zero, and the nonlinear stability/instability cannot be deduced in our case. Therefore, we treat the eigenvalue problem (4.2) related to linear stability, when the unstable eigenfunctions are determined by real values of η (positive η^2).

It can be seen that Eq.(4.2) is a system of two coupled linear non-autonomous second order differential equations, and the dependence on η is entirely through η^2. We denote $\alpha = \rho\eta^2$. Eq. (4.2) then can be written in the form

$$\mathbf{y}' = \mathcal{M}\mathbf{y}, \tag{4.4}$$

where $\mathbf{y} = (\Psi, \Psi', \Phi, \Phi')^T$ and $\mathcal{M}$ is a 4×4 matrix whose components are not written here for brevity. Eq. (4.4) subject to the decay conditions $\mathbf{y} \to \mathbf{0}$ as $Z \to \pm\infty$ is an eigenvalue problem for α. The aneurysm solution is said to be unstable if this eigenvalue problem has a positive eigenvalue.

We denote by $\mathcal{M}_\infty$ the limit of $\mathcal{M}$ as $Z \to \pm\infty$. Since in the same limit the localized bulging solution has the asymptotic behaviour $\overline{r}(Z) \to r_\infty$, $\overline{z}'(Z) \to z_\infty$, we find, using (2.8), that the matrix $\mathcal{M}_\infty$ takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{r_\infty z_\infty(\alpha + W_{11}^{(\infty)}) - z_\infty}{r_\infty W_2^{(\infty)}} & 0 & 0 & \frac{z_\infty W_{12}^{(\infty)} - W_1^{(\infty)}}{W_2^{(\infty)}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{W_1^{(\infty)} - z_\infty W_{12}^{(\infty)}}{z_\infty W_{22}^{(\infty)}} & \frac{\alpha}{W_{22}^{(\infty)}} & 0 \end{pmatrix}. \tag{4.5}$$

Eq. (4.4) asymptotes to a constant coefficient problem with exponential solutions $\exp(kZ)$, for values of k related to the parameter α by the equation

$$\det\left(\mathcal{M}_\infty - kI\right) = 0,$$

where I is the 4×4 identity matrix. On evaluating the determinant, we obtain

$$\alpha_0\gamma_1\hat{k}^4 + [(\alpha_1 - \beta_0)^2 - (\beta_1 - \beta_0)\gamma_1 - \hat{c}^2(\alpha_0 + \gamma_1)]\hat{k}^2 + (\hat{c}^2 + \beta_1 - \beta_0)\hat{c}^2 = 0, \tag{4.6}$$

where

$$\hat{c}^2 = \alpha r_\infty^2 = \rho\eta^2 r_\infty^2, \quad \hat{k} = \frac{r_\infty}{z_\infty}k,$$

and the material constants α_0, α_1, β_0, β_1, γ_1 are defined in [1].

It can easily be seen that the four eigenvalues of $\mathcal{M}_\infty$ take the form $\pm k_1$, $\pm k_2$. It can be proved by the arguments similar to [6] that $\hat{k}$ can be pure imaginary if and only if $\hat{c}$ is pure imaginary (or equivalently, if $\hat{c}^2$ is real and negative). Therefore, for our construction of unstable eigenfunctions ($\hat{c}^2$ is positive) the roots $\hat{k}_1$ and $\hat{k}_2$ cannot cross the imaginary axis of the complex plane, and the four eigenvalues are symmetric with respect to both the real and imaginary axes. Without loss of generality, we assume that it is the k_1 and k_2 that have a negative real part. The system of equations (4.4) then has two independent solutions, $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ say, that decay as $Z \to \infty$ like $e^{k_1 Z}$ and $e^{k_2 Z}$, respectively, and another two independent solutions, $\boldsymbol{y}_1^-$, $\boldsymbol{y}_2^-$ say, that decay as $Z \to -\infty$ like $e^{-k_1 Z}$ and $e^{-k_2 Z}$, respectively.

The eigenvalue problem (4.4) can be solved in a number of ways. The most straightforward approach is the so-called determinant method, which determines α by solving the equation

$$\det\left(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_1^-, \boldsymbol{y}_2^-\right) = 0, \tag{4.7}$$

where the left hand can be evaluated at any appropriate matching point on the real line. The method suffers from the "stiffness" problem in the sense that one column can get dominated by another column due to different exponential behaviour. A better method is the compound matrix method. One version of this method is used in [7]. In this paper we employ another version of this method which is usually called the Evans function method in the nonlinear waves community.

To solve the eigenvalue problem (4.4) using the Evans function method, we first define the following adjoint of (4.4):

$$\mathbf{x}' = -\mathcal{M}^T\mathbf{x}, \tag{4.8}$$

For each $\mathbf{y}(\alpha, Z)$ and $\mathbf{x}(\alpha, Z)$ that satisfy (4.4), (4.8), respectively, it can easily be verified that

$$\frac{\partial}{\partial Z}\big(\mathbf{x}(\alpha, Z)\cdot\mathbf{y}(\alpha, Z)\big) = 0, \tag{4.9}$$

where " $\cdot$ " denotes the usual scalar product between two vectors.

Denote by $\mathbf{a}_1$, $\mathbf{a}_2$ the right eigenvectors of $\mathcal{M}_\infty$ associated with the eigenvalues k_1 and k_2, and by $\mathbf{b}_1$, $\mathbf{b}_2$ the left eigenvectors of $\mathcal{M}_\infty$ associated with the eigenvalues $-k_1$ and $-k_2$. From the general theory of ordinary differential equations it follows (see [13]) that there exist the solutions of (4.4), (4.8), such that

$$\lim_{Z\to\infty} \mathrm{e}^{-k_i Z}\mathbf{y}_i(\alpha, Z) = \mathbf{a}_k(\lambda), \qquad \lim_{Z\to-\infty} \mathrm{e}^{k_i Z}\mathbf{x}_i(\alpha, Z) = \mathbf{b}_k(\lambda), \quad i = 1, 2. \tag{4.10}$$

A general solution that decays exponentially as $Z \to \infty$ is given by

$$\mathbf{y} = c_1\mathbf{y}_1(\alpha, Z) + c_2\mathbf{y}_2(\alpha, Z) = (\mathbf{y}_1, \mathbf{y}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{4.11}$$

where c_1 and c_2 are constants. It follows from (4.9) and the decay behaviour of $\mathbf{y}$ that

$$\mathbf{0} = \begin{pmatrix} \mathbf{x}_1\cdot\mathbf{y} \\ \mathbf{x}_2\cdot\mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{pmatrix}(\mathbf{y}_1, \mathbf{y}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{4.12}$$

and so for a non-trivial solution we must have

$$\det\begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{pmatrix}(\mathbf{y}_1, \mathbf{y}_2) = 0. \tag{4.13}$$

This condition is equivalent to (4.7), and its direct numerical computation would suffer from the same stiffness problem. In the following, this determinant is evaluated with the aid of the associated exterior systems (or in terms of the compound matrices).

4.1 Exterior systems

Consider the vectors $\mathbf{y}^\wedge(\lambda, \zeta)$, $\mathbf{x}^\wedge(\lambda, \zeta)$ with components defined by

$$y^\wedge_{\beta\wedge\gamma} = y_{\mathbf{1}\beta}y_{\mathbf{2}\gamma} - y_{\mathbf{1}\gamma}y_{\mathbf{2}\beta}, \quad x^\wedge_{\beta\wedge\gamma} = x_{\mathbf{1}\beta}x_{\mathbf{2}\gamma} - x_{\mathbf{1}\gamma}x_{\mathbf{2}\beta}, \quad \beta < \gamma, \tag{4.14}$$

where $\beta, \gamma = 1, 2, 3, 4$, and $y_{\mathbf{k}\beta}$, $x_{\mathbf{k}\beta}$ are the β-th components of the vectors $\mathbf{y}_k$ and $\mathbf{x}_k$, respectively. We use the following correspondence between $\alpha \wedge \beta$ and the numbers:

$$1 \wedge 2 \to 1,\ 1 \wedge 3 \to 2,\ 1 \wedge 4 \to 3,\ 2 \wedge 3 \to 4,\ 2 \wedge 4 \to 5,\ 3 \wedge 4 \to 6.$$

The vectors $\mathbf{y}^{\wedge}(\alpha, Z)$, $\mathbf{x}^{\wedge}(\nu, Z)$ satisfy the linear systems

$$\mathbf{y}^{\wedge\prime} = \mathcal{M}^{\wedge}(\alpha, Z)\mathbf{y}^{\wedge}, \quad \mathbf{x}^{\wedge\prime} = -\left[\mathcal{M}^{\wedge}(\alpha, Z)\right]^T \mathbf{x}^{\wedge}. \tag{4.15}$$

We define the asymptotic matrix

$$\mathcal{M}^{\wedge}_{\infty}(\alpha) = \lim_{Z \to \pm\infty} \mathcal{M}^{\wedge}(\alpha, Z).$$

It is well-known that the six eigenvalues of $\mathcal{M}^{\wedge}_{\infty}(\alpha)$ are given by

$$k_\alpha(\alpha) + k_\beta(\alpha), \quad 1 \le \alpha < \beta \le 4.$$

4.2 Evans function

For η in the right complex half-plane, the matrix $\mathcal{M}_{\infty}(\alpha)$ has two eigenvalues $k_1(\alpha)$ and $k_2(\alpha)$ in the left half-plane (recall that $\alpha = \rho\eta^2$). Thus the matrix $\mathcal{M}^{\wedge}_{\infty}(\alpha)$ has simple (hence analytic) left-most eigenvalue $k^{\wedge}(\alpha) = k_1(\alpha) + k_2(\alpha)$ for η in the right half-plane. By exact analogy with (4.10), there are solutions of (4.15) such that

$$\lim_{Z \to \infty} \mathrm{e}^{-k^{\wedge}(\alpha)Z} \mathbf{y}^{\wedge}(\alpha, Z) = \mathbf{a}^{\wedge}(\alpha),$$
$$\lim_{Z \to -\infty} \mathrm{e}^{k^{\wedge}(\alpha)Z} \mathbf{x}^{\wedge}(\alpha, Z) = \mathbf{b}^{\wedge}(\alpha),$$

where $\mathbf{a}^{\wedge}(\alpha)$ is the left eigenvectors of $\mathcal{M}^{\wedge}_{\infty}(\alpha)$ associated with the eigenvalue $k^{\wedge}(\alpha)$, and $\mathbf{b}^{\wedge}(\alpha)$ is the right eigenvectors of $\mathcal{M}^{\wedge}_{\infty}(\alpha)$ associated with the eigenvalue $-k^{\wedge}(\alpha)$.

We define the Evans function by

$$D(\alpha) = \mathbf{x}^{\wedge} \cdot \mathbf{y}^{\wedge}. \tag{4.16}$$

It is a standard result that $\mathbf{x}^{\wedge} \cdot \mathbf{y}^{\wedge}$ defined above is equal to the determinant on the left hand side of (4.13). Thus, the above construction is simply an alternative way to evaluate (4.13) that avoids any stiffness behaviour. Since the eigenvalue $k^{\wedge}(\eta)$ is simple, the argument of Alexander and Sachs [14] can be used to show that the Evans function, defined by (4.16), is analytic in the entire complex right half-plane of η and it is real for real η.

For $\mathrm{Re}\,\eta > 0$ the function $D(\alpha)$ is zero if and only if there is a solution of (4.4) (or unstable eigenfunction) which decays exponentially at $Z \to \pm\infty$ (see, for example, [14]).

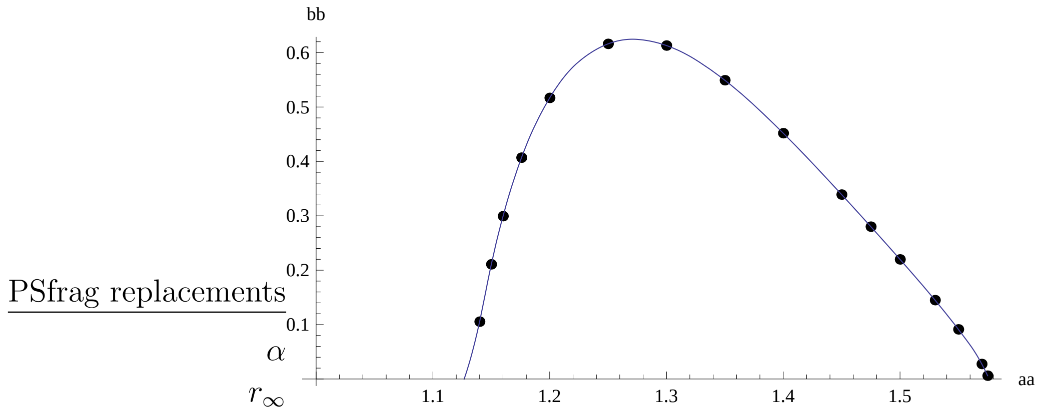

Figure 2: Dependence of the single eigenvalue of $\alpha = \rho\eta^2$ on r_∞. The solid line is the quadratic spline interpolation of the finite set of numerical results represented by the dots.

4.3 Numerical results

For the different bulging solutions represented by the bifurcation diagram in Figure 1, no unstable eigenvalues are found for any solution corresponding to the lower branch. For each solution corresponding to the upper branch, we found a single unstable eigenvalue. The dependence of α on r_∞ is displayed in Figure 2. It is seen that the growth rate of the single unstable mode tends to zero in two limits. The right one corresponds to the turning point in Figure 1, whereas the left limit corresponds to the case when the bulging solution becomes a "hat" solution. The hat solution has the property that at its centre $r''(0)$ is zero as well as $r'(0)$, and therefore, it can be viewed as two kink solutions joined together. The growth rate of the unstable mode tending to zero in the latter limit is consistent with the fact that the kink solution is usually observed to be stable in experiments.

5 Conclusion and discussion

In order to adopt the spectral stability analysis of the aneurysm solutions we linearize (2.6) about $\boldsymbol{\phi}$. Defining

$$\delta \mathbf{u} = \mathbf{u}(x,t) - \boldsymbol{\phi}$$

we get

$$\frac{d\delta\mathbf{u}}{dt} = \mathcal{J}DE\big(\boldsymbol{\phi} + \delta\mathbf{u}\big) = \mathcal{J}\mathcal{H}\,\delta\mathbf{u} + O(||\delta\mathbf{u}||^2).$$

Thus the linearized equation (2.6) is

$$\frac{d\delta\mathbf{u}}{dt} = \mathcal{J}\mathcal{H}\,\delta\mathbf{u}.$$

The operator $\mathcal{J}\mathcal{H}$ is a product of skew-adjoint and self adjoint operators and its continuous spectrum fills the imaginary axis and the spectrum is symmetric with respect to both axes.

The question is whether this operator has any spectrum off the imaginary axis. If so, we can expect instability.

From (4.3) it follows that the spectral problem for linear stability

$$\mathcal{J}\mathcal{H}\boldsymbol{\psi} = \eta\boldsymbol{\psi}, \quad \boldsymbol{\psi} = \{\psi_1, \psi_2, \psi_3, \psi_4\}^T$$

is reduced to (4.2).

The nonlinear (Lyapunov) stability is reduced to the spectral problem (where the spectral parameter is α)

$$\mathcal{H}\boldsymbol{\psi} = -\alpha\boldsymbol{\psi}. \tag{5.1}$$

The operator $\mathcal{H}$ is self-adjoint and, therefore, $\mathcal{L}$ in (4.3) is also self-adjoint, and $\alpha = \rho\eta^2$. Thus, the spectrum α of $\mathcal{L}$ is real, and, therefore, η can be either real or imaginary. While the η-spectrum fills the entire imaginary axis, the continuous spectrum α in (5.1) is positive and it is not separated from zero, but it starts from zero.

We deduce that spectral (linear) instability is stipulated by the existence of negative eigenvalues η of the linearized problem or the same any eigenvalue of the operator $\mathcal{L}$ (which is positive, because it is quadratic in η and η itself is the real number). In case of existence of any eigenvalue of $\mathcal{L}$ we have the negative eigenvalue of $\mathcal{H}$ with the same absolute value. Hence, the bulge under consideration is also non-linearly unstable (in the sense of Lyapunov's theory: the energy has not the local minimum). If there are no eigenvalues of $\mathcal{L}$ and the bulge in question is spectrally stable, we have only positive continuous spectrum of $\mathcal{H}$ which is not separated from zero. So, the sufficient condition for Lyapunov's stability is violated and we can say nothing about nonlinear stability of the bulge.

But in practice we have no infinitely long tubes, they are finite, though they can have a long extension. In this case the positive continuous spectrum transforms into positive discrete one and the operator $\mathcal{H}$ has only positive eigenvalues. At that, zero is not instant eigenvalue of $\mathcal{H}$ because there is no translational invariance of the problem. The calculations show (see [18]), that the zero is not limit point of the sequence of eigenvalues of $\mathcal{H}$. Therefore, the spectrum is separated from zero, and linear stability of the bulge in question implies the non-linear one.

We have established that the upper branch of the bifurcation curve (see Fig. 1) is linearly (and also non-linearly) unstable, though the lowest branch is linearly stable. This prompts the stability of the aneurysms on the lowest branch of the bifurcation curve and, therefore, their existence in practice. The lowest branch corresponds to aneurysms which amplitude grows with increase of r_∞ $\big($internal pressure P, according to (2.8)$\big)$. In practice, the localization of the imperfection (the thinning) of the wall of the tube may be more or less arbitrary, and the turning point in the bifurcation diagram (Fig. 1) may correspond to considerable amplitudes of the bulge in question.

The existence of the stable aneurism is supported even by the simple model for the inner fluid (inviscid fluid, average and axisymmetric flow) which is used here. In practice, of course we must take into account the viscosity of the inner fluid, the transversal size of the tube, etc. But it seems, that the local effects associated with formation of the stable aneurysm on the membrane tube can be qualitatively described by our model

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