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ON A GENERALIZATION OF SZEMERÉDI'S THEOREM.

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Abstract

Let N be a natural number and $A \subseteq \{1,\dots,N\}^2$ be a set of cardinality at least $N^2/(\log\log N)^c$, where $c>0$ is an absolute constant. We prove that A contains a triple $\{(k,m),(k+d,m),(k,m+d)\}$, where $d>0$. This theorem is a two-dimensional generalization of Szemerédi's theorem on arithmetic progressions.

1. *Introduction.*

Let N be a natural number. We set

$$a_k(N) = \frac{1}{N}\max\{|A| \ : \ A \subseteq [1,N],$$

$$A \text{ contains no arithmetic progressions of length } k\},$$

where $|A|$ denotes the cardinality of A. In [**5**], Erdös and Turán conjectured that any set of positive density contains an arithmetic progression of given length. In other words, they supposed that, for any $k \ge 3$

$$a_k(N) \to 0 \text{ as } N \to \infty \tag{1.1}$$

Clearly, this conjecture implies van der Waerden's theorem [**22**].

In the simplest case of $k=3$ conjecture (1.1) was proven in [**14**] by K.F. Roth, who applied the Hardy – Littlewood method to show that

$$a_3(N) \ll \frac{1}{\log\log N}.$$

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At present, the best upper bound for $a_3(N)$ is due to J. Bourgain. He proved that

$$a_3(N) \ll \sqrt{\frac{\log\log N}{\log N}}. \tag{1.2}$$

For an arbitrary k conjecture (1.1) was proven by E. Szemerédi [**19**] in 1975. Szemerédi's proof uses difficult combinatorial arguments.

An alternative proof was suggested by Furstenberg in [**7**]. His approach uses the methods of ergodic theory. Furstenberg showed that Szemerédi's theorem is equivalent to the multiple recurrence of almost all points in any dynamical system.

A. Behrend [**2**] obtained the following lower bound for $a_3(N)$

$$a_3(N) \gg \exp(-C(\log N)^{\frac{1}{2}}) \,,$$

where C is an absolute constant. A lower bound on $a_k(N)$ for an arbitrary k was given in [**13**].

Unfortunately, Szemerédi's methods give very weak upper estimates for $a_k(N)$. The ergodic approach gives no estimates at all. Only in 2001 W.T. Gowers [**8**] obtained a quantitative result concerning the rate at which $a_k(N)$ approaches zero for $k \geq 4$. He proved the following theorem.

THEOREM 1.1. *Let $\delta > 0$, $k \geq 4$ and $N \gg \exp\exp(C\delta^{-K})$, where $C, K > 0$ is absolute constants. Let $A \subseteq \{1, 2, \dots, N\}$ be a set of cardinality at least δN. Then A contains an arithmetic progression of length k.*

In other words, W.T. Gowers proved that, for any $k \geq 4$, we have $a_k(N) \ll 1/(\log\log N)^{c_k}$, where constant c_k depends on k only.

In this paper, we solve the following problem. Consider the two–dimensional lattice $[1, N]^2$ with basis $\{(1, 0),\ (0, 1)\}$. Let

$$L(N) = \frac{1}{N^2} \max\{\ |A| \ :\ A \subseteq [1, N]^2 \ \text{ and}$$

$$A \text{ contains no triples of the form } \{(k, m),\ (k+d, m),\ (k, m+d)\}$$

$$\text{with positive } d\}. \tag{1.3}$$

A triple from (1.3) will be called a *"corner"*. In [**1**, **7**], it was proven that $L(N)$

tends to 0 as N tends to infinity. W.T. Gowers (see [**8**]) asked the question of what is the rate of convergence of $L(N)$ to 0.

The following theorem was proven in [**16**, **17**] (see also [**18**, **21**]).

THEOREM 1.2. *Let $\delta > 0$ and $N \gg \exp\exp\exp(\delta^{-C})$, where > 0 is an absolute constant. Let A be a subset of $\{1,\dots,N\}^2$ of cardinality at least δN^2. Then A contains a corner.*

The question on upper estimates for $L(N)$ in the group $\mathbf{F}_3^n$ was considered in [**11**]. The main result of this paper is the following theorem.

THEOREM 1.3. *Let $\delta > 0$, and $N \gg \exp\exp(\delta^{-c})$, where $c > 0$ is an absolute constant. Let A be a subset of $\{1,\dots,N\}^2$ of cardinality at least δN^2. Then A contains a corner.*

Thus, we prove the estimate $L(N) \ll 1/(\log\log N)^{C_1}$, where $C_1 = 1/c$.

NOTE. The constant c in Theorem 1.3 might be taken as 73.

The constructions which we use develop the approach of [**3**, **8**, **16**].

The proof of Theorem 1.3 is contained in §3,4,5,6 and proceeds by an iteration scheme.

Let A be a set, $A \subseteq E_1 \times E_2$, where $E_1, E_2 \subseteq \mathbf{Z}^2$. At each step of our procedure we prove the following : either A is "sufficiently regular" or its "density" can be increased. A suitable definition of "sufficiently regular" sets (so–called uniform sets) is one of the main aims of our proof.

If A is a random set and A has cardinality δN^2, then A contains approximately $\delta^3 N^3$ corners. We shall say A is regular (or in other words α*–uniform*) if A contains the same approximate number of corners.

Let us consider the following example. Let A be a set of the form $E_1 \times E_2$, where E_1, E_2 are two random sets. Denote by β_i the density of the set E_i, and set $\beta_1\beta_2 = \delta$. Since each E_i has small Fourier coefficients, so does A. On the other hand, the number of corners in A equals $\beta_1^2\beta_2^2N^3 = \delta^2N^3 \neq \delta^3N^3$. So, if A has small Fourier coefficients, then A might not be regular (uniform).

Let E_1, E_2 be subsets of Λ, where $\Lambda \subseteq \mathbf{Z}$ to be chosen later. Let A be a subset of $E_1 \times E_2$ of cardinality $\delta|E_1||E_2|$. We shall say that A is *rectilinearly α–uniform* if, roughly speaking, the number of quadruples $\{(x,y),(x+d,y),(x,y+s),(x+d,y+s)\}$ in A^4 is at most $(\delta^4+\alpha)|E_1|^2|E_2|^2$ (in fact we need a slightly different definition of α–uniformity, which depends on the set Λ). In §3 we prove that if E_1, E_2 has small Fourier coefficients and A is rectilinearly α–uniform, then A has about the expected number of corners.

Suppose A fails to be rectilinearly α–uniform. We shall show in §4 that A has increased density $\delta + c(\delta)$ on some product set $F_1 \times F_2$, $F_1 \subseteq E_1$, $F_2 \subseteq E_2$. To obtain this we need Proposition 4.1, which was proven by Ben Green in [**11**]. A similar proposition was proven in [**16**] with worse bounds (see section 4).

Unfortunately, the structure of $F_1 \times F_2$ need not be regular. To make it regular, we pass to a subset of Λ, say, Λ' and an integer vector $\vec{t} = (t_1, t_2)$ such that $(F_1 - t_1) \cap \Lambda'$, $(F_2 - t_2) \cap \Lambda'$ has small Fourier coefficients.

This is attained by a further iteration procedure. Suppose that $F_1 \times F_2$ is not good; then either F_1 or F_2 has a large Fourier coefficient. This may be used to find a subset of Λ, say, Λ_1 such that some sort of density (so–called *index*, see §5) of $F_1 \times F_2$ in $\Lambda_1 \times \Lambda_1$ increases. This can only occur finitely many times.

We are now in the situation we started with, but A has a larger density and we iterate the procedure. This also can only occur finitely many times. In §6 we combine the arguments from the earlier sections and show that they give the bound that we stated in Theorem 1.3.

The main difference between this paper and [**16**] consists in the following: in [**16**] we chose Λ to be an *arithmetic progression*, whereas here we put Λ to be a so–called *Bohr set* (see [**3**, **10**] and others). This choice turns out to be more economical than dealing with progressions. Note that the best upper bound for $a_3(N)$ was proven by J. Bourgain in [**3**] using exactly these very sets. The properties of Bohr sets will be considered in §2.

Note, finally, that our results were formulated for subsets of $\{1,\dots,N\}\times\{1,\dots,N\}$, but the arguments work in $G\times G$, where G is an finite abelian group G. The author is going to obtain such result.

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2. *On Bohr sets.*

Let A be a subset of $\mathbf{Z}$. It is very convenient to write $A(x)$ for such a function. Thus $A(x)=1$ if $x\in A$ and $A(x)=0$ otherwise.

One of the crucial moments in [**3**] was the notion of Bohr set.

Let N and d be natural numbers, $\varepsilon>0$ be a real number and $\theta=(\theta_1,\dots,\theta_d)\in\mathbf{T}^d$.

Definition 2.1. Define the Bohr set $\Lambda=\Lambda_{\theta,\varepsilon,N}$ by

$$\Lambda_{\theta,\varepsilon,N}=\{n\in\mathbf{Z}\mid |n|\le N,\ \|n\theta_j\|<\varepsilon \text{ for } j=1,\dots,d\}$$

We shall say that the vector $\theta\in\mathbf{T}^d$ is *generative vector* of Bohr set Λ. The number d is called *dimension* of Bohr set Λ and is denoted by $\dim\Lambda$. If $M=\Lambda+n$, $n\in\mathbf{Z}$ is a translation of Λ, then, by definition, put $\dim M=\dim\Lambda$.

Another construction of Bohr set (so–called *smoothed* Bohr set) was given in [**20**] and [**10**].

Definition 2.2. Let $0<\kappa<1$ be a real number. A Bohr set $\Lambda=\Lambda_{\theta,\varepsilon,N}$ is called *regular*, if for an arbitrary ε', N' such that

$$|\varepsilon-\varepsilon'|<\frac{\kappa}{100d}\varepsilon \quad\text{and}\quad |N-N'|<\frac{\kappa}{100d}N$$

we have

$$1-\kappa < \frac{|\Lambda_{\theta,\varepsilon',N'}|}{|\Lambda_{\theta,\varepsilon,N}|} < 1+\kappa\,.$$

We need several results concerning Bohr sets (see [**3**]).

LEMMA 2.1. *Let $\Lambda_{\theta,\varepsilon,N}$ be a Bohr set, $\theta \in \mathbf{T}^d$. Then*

$$|\Lambda_{\theta,\varepsilon,N}| \ge \frac{1}{2}\varepsilon^d N\,.$$

LEMMA 2.2. *Let $0<\kappa<1$ be a real number, and $\Lambda_{\theta,\varepsilon,N}$ be a Bohr set. Then there exists a pair (ε_1, N_1) such that*

$$\frac{\varepsilon}{2} < \varepsilon_1 < \varepsilon \quad \text{and} \quad \frac{N}{2} < N_1 < N\,,$$

and such that $\Lambda_{\theta,\varepsilon_1,N_1}$ is a regular Bohr set.

All Bohr sets will be regular in the article.

DEFINITION 2.3. Let f, g be functions from $\mathbf{Z}$ to $\mathbf{C}$. By $f*g$ define the function

$$(f*g)(n) = \sum_{s\in\mathbf{Z}} f(s)\overline{g(n-s)}$$

DEFINITION 2.4. Let $\varepsilon \in (0,1]$ be a real number, and $\Lambda_{\theta,\varepsilon_0,N_0}$ be a Bohr set, $\theta = (\theta_1,\dots,\theta_d)$. A regular Bohr set $\Lambda' = \Lambda_{\theta',\varepsilon',N'}$ is called ε–*attendant* of Λ if $\theta' = (\theta_1,\dots,\theta_d,\theta_{d+1},\dots,\theta_{d+k})$, $k\ge 0$, $\varepsilon\varepsilon_0/2 \le \varepsilon' \le \varepsilon\varepsilon_0$, $\varepsilon N_0/2 \le N' \le \varepsilon N_0$.

Lemma 2.2 implies that for an arbitrary Bohr set there exists its ε–attendant.

We shall consider that $k=0$ unless stated otherwise.

Let n be a natural number, and Λ be a Bohr set. We shall say that a Bohr set Λ' is ε–attendant of $\Lambda + n$, if Λ' is ε–attendant of Λ.

The following lemma is also due to J. Bourgain [**3**]. We give his proof for the sake of completeness.

LEMMA 2.3. *Let $\kappa > 0$ be a real number, $\theta \in \mathbf{T}^d$, $\Lambda = \Lambda_{\theta,\varepsilon,N}$ be a regular Bohr* *set, and* $\Lambda' = \Lambda_{\theta,\varepsilon',N'}$ *its* $\kappa/(100d)$*–attendant. Then the number of* n'*s such that* $(\Lambda * \Lambda')(n) > 0$ *does not exceed* $|\Lambda|(1+\kappa)$*, the number of* n'*s such that* $(\Lambda * \Lambda')(n) = |\Lambda'|$ *is greater than* $|\Lambda|(1-\kappa)$ *and*

$$\left\| \frac{1}{|\Lambda'|}(\Lambda * \Lambda')(n) - \Lambda(n) \right\|_1 < 2\kappa|\Lambda| \,. \tag{2.1}$$

Proof. If $(\Lambda * \Lambda^{'})(n) > 0$, then there exists m such that

$$|m| \le \frac{\kappa}{100d}N, \quad |n-m| \le N \tag{2.2}$$

and

$$\|m\theta_j\| < \frac{\kappa}{100d}\varepsilon, \quad \|(n-m)\theta_j\| < \varepsilon, \; j = 1, \ldots, d \tag{2.3}$$

Using (2.2) and (2.3), we get

$$|n| \le \Big(1 + \frac{\kappa}{100d}\Big)N \quad \text{and} \quad \|n\theta_j\| < \Big(1 + \frac{\kappa}{100d}\Big)\varepsilon, \; j = 1, \ldots, d \tag{2.4}$$

It follows that

$$n \in \Lambda^+ := \Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N} \,. \tag{2.5}$$

By Lemma 2.2 we have $|\Lambda^+| \le (1+\kappa)|\Lambda|$.

On the other hand, if

$$n \in \Lambda^- := \Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N} \,, \tag{2.6}$$

then $(\Lambda * \Lambda^{'})(n) = |\Lambda'|$. Using Lemma 2.2, we obtain $|\Lambda^-| \ge (1-\kappa)|\Lambda|$.

Let us prove (2.1). We have

$$\begin{aligned} \left\| \frac{1}{|\Lambda'|}(\Lambda * \Lambda')(n) - \Lambda(n) \right\|_1 &= \left\| \frac{1}{|\Lambda'|}(\Lambda * \Lambda')(n) - \Lambda(n) \right\|_{l^1(\Lambda^+ \setminus \Lambda^-)} \\ &\le |\Lambda^+| - |\Lambda^-| < 2\kappa|\Lambda| \end{aligned}$$

as required.

Corollary. *Lemma 2.3 implies that* $|\Lambda| \le |\Lambda + \Lambda'| \le (1+2\kappa)|\Lambda|$.

Note. Let $\Lambda^x(n) = \Lambda(n-x)$. Since $(\Lambda^x * \Lambda')(n) = (\Lambda * \Lambda')(n-x)$, it follows that (2.1) takes place for translations $\Lambda + x$.

Definition 2.5. By Λ^+ and Λ^- denote the Bohr sets defined in (2.5) and (2.6), respectively, $\Lambda^- \subseteq \Lambda \subseteq \Lambda^+$.

By Lemma 2.3 we have $|\Lambda^+| \le |\Lambda|(1+\kappa)$ and $|\Lambda^-| \ge |\Lambda|(1-\kappa)$. Note that for any $s \in \Lambda^{'}$, we get $\Lambda^- \subseteq \Lambda + s$.

Suppose $\Lambda \subseteq \mathbf{Z}$ is a Bohr set, and $\vec{x} = (x_1, x_2)$ belongs to $\mathbf{Z^2}$. By $\Lambda + \vec{x}$ denote the set $(\Lambda + x_1) \times (\Lambda + x_2) \subseteq \mathbf{Z^2}$. Let $\vec{n} \in \mathbf{Z^2}$. Let $\Lambda(\vec{n})$ denote the characteristic function of $\Lambda \times \Lambda$. We shall write $\vec{s} \in \Lambda$, $\vec{s} = (s_1, s_2)$, if $s_1 \in \Lambda$ and $s_2 \in \Lambda$.

Lemma 2.4. *Suppose Λ is a Bohr set, $\Lambda^{'}$ is its ε–attendant, $\varepsilon = \kappa/(100d)$, $\vec{x}$ is a vector, and $E \subseteq \mathbf{Z^2}$. Then*

$$\left| \delta_{\Lambda + \vec{x}}(E) - \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda + \vec{x}} \delta_{\Lambda^{'} + \vec{n}}(E) \right| \le 4\kappa \,. \tag{2.7}$$

Proof. We have

$$\sigma = \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda + \vec{x}} \delta_{\Lambda^{'} + \vec{n}}(E) = \frac{1}{|\Lambda|^2 |\Lambda^{'}|^2} \sum_{\vec{s}} E(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n} - \vec{x}) \Lambda^{'}(\vec{s} - \vec{n})$$

$$= \frac{1}{|\Lambda|^2 |\Lambda^{'}|^2} \sum_{\vec{s}} E(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n}) \Lambda^{'}(\vec{s} - \vec{x} - \vec{n})$$

Using Lemma 2.3, we get

$$\sigma = \frac{1}{|\Lambda|^2} \sum_{\vec{s}} E(\vec{s}) \Lambda(\vec{s} - \vec{x}) + 4\vartheta\kappa = \delta_{\Lambda + \vec{x}}(E) + 4\vartheta\kappa \,,$$

where $|\vartheta| \le 1$. This completes the proof.

Note. Clearly, the one–dimension analog of Lemma 2.4 takes place.

Let $\Lambda_1 = \Lambda_{\theta_1, \varepsilon_1, N_1}$, $\Lambda_2 = \Lambda_{\theta_2, \varepsilon_2, N_2}$ be two Bohr sets. We shall write $\Lambda_1 \le \Lambda_2$, if $\theta_1 = \theta_2$, $\varepsilon_1 \le \varepsilon_2$ and $N_1 \le N_2$.

Note that if $\Lambda_1 \le \Lambda_2$, then an arbitrary ε–attendant of Λ_1 is ε attedant of Λ_2.

3. *On α–uniformity.*

Let f be a function from $\mathbf{Z}$ to $\mathbf{C}$. By $\widehat{f}(x)$ denote the Fourier transformation of f

$$\widehat{f}(x) = \sum_{s \in \mathbf{Z}} f(s) e(-sx),$$

where $e(x) = e^{2\pi i x}$.

We shall use the following basic facts

$$\sum_{s \in \mathbf{Z}} |f(s)|^2 = \int_0^1 |\widehat{f}(x)|^2 dx \tag{3.1}$$

$$\sum_{s \in \mathbf{Z}} f(s)\overline{g(s)} = \int_0^1 \widehat{f}(x)\overline{\widehat{g}(x)} dx \tag{3.2}$$

$$\sum_{k \in \mathbf{Z}} | \sum_{s \in \mathbf{Z}} f(s)\overline{g(s-k)}|^2 = \int_0^1 |\widehat{f}(x)|^2 |\widehat{g}(x)|^2 dx \tag{3.3}$$

Let Λ be a Bohr set, and A be an arbitrary subset of Λ. Let $|A| = \delta|\Lambda|$. Define the *balanced* function of A to be $f(s) = (A(s) - \delta)\Lambda(s) = A(s) - \delta\Lambda(s)$.

Let $\mathbf{D}$ denote the closed disk of radius 1 centered at 0 in the complex plane. Let R be an arbitrary set. We write $f : R \to \mathbf{D}$ if f is zero outside R.

The following definition is due to Gowers [**8**].

Definition 3.1. A function $f : \Lambda \to \mathbf{D}$ is called α–uniform if

$$\|\widehat{f}\|_\infty \le \alpha|\Lambda| \tag{3.4}$$

We say that A is α–uniform if its balanced function is.

We shall write $\int$ instead of $\int_0^1$ and $\sum_s$ instead of $\sum_{s \in \mathbf{Z}}$.

Let us prove an analog of Lemma 2.2 from [**8**].

Lemma 3.1. *Let Λ be a Bohr set, and let $f : \Lambda \to \mathbf{D}$ be α–uniform function.*

Then we have

$$\sum_k |\sum_s f(s)\overline{g(s-k)}|^2 \le \alpha^2 |\Lambda|^2 \|g\|_2^2,$$

for an arbitrary function g, $g : \mathbf{Z} \to \mathbf{D}$.

Proof. By (3.3) we get

$$\sum_k |\sum_s f(s)\overline{g(s-k)}|^2 = \int |\widehat{f}(x)|^2 |\widehat{g}(x)|^2 dx. \tag{3.5}$$

Since the function f is α–uniform, it follows that $\|\widehat{f}\|_\infty \le \alpha|\Lambda|$. Using this inequality and (3.2), we have

$$\sum_k |\sum_s f(s)\overline{g(s-k)}|^2 \le \alpha^2 |\Lambda|^2 \int |\widehat{g}(x)|^2 dx \le \alpha^2 |\Lambda|^2 \|g\|_2^2 \tag{3.6}$$

This completes the proof.

Corollary 3.1. *Let S be a set, and $\Lambda^{'}$ be a Bohr set. Suppose $E \subseteq \Lambda^{'}$ is α–uniform, and E have the cardinality $\delta|\Lambda^{'}|$. Let g be a function from S to $\mathbf{D}$. Then for all but $\alpha^{2/3}|S|$ choices of k we have*

$$\Big|(E * g)(k) - \delta(\Lambda^{'} * g)(k)\Big| \le \alpha^{2/3} |\Lambda^{'}|\,.$$

Proof. Let f be the balanced function of $E \cap \Lambda^{'}$. Using Lemma 3.1, we get

$$\sum_k |(E * g)(k) - \delta(\Lambda^{'} * g)(k)|^2 = \sum_k |\sum_s f(s)\overline{g(s-k)}|^2 \le \tag{3.7}$$

$$\le \alpha^2 |\Lambda^{'}|^2 \|g\|_2^2 \le \alpha^2 |\Lambda^{'}|^2 |S|\,. \tag{3.8}$$

This concludes the proof.

By $\vec{e}_1$ and $\vec{e}_2$ define the vectors $(1,0)$ and $(0,-1)$.

Let Λ_1 and Λ_2 be Bohr sets, and $E_1 \times E_2$ be a subset of $\Lambda_1 \times \Lambda_2$. Suppose $f : \Lambda_1 \times \Lambda_2 \to \mathbf{D}$ is a function.

Definition 3.2. Let α be a real number, $\alpha \in [0,1]$. A function $f : E_1 \times E_2 \to \mathbf{D}$

is called *rectilinearly α–uniform* if

$$\sum_{\vec{s},u}\sum_{r} f(\vec{s})\overline{f(\vec{s}+u\vec{e}_2)f(\vec{s}+r\vec{e}_1)}f(\vec{s}+u\vec{e}_2+r\vec{e}_1) \le \alpha |E_1|^2|E_2|^2. \qquad (3.9)$$

Let $f(k,m) = f(k\vec{e}_1 + m\vec{e}_2)$. Note that the function f is α–uniform iff

$$\sum_{m,p} |\sum_{k} f(k,m)\overline{f(k,p)}|^2 \le \alpha |E_1|^2|E_2|^2. \qquad (3.10)$$

Let A be a subset of $E_1 \times E_2$, $|A| = \delta|E_1||E_2|$. Define the *balanced* function of A to be $f(\vec{s}) = (A(\vec{s}) - \delta)\cdot(E_1 \times E_2)(\vec{s})$. We say that $A \subseteq E_1 \times E_2$ is rectilinearly α–uniform if its balanced function is.

Let f be an arbitrary function, $f : \mathbf{Z}^2 \to \mathbf{C}$. Define $\|f\|$ by the formula

$$\|f\| = |\sum_{\vec{s},u}\sum_{r} f(\vec{s})\overline{f(\vec{s}+u\vec{e}_2)f(\vec{s}+r\vec{e}_1)}f(\vec{s}+u\vec{e}_2+r\vec{e}_1)|^{\frac{1}{4}} \qquad (3.11)$$

Lemma 3.2. *$\|\cdot\|$ is a norm.*

Proof. See [**16**].

Definition 3.3. Let Λ be a Bohr set, $Q \subseteq \Lambda$, $|Q| = \delta|\Lambda|$, α, ε are positive numbers, and Λ' be ε–attendant set of Λ. Consider the set

$$B = \{m \in \Lambda \mid \|(Q \cap (\Lambda' + m) - \delta(\Lambda' + m)\widehat{\ }\|_\infty \ge \alpha|\Lambda'|\}\,.$$

A set Q is called *(α,ε)–uniform* if

$$|B| \le \alpha|\Lambda|\,, \qquad (3.12)$$

$$\frac{1}{|\Lambda|}\sum_{m\in\Lambda} |\delta_{\Lambda'+m}(Q) - \delta|^2 \le \alpha^2\,. \qquad (3.13)$$

and

$$\|(Q \cap \Lambda - \delta\Lambda\widehat{\ }\|_\infty \le \alpha|\Lambda|\,. \qquad (3.14)$$

Certainly, this definition depends on Λ and Λ'. We do not assume that Λ' has the same generative vector as Λ.

NOTE. Let

$$B^* = \{m \in \Lambda \mid |\delta_{\Lambda'+m}(Q) - \delta| \ge \alpha^{2/3}\}\,.$$

Condition (3.13) implies that $|B^*| \le \alpha^{2/3}|\Lambda|$.

NOTE. Condition (3.14) is not so important as (3.12) and (3.13). The inequality

$$\|(Q \cap \Lambda - \delta\Lambda)\widehat{\ }\|_\infty \le 4\alpha|\Lambda|$$

follows from (3.12), (3.13) (see Proposition 3.1).

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, $\varepsilon > 0$ be a real number, and Λ' be ε–attendant of Λ_1. Let also E_1, E_2 be subsets of Λ_1, Λ_2, respectively, and $|E_1| = \beta_1|\Lambda_1|$, $E_2 = \beta_2|\Lambda_2|$.

DEFINITION 3.4. A function $f : \Lambda_1 \times \Lambda_2 \to \mathbf{D}$ is called *rectilinearly* (α, ε)–*uniform* if

$$\|f\|^4_{\Lambda_1\times\Lambda_2,\varepsilon} = \sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m,u}\Lambda'(m-k-i)\Lambda'(u-k-i)\,\times$$

$$|\sum_r \Lambda'(k+r-j)f(r,m)f(r,u)|^2 \le \alpha\beta_1^2\beta_2^2|\Lambda'|^4|\Lambda_1|^2|\Lambda_2|\,. \qquad (3.15)$$

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, and Λ' be ε–attendant of Λ_1. Suppose that Λ'_ε is ε–attendant of Λ'. Let also E_1, E_2 be subsets of Λ_1, Λ_2, respectively, and $|E_1| = \beta_1|\Lambda_1|$, $E_2 = \beta_2|\Lambda_2|$.

DEFINITION 3.5. Let $A \subseteq E_1 \times E_2$, $|A| = \delta\beta_1\beta_2|\Lambda_1||\Lambda_2|$, and $f(\vec{s}) = A(\vec{s}) - \delta(E_1\times E_2)(\vec{s})$. Let $f_l(\vec{s}) = f(s_1 + l, s_2)\Lambda'(s_1)$, $l \in \Lambda_1$. Consider the set

$$B = \{l \in \Lambda_1 \mid \|f_l\|^4_{\Lambda'\times\Lambda_2,\varepsilon} > \alpha\beta_1^2\beta_2^2|\Lambda'_\varepsilon|^4|\Lambda'|^2|\Lambda_2|\}\,.$$

A is called *rectilinearly* $(\alpha, \alpha_1, \varepsilon)$–*uniform* if $|B| \le \alpha_1|\Lambda_1|$.

Note that

$$\|f_l\|^4_{\Lambda'\times\Lambda_2,\varepsilon} = \sum_{i\in\Lambda'}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m,u}\Lambda''(m-k-i)\Lambda''(u-k-i)|\sum_r \Lambda''(k+r-j)f_l(r,m)f_l(r,u)|^2$$

$$= \sum_{i\in\Lambda'+l} \sum_{j\in\Lambda_2} \sum_{k} \sum_{m,u} \Lambda''(m-k-i)\Lambda''(u-k-i) \times |\sum_{r} \Lambda''(k+r-j)\tilde{f}_l(r,m)\tilde{f}_l(r,u)|^2 \,,$$

where $\Lambda'' = \Lambda'(\varepsilon)$ and $\tilde{f}$ is a restriction of f to $(\Lambda'+l)\times\Lambda_2$.

NOTE. We need parameter α_1 to decrease the constant c in Theorem 1.3. To obtain Theorem 1.3 with c equals, say, 1000, one can put $\alpha_1 = \alpha$.

LEMMA 3.3. *Let Λ be a Bohr set. Suppose Λ' is ε–attendant of Λ, Λ'' is ε–attendant Λ' and ε^2–attendant of Λ, $\varepsilon = \alpha^2/4(100d)$, $Q \subseteq \Lambda$, $|Q| = \delta\Lambda$, and $\alpha > 0$. Let*

$$\Omega_1 = \{ s\in\Lambda ~|~ |\delta_{\Lambda'+s}(Q) - \delta| \ge 4\alpha^{1/2} ~~or~~ \frac{1}{|\Lambda'|} \sum_{n\in\Lambda'+s} |\delta_{\Lambda''+n}(Q)-\delta|^2 \ge 4\alpha^{1/2} \} \,.$$

$$\Omega_2 = \{ s\in\Lambda ~|~ \|(Q\cap(\Lambda'+s) - \delta(\Lambda'+s))\widehat{~}\|_\infty \ge 4\alpha^{1/4}|\Lambda'| \} \,.$$

1) If

$$\frac{1}{|\Lambda|} \sum_{n\in\Lambda} |\delta_{\Lambda''+n}(Q)-\delta|^2 \le \alpha^2 \,, \tag{3.16}$$

then $|\Omega_1| \le 4\alpha^{1/2}|\Lambda|$.

2) If

$$\Omega^* = \{ s\in\Lambda ~|~ \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{~}\|_\infty \ge \alpha|\Lambda''| \} \tag{3.17}$$

has the cardinality at most $\alpha|\Lambda|$, then $|\Omega_2| \le 4\alpha^{1/2}|\Lambda|$.

3) Suppose Q is (α,ε^2)–uniform subset of Λ. Let

$$\tilde{\Omega} = \{ s\in\Lambda ~|~ \text{ Set } (Q-s)\cap\Lambda' \text{ is not } (8\alpha^{1/4},\varepsilon)\text{–uniform } \} \,.$$

Then $|\tilde{\Omega}| \le 8\alpha^{1/2}|\Lambda|$.

Proof. Let us prove 1). Let $\delta'_n = \delta_{\Lambda'+n}(Q)$, $\delta''_n = \delta_{\Lambda''+n}(Q)$, $\kappa = \alpha^2/4$, and $\epsilon = \alpha^{1/2}$. onsider the sets

$$B_s = \{ n\in\Lambda'+s ~|~ |\delta''_n - \delta| \ge \epsilon \} \,, G_s = \{ n\in\Lambda'+s ~|~ |\delta''_n - \delta| < \epsilon \} \,, s\in\Lambda$$

and sets

$$B = \{ s\in\Lambda ~|~ |B_s| \ge \epsilon|\Lambda'| \} \,, G = \{ s\in\Lambda ~|~ |B_s| < \epsilon|\Lambda'| \}$$

If $s \in G$, then $|B_s| < \epsilon|\Lambda'|$. Using Lemma 2.4, we have

$$|\delta'_s - \delta| \le \Big| \frac{1}{|\Lambda'|} \sum_{x\in\Lambda'+s} \delta''_x - \delta \Big| + 4\kappa \le \frac{1}{|\Lambda'|} \sum_{x\in\Lambda'+s} |\delta''_x - \delta| + 4\kappa \le$$

$$\le \frac{1}{|\Lambda'|} \sum_{x\in B_s} |\delta''_x - \delta| + \frac{1}{|\Lambda'|} \sum_{x\in G_s} |\delta''_x - \delta| + 4\kappa < \epsilon + \frac{\epsilon|G_s|}{|\Lambda'|} + 4\kappa \le 4\epsilon\,. \qquad (3.18)$$

Besides that for $s \in G$, we get

$$\frac{1}{|\Lambda'|} \sum_{x\in\Lambda'+s} |\delta''_x - \delta|^2 \le \frac{1}{|\Lambda'|} \sum_{x\in B_s} |\delta''_x - \delta|^2 + \frac{1}{|\Lambda'|} \sum_{x\in G_s} |\delta''_x - \delta|^2 \le \epsilon + \epsilon^2 \le 2\epsilon\,. \quad (3.19)$$

Let us estimate the cardinality of B. We have

$$\alpha^2 \ge \frac{1}{|\Lambda|} \sum_{s\in B} |\delta''_s - \delta|^2 \ge \frac{1}{|\Lambda'||\Lambda|} \sum_{s\in B} \sum_{n\in\Lambda'+s} |\delta''_n - \delta|^2 - 4\kappa \ge$$

$$\ge \frac{1}{|\Lambda'||\Lambda|} \sum_{s\in B} \sum_{n\in B_s} |\delta''_n - \delta|^2 - 4\kappa \ge \frac{|B|\epsilon^3|\Lambda'|}{|\Lambda'||\Lambda|} - 4\kappa\,.$$

It follows that, $|B| \le 4\alpha^{1/2}|\Lambda|$. Using (3.18), (3.19) we get $\Omega_1 \subseteq B$ and 1) is proven. To prove 2) it suffices to note that

$$\frac{1}{|\Lambda||\Lambda'|} \sum_{s\in\Lambda} \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{\ }\|_\infty = \frac{1}{|\Lambda||\Lambda'|} \sum_{s\in\Omega^*} \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{\ }\|_\infty +$$

$$+ \frac{1}{|\Lambda||\Lambda'|} \sum_{s\in(\Lambda\setminus\Omega^*)} \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{\ }\|_\infty \le \alpha + \frac{\alpha|\Lambda'|}{|\Lambda||\Lambda'|}|\Lambda\setminus\Omega^*| \le 2\alpha\,.$$

and define the sets B'_s, G'_s, B', G' :

$$B'_s = \{n \in \Lambda' + s \mid \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{\ }\|_\infty \ge \epsilon_1|\Lambda''|\}\,,$$

$$G'_s = \{n \in \Lambda' + s \mid \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{\ }\|_\infty < \epsilon_1|\Lambda''|\}, \ \ s\in\Lambda\,.$$

$$B' = \{s \in \Lambda \mid |B_s| \ge \epsilon_1|\Lambda'|\} \quad \text{ and } \quad G' = \{s\in\Lambda \mid |B_s| < \epsilon_1|\Lambda'|\}\,,$$

where $\epsilon_1 = \alpha^{1/4}$. After that we can apply the same arguments as above, using Lemma 2.3 instead of Lemma 2.4.

Let us prove 3). Since Q is (α, ε^2)–uniform subset of Λ, it follows that Q satisfies (3.16). Also we have $|\Omega^*| \le \alpha|\Lambda|$ and $|B|, |B'| \le 4\alpha^{1/2}|\Lambda|$. It is easily shown that for all $s \notin B \cup B'$ the set $(Q-s)\cap\Lambda'$ is $(8\alpha^{1/4}, \varepsilon)$–uniform. This completes the proof.

In the same way we can prove

PROPOSITION 3.1. *Let Λ be a Bohr set, and $E \subseteq \Lambda$, $|Q| = \delta|\Lambda|$ be (α, ε)–uniform, $\varepsilon = \alpha/4(100d)$. Then*

$$\|(Q \cap \Lambda - \delta\Lambda)\widehat{\ }\|_\infty < 4\alpha|\Lambda|\,. \tag{3.20}$$

We will not, however, use this fact.

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, and $E_1 \subseteq \Lambda_1$, $E_2 \subseteq \Lambda_2$, $|E_1| = \beta_1|\Lambda_1|$, $|E_2| = \beta_2|E_2|$. By $\mathcal{P}$ denote the $E_1 \times E_2$. Let H, G be subsets of $\mathcal{P}$.

THEOREM 3.1. *Let $f : \mathcal{P} \to \mathbf{D}$ be a function. Suppose that f is rectilinearly (α, ε)–uniform, and the sets E_1, E_2 are (α_0, ε)–uniform, $\alpha_0 = 2^{-50}\alpha^2\beta_1^{12}\beta_2^{12}$, $\varepsilon = 2^{-10}\varepsilon_0^2$, $\varepsilon_0 = (2^{-10}\alpha_0^2)/(100d)$. Let also Λ_1 be ε_0–attendant of Λ_2. Then*

$$|\sum_{\vec{s}\in\mathbf{Z}^2}\sum_{r\in\mathbf{Z}} H(\vec{s})G(\vec{s}+r\vec{e})f(\vec{s}+r\vec{e}_2)| \le 2^5\alpha^{1/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2|\,. \tag{3.21}$$

Proof. Let $\vec{e} = \vec{e}_1 + \vec{e}_2$, $\vec{s} = k\vec{e}_1 + m\vec{e}_2$, and Λ' be ε–attendant of Λ_1.

Let

$$\Omega_1^{(1)} = \{s \in \Lambda_1 \mid \|(E_1 \cap (\Lambda' + s) - \delta(\Lambda' + s)\widehat{\ }\|_\infty \ge \alpha_0\}\,,$$

$$\Omega_2^{(1)} = \{s \in \Lambda_1 \mid |\delta_{\Lambda'+s}(E_1) - \beta_1| \ge \alpha_0^{2/3}\}\,,$$

and

$$\Omega_1^{(2)} = \{s \in \Lambda_2 \mid \|(E_2 \cap (\Lambda' + s) - \delta(\Lambda' + s)\widehat{\ }\|_\infty \ge \alpha_0\}\,,$$

$$\Omega_2^{(2)} = \{s \in \Lambda_2 \mid |\delta_{\Lambda'+s}(E_2) - \beta_2| \ge \alpha_0^{2/3}\}\,,$$

Let also $\Omega_1 = \Omega_1^{(1)} \cup \Omega_2^{(1)}$, and $\Omega_2 = \Omega_1^{(2)} \cup \Omega_2^{(2)}$. By assumption the sets E_1, E_2 are (α_0, ε)–uniform. It follows that $|\Omega_l^{(1)}| \le \alpha_0^{2/3}|\Lambda_1|$, $|\Omega_l^{(2)}| \le \alpha_0^{2/3}|\Lambda_2|$, $l = 1, 2$. Hence, $|\Omega_1| \le 2\alpha_0^{2/3}|\Lambda_1|$ and $|\Omega_2| \le 2\alpha_0^{2/3}|\Lambda_2|$.

Let $g_i(\vec{s}) = g_i(k, m) = G(k, m)\Lambda'(k - i)$, $i \in \Lambda_1$, and $h_j(\vec{s}) = h_j(k, m) = H(k, m)\Lambda'(m - j)$, $j \in \Lambda_2$. We have $k \in \Lambda_1$, $m \in \Lambda_2$ and $k + r \in \Lambda_1$ in (3.21).

It follows that the sum (3.21) does not exceed $|\Lambda_1|^2|\Lambda_2|$. Let also $\lambda_i = \Lambda' + i$, and $\mu_j = \Lambda' + j$. Using Lemma 2.3, we get

$$\sigma_0 = \sum_{\vec{s}\in\mathbf{Z^2}} \sum_{r\in\mathbf{Z}} H(\vec{s})G(\vec{s}+r\vec{e})f(\vec{s}+r\vec{e}_2) =$$

$$= \sum_{k,m}\sum_{r} H(k,m)G(k+r,m+r)f(k,m+r)\Lambda_1(k+r)\Lambda_2(m) =$$

$$\frac{1}{|\Lambda'|^2}\sum_{k,m}\sum_{r} H(k,m)G(k+r,m+r)f(k,m+r)(\Lambda_1*\Lambda')(k+r)(\Lambda_2*\Lambda')(m)+16\vartheta_0\kappa|\Lambda_1|^2|\Lambda_2| =$$

$$\frac{1}{|\Lambda'|^2}\sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2}\sum_{k,m}\sum_{r} h_j(k,m)g_i(k+r,m+r)f(k,m+r)+16\vartheta_0\kappa|\Lambda_1|^2|\Lambda_2|\,, \quad (3.22)$$

where $|\vartheta_0| \le 1$ and $\kappa \le 2^{-10}\alpha_0^2$. Split the sum σ_0 as

$$\sigma_0 = \widetilde{\sigma}_0 + \sigma_0^{'} + \sigma_0^{''} + \sigma_0^{'''} + R\,, \quad (3.23)$$

The sum $\widetilde{\sigma}_0$ is taken over $i \notin \Omega_1, j \notin \Omega_2$, the sum $\sigma_0^{'}$ is taken over $i \in \Omega_1, j \notin \Omega_2$, the sum $\sigma_0^{''}$ is taken over $i \notin \Omega_1, j \in \Omega_2$, the sum $\sigma_0^{'''}$ is taken over $i \in \Omega_1, j \in \Omega_2$ and $|R| \le 16\varepsilon|\Lambda_1|^2|\Lambda_2|$. Let us estimate $\sigma_0^{'}$, $\sigma_0^{''}$ and $\sigma_0^{'''}$. Rewrite σ_0 as

$$\sigma_0 = \frac{1}{|\Lambda'|^2}\sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2}\sum_{k,m}\sum_{r} h_j(k-r,m)g_i(k,m+r)f(k-r,m+r) + R\,. \quad (3.24)$$

Let i and j in the sum (3.24) be fixed. We have $k \in \lambda_i$ and $m \in \mu_j$. Further if $f(k-r,m+r)$ is not zero, then $k-r \in \Lambda_1$. It follows that $r \in \lambda_i - \Lambda_1 = \Lambda' - \Lambda_1 + i$. The set Λ' is ε–attendant of Λ_1. Using Lemma 2.3, we obtain that r belongs to a set of cardinality at most $2|\Lambda_1|$. Hence

$$|\sigma_0^{'}| \le \frac{1}{|\Lambda'|^2} 2|\Omega_1|\cdot|\Lambda_2|\cdot|\Lambda'|^2|\Lambda_1| \le 2\alpha_0^{2/3}|\Lambda_1|^2|\Lambda_2|\,. \quad (3.25)$$

In the same way $|\sigma_0^{''}| \le 2\alpha_0^{2/3}|\Lambda_1|^2|\Lambda_2|$ and $|\sigma_0^{'''}| \le 2\alpha_0^{2/3}|\Lambda_1|^2|\Lambda_2|$.

Take i and j such that $i \notin \Omega_1$, $j \notin \Omega_2$. Let $g(\vec{s}) = g_i(\vec{s})$, $h(\vec{s}) = h_j(\vec{s})$, and $\Lambda_1 \times \mu_j = \Lambda_1^{(1)} \times \Lambda_2^{(1)}$, $\lambda_i \times \Lambda_2 = \Lambda_1^{(2)} \times \Lambda_2^{(2)}$. Let $E_2^{(1)} = E_2 \cap \Lambda_2^{(1)}$, $E_1^{(2)} = E_1 \cap \Lambda_1^{(2)}$, $\beta_2^{(1)} = |E_2^{(1)}|/|\Lambda_2^{(1)}|$, and $\beta_1^{(2)} = |E_1^{(2)}|/|\Lambda_1^{(2)}|$. We have

$$\sigma = \sigma_{i,j} = \sum_{\vec{s}\in\mathbf{Z^2}}\sum_{r\in\mathbf{Z}} h(\vec{s})g(\vec{s}+r(\vec{e}_1+\vec{e}_2))f(\vec{s}+r\vec{e}_2) = \quad (3.26)$$

$$= \sum_{k,m} h(k,m)E_2^{(1)}(m)\sum_{r} g(k+r,m+r)f(k,m+r) \quad (3.27)$$

Using the Cauchy–Schwarz inequality, we obtain

$$|\sigma|^2 \le \|h\|_2^2 \sum_{k,m} E_2^{(1)}(m) | \sum_r g(k+r, m+r) f(k, m+r)|^2 = \tag{3.28}$$

$$= \|h\|_2^2 \sum_{k,m} E_2^{(1)}(m) \sum_{r,p} g(k+r, m+r) f(k, m+r) g(k+p, m+p) f(k, m+p) =$$

$$= \|h\|_2^2 \sum_{k,m} E_2^{(1)}(m-r) \sum_{r,u} g(k, m) f(k-r, m) g(k+u, m+u) f(k-r, m+u) =$$

$$= \|h\|_2^2 \sum_{k,m} \sum_u g(k,m) g(k+u, m+u) \sum_r E_2^{(1)}(m-r) f(k-r, m) f(k-r, m+u) =$$

$$= \|h\|_2^2 \sum_{k,m} \sum_u g(k,m) g(k+u, m+u) E_1^{(2)}(k) E_1^{(2)}(k+u)$$

$$\cdot \sum_r E_2^{(1)}(m-r) f(k-r, m) f(k-r, m+u) \tag{3.29}$$

We have $k \in \Lambda_1^{(2)}$ and $k - r \in \Lambda_1$. It follows that $r \in k - \Lambda_1 \in \Lambda_1^{(2)} - \Lambda_1$. Since $m - r \in \Lambda_2^{(1)}$, it follows that $m \in \Lambda_2^{(1)} + r \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1$. On the other hand $k + u \in \Lambda_1^{(2)}$. Hence $u \in \Lambda_1^{(2)} - \Lambda_1^{(2)}$ and $m + u \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1 + \Lambda_1^{(2)} - \Lambda_1^{(2)}$. Let $\tilde{\Lambda}_i = \Lambda' + \Lambda' + \Lambda' + \Lambda' + \Lambda_1 + i$. Then $m, m+u \in \tilde{\Lambda}_i + j = Q_{ij} = Q$. Using Lemma 2.3 for Bohr set Λ_1 and its ε–attendant Λ', we obtain that the cardinality of $\tilde{\Lambda}_i$ does not exceed $5|\Lambda_1|$. Using the Cauchy–Schwarz inequality, we get

$$|\sigma|^4 \le \|h\|_2^4 \Big(\sum_k \sum_{m,u} g(k,m) g(k+u, m+u) \Big) \tag{3.30}$$

$$\cdot \Big(\sum_{k,m,u} E_1^{(2)}(k) E_1^{(2)}(k+u) \sum_{r,r'} E_2^{(1)}(m-r) E_2^{(1)}(m-r') \times$$

$$\times f(k-r, m) f(k-r, m+u) f(k-r', m) f(k-r', m+u) \Big)$$

Let us estimate $\sigma^* = \sigma^*_{ij} = \sum_{k,m,u} g(k,m) g(k+u, m+u)$. Let $\tilde{E}_2^{(2)} = E_2 \cap Q$. We have

$$\sigma^* = \sum_{k,m,u} g(k,m) g(k+u, m+u) \le \sum_{k,m,u} E_1^{(2)}(k) E_1^{(2)}(k+u) \tilde{E}_2^{(2)}(m) \tilde{E}_2^{(2)}(m+u)$$

$$= \sum_{k,m,u} E_1^{(2)}(k) E_1^{(2)}(u) \tilde{E}_2^{(2)}(m) \tilde{E}_2^{(2)}(m+u-k) = \tag{3.31}$$

$$= \sum_{k,m,u} E_1^{(2)}(k)\tilde{E}_2^{(2)}(m+k)E_1^{(2)}(u)\tilde{E}_2^{(2)}(m+u) = \qquad (3.32)$$

$$= \sum_m (E_1^{(2)} * \tilde{E}_2^{(2)})^2(m)\,. \qquad (3.33)$$

Recall that $|Q_{ij}| \le 5|\Lambda_1|$. Lemma 2.3 implies that m in the sum (3.33) belongs to a set of cardinality at most $8|\Lambda_1|$. The expression (3.33) implies that for all i,j we have

$$|\sigma^*_{ij}| \le 8|\Lambda'|^2|\Lambda_1|\,. \qquad (3.34)$$

We need a stronger upper bound for σ^*_{ij}. Let

$$\Omega^{'} = \{ s \in \Lambda_2 \mid |\delta_{\Lambda_1+s}(E_2) - \beta_2| \ge 4\alpha_0^{1/2} \text{ or}$$

$$\frac{1}{|\Lambda_1|} \sum_{n\in\Lambda_1+s} |\delta_{\Lambda'+n}(E_2) - \beta_2|^2 \ge 4\alpha_0^{1/2} \},\ \text{and } G^{'} = \Lambda_2 \setminus \Omega^{'}\,.$$

By assumption Λ_1 is ε_0–attendant of Λ_2 and E_2 is (α_0,ε)–uniform subset of Λ_2. Using Lemma 3.3, we get $|\Omega^{'}| \le 8\alpha_0^{1/2}|\Lambda_2|$. Let $\tilde{\Lambda} = \Lambda' + \Lambda' + \Lambda' + \Lambda' + \Lambda_1$. Since Λ' is ε–attendant of Λ_1, it follows that for any $s \in G^{'}$ we have $|\delta_{\tilde{\Lambda}+s}(E_2) - \beta_2| < 8\alpha_0^{1/2}$ and $\sum_{n\in\tilde{\Lambda}+s} |\delta_{\Lambda'+n}(E_2) - \beta_2|^2 < 8\alpha_0^{1/2}|\tilde{\Lambda}|$. For an arbitrary $i \in \Lambda_1$ consider the set

$$\Omega^* = \Omega^*_i = \{\ j \in \Lambda_2 \mid |\delta_{\tilde{\Lambda}_i+j}(E_2) - \beta_2| \ge 8\alpha_0^{1/2} \text{ or}$$

$$\frac{1}{|\tilde{\Lambda}_i|} \sum_{n\in\tilde{\Lambda}_i+j} |\delta_{\Lambda'+n}(E_2) - \beta_2|^2 \ge 8\alpha_0^{1/2}\ \}\,. \qquad (3.35)$$

Since $(\Lambda_2 \setminus \Omega^*_i) \supseteq (\Lambda_2 \cap (G^{'} - i))$, it follows that $\Omega^*_i \subseteq (\Lambda_2 \setminus (G^{'} - i))$. Since Λ_1 is ε_0–attendant of Λ_2, it follows that $|\Lambda_2 \setminus (G^{'} - i)| = |(\Lambda_2 + i) \setminus G^{'}| \ge |\Lambda_2^- \cap G^{'}| \ge (1 - 8\alpha_0^{1/2} - 8\kappa_0)|\Lambda_2|$, $\kappa_0 \le \alpha_0^2$. Hence $|\Omega^*_i| \le 8\alpha_0^{1/2}|\Lambda_2| + 8\kappa_0|\Lambda_2| \le 16\alpha_0^{1/2}|\Lambda_2|$. This yields

$$\frac{1}{|\Lambda'|^2} \sum_{i\notin\Omega_1, j\in\Omega^*_i} |\sigma_{ij}| \le \frac{1}{|\Lambda'|^2} \sum_{i\notin\Omega_1} (16\alpha_0^{1/2}|\Lambda_2|2|\Lambda'|^2|\Lambda_1|) \le 32\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2|\,. \quad (3.36)$$

We have $j \notin \Omega_2$. Suppose in addition that $j \notin \Omega^*_i$. Using (3.3), we get

$$\sigma^*_{ij} \le \int_0^1 |\widehat{E}_1^{(2)}(x)|^2 |\widehat{\tilde{E}}_2^{(2)}(x)|^2 dx\,. \qquad (3.37)$$

Since $E_1^{(2)}$ is α_0–uniform, it follows that

$$\widehat{E}_1^{(2)}(x) = \beta_1^{(2)} \widehat{\Lambda}_1^{(2)}(x) + \vartheta_1 \alpha_0 |\Lambda'|\,, \qquad (3.38)$$

where $|\vartheta_1| \le 1$. We have $|\widehat{\Lambda}_1^{(2)}(x)| \le |\Lambda'|$. Combining (3.38) and (3.37), we obtain

$$\sigma^*_{ij} \le (\beta_1^{(2)})^2 \int_0^1 |\widehat{\Lambda}_1^{(2)}(x)|^2 |\widehat{\tilde{E}}_2^{(2)}(x)|^2 dx + 3\alpha_0 |\Lambda'|^2 \int_0^1 |\widehat{\tilde{E}}_2^{(2)}(x)|^2 dx \le$$

$$\le (\beta_1^{(2)})^2 \sum_m (\Lambda_1^{(2)} * \tilde{E}_2^{(2)})^2(m) + 15\alpha_0 |\Lambda'|^2 |\Lambda_1| \,. \tag{3.39}$$

Let $\tilde{\beta}_2^j = \delta_Q(\tilde{E}_2^{(2)})$. Since $j \notin \Omega_i^*$, it follows that $|\tilde{\beta}_2^j - \beta_2| \le 8\alpha_0^{1/2}$ and

$$\sum_m (\Lambda_1^{(2)} * \tilde{E}_2^{(2)})^2(m) \le 4(\beta_1^{(2)})^2 \beta_2^2 |\Lambda'|^2 |\Lambda_1| + 200\alpha_0^{1/2} |\Lambda'|^2 |\Lambda_1| \,.$$

This implies that

$$\sigma^*_{ij} \le 4(\beta_1^{(2)})^2 \beta_2^2 |\Lambda'|^2 |\Lambda_1| + 200\alpha_0^{1/2} |\Lambda'|^2 |\Lambda_1| + 15\alpha_0 |\Lambda'|^2 |\Lambda_1| \le$$

$$\le 4(\beta_1^{(2)})^2 \beta_2^2 |\Lambda'|^2 |\Lambda_1| + 256\alpha_0^{1/2} |\Lambda'|^2 |\Lambda_1| \,. \tag{3.40}$$

Since $i \notin \Omega_1$, it follows that $\beta_1/2 \le \beta_1^{(2)} \le 2\beta_1$. Hence $256\alpha_0^{1/2} \le (\beta_1^{(2)})^2 \beta_2^2$. Consequently for all $i \notin \Omega_1$, $j \notin \Omega_2 \cup \Omega_i^*$, we obtain

$$\sigma^*_{ij} \le 128\ \beta_1^2 \beta_2^2 |\Lambda'|^2 |\Lambda_1| \,. \tag{3.41}$$

We have

$$|\sigma|^4 \le \|h\|_2^4 \cdot \sigma^* \cdot \sum_{m,u} \sum_{r,r'} f(r,m) f(r,u) f(r',m) f(r',u) \ \cdot \tag{3.42}$$

$$\sum_k E_1^{(2)}(k) E_1^{(2)}(k-m+u) E_2^{(1)}(m-k+r) E_2^{(1)}(m-k+r') = \tag{3.43}$$

$$= \|h\|_2^4 \cdot \sigma^* \cdot \sum_{m,u} \sum_{r,r'} f(r,m) f(r,u) f(r',m) f(r',u) \ \cdot \tag{3.44}$$

$$\sum_k E_1^{(2)}(m-k) E_1^{(2)}(u-k) E_2^{(1)}(k+r) E_2^{(1)}(k+r') = \|h\|_2^4 \cdot \sigma^* \cdot \sigma' \,. \tag{3.45}$$

Rewrite σ' as

$$\sigma' = \sum_k \sum_{r,r'} E_2^{(1)}(k+r) E_2^{(1)}(k+r') |\sum_m E_1^{(2)}(m-k) f(r,m) f(r',m)|^2 \tag{3.46}$$

We have $r \in \Lambda_1$ and $k + r \in \Lambda_2^{(1)}$. It follows that $k \in \Lambda_2^{(1)} - \Lambda_1$. On the other hand $m - k \in \Lambda_1^{(2)}$. Hence $m \in \Lambda_1^{(2)} + k \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1$. By symmetry u belongs to $\Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1$. Using Lemma 2.3 for Λ_1 and its ε–attendant Λ', we obtain that

k and m, u belongs to some translations of *Bohr* sets $W_1 = \Lambda_1^+$ and $W_2 = W_1^+$, respectively, and the cardinalities of these sets do not exceed $3|\Lambda_1|$.

If k is fixed, then m, u, r, r' in (3.45) run some sets of the cardinalities at most $|\Lambda'|$.

Let $\Phi^1_{r,r'}(m) = f(r,-m)f(r',-m)W_2(m-i-j)$,
$\Phi^2_{r,r'}(u) = f(r,-u)f(r',-u)W_2(u-i-j)$, $\Phi^3_{m,u}(r) = f(-r,m)f(-r,u)$, and
$\Phi^4_{m,u}(r') = f(r',m)f(r',u)$. Consider the sets

$$B_1 = \{k \mid |(\Phi^1_{r,r'} * E_1^{(2)})(-k) - \beta_1^{(2)}(\Phi^1_{r,r'} * \Lambda_1^{(2)})(-k)| \ge \alpha_0^{2/3}|\Lambda'|\}$$

$$B_2 = \{k \mid |(\Phi^2_{r,r'} * E_1^{(2)})(-k) - \beta_1^{(2)}(\Phi^2_{r,r'} * \Lambda_1^{(2)})(-k)| \ge \alpha_0^{2/3}|\Lambda'|\}$$

$$B_3 = \{k \in \Lambda_1 \mid |(\Phi^3_{m,u} * E_2^{(1)})(k) - \beta_2^{(1)}(\Phi^3_{m,u} * \Lambda_2^{(1)})(k)| \ge \alpha_0^{2/3}|\Lambda'|\}$$

$$B_4 = \{k \in \Lambda_1 \mid |(\Phi^4_{m,u} * E_2^{(1)})(k) - \beta_2^{(1)}(\Phi^4_{m,u} * \Lambda_2^{(1)})(k)| \ge \alpha_0^{2/3}|\Lambda'|\}\,.$$

We have $i \notin \Omega_1$, $j \notin \Omega_2$. Using Corollary 3.1, we get $|B_1|, |B_2| \le 3\alpha_0^{2/3}|\Lambda_1|$ and $|B_3|, |B_4| \le \alpha_0^{2/3}|\Lambda_1|$. Let $B = B_1 \bigcup B_2 \bigcup B_3 \bigcup B_4$. Then $|B| \le 8\alpha_0^{2/3}|\Lambda_1|$. Split σ' as

$$\sigma' = \sum_{k\in B}\sum_{r,r'} E_2^{(1)}(k+r)E_2^{(1)}(k+r')|\sum_m E_1^{(2)}(m-k)f(r,m)f(r',m)|^2 +$$

$$+ \sum_{k\notin B}\sum_{r,r'} E_2^{(1)}(k+r)E_2^{(1)}(k+r')|\sum_m E_1^{(2)}(m-k)f(r,m)f(r',m)|^2 = \sigma_1 + \sigma_2$$

Let us estimate σ_1. Since $|B| \le 8\alpha_0^{2/3}|\Lambda_1|$, it follows that

$$|\sigma_1| \le 8\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1|\,. \tag{3.47}$$

If $k \notin B$, then $k \notin B_1$. This implies that

$$\sigma_2 = \sum_{k\notin B}\sum_u\sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r')\,\cdot$$

$$\sum_m f(r,m)f(r',m)E_1^{(2)}(m-k) =$$

$$= \sum_{k\notin B}\sum_u\sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r')(\Phi^1_{r,r'} * E_1^{(2)})(-k)$$

$$= \beta_1^{(2)}\sum_{k\notin B}\sum_u\sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r')\,\cdot$$

$$\sum_m f(r,m)f(r',m)\Lambda_1^{(2)}(m-k)+$$

$$+\vartheta\alpha_0^{2/3}|\Lambda'|\sum_{k\notin B}\sum_u\sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r')$$

$$=\beta_1^{(2)}\sum_{k\notin B}\sum_u\sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r')\cdot$$

$$\sum_m f(r,m)f(r',m)\Lambda_1^{(2)}(m-k)+4\vartheta\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1|\,, \tag{3.48}$$

where $|\vartheta|\le 1$. Using these arguments for B_2, B_3 and B_4, we get

$$|\sigma_2|\le(\beta_1^{(2)})^2(\beta_2^{(1)})^2\sum_{m,u}\sum_{r,r'} f(r,m)f(r,u)f(r',m)f(r',u)\cdot$$

$$\sum_k \Lambda_1^{(2)}(m-k)\Lambda_1^{(2)}(u-k)\Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r')+16\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1|\,, \tag{3.49}$$

It follows that

$$|\sigma'|\le|\sigma_1|+|\sigma_2|\le(\beta_1^{(2)})^2(\beta_2^{(1)})^2\sum_{m,u}\sum_{r,r'} f(r,m)f(r,u)f(r',m)f(r',u)\cdot$$

$$\sum_k \Lambda_1^{(2)}(m-k)\Lambda_1^{(2)}(u-k)\Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r')+32\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1|\,. \tag{3.50}$$

Using (3.45), we obtain

$$|\sigma|^4\le\|h\|_2^4\cdot\sigma^*\cdot(\beta_1^{(2)})^2(\beta_2^{(1)})^2\sum_k\sum_{r,r'}\Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r')\cdot$$

$$\Big|\sum_m \Lambda_1^{(2)}(m-k)f(r,m)f(r',m)\Big|^2+32\|h\|_2^4\cdot\sigma^*\cdot\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1| \tag{3.51}$$

We have

$$\|h\|_2^2=\sum_{k,m}h(k,m)\le\sum_{k,m}E_1^{(1)}(k)E_2^{(1)}(m)=\beta_1\beta_2^{(1)}|\Lambda'||\Lambda_1|\,. \tag{3.52}$$

Since $i\notin\Omega_1$, $j\notin\Omega_2$, it follows that $\beta_1^{(2)}\le 2\beta_1$ and $\beta_2^{(1)}\le 2\beta_2$. Combining the estimates of $\|h\|_2^2$ and σ^* with (3.51), we get

$$|\sigma_{ij}|^4\le 2^{15}\beta_1^6\beta_2^6|\Lambda'|^4|\Lambda_1|^3\sum_k\sum_{r,r'}\Lambda_2^{(1)}(k+r)\Lambda_2^{(1)}(k+r')\cdot$$

$$\Big|\sum_m \Lambda_1^{(2)}(m-k)f(r,m)f(r',m)\Big|^2+2^{15}\alpha_0^{2/3}|\Lambda'|^8|\Lambda_1|^4\,. \tag{3.53}$$

Let $\Omega_2'=\Omega_2'(i)=\Omega_2\cup\Omega_i^*$. We have

$$\Big(\sum_{i\notin\Omega_1,j\notin\Omega_2'(i)}|\sigma_{i,j}|\Big)^4\le(|\Lambda_1||\Lambda_2|)^3\sum_{i\notin\Omega_1,j\notin\Omega_2'(i)}|\sigma_{i,j}|^4\le$$

$$\leq 2^{15}\beta_1^6\beta_2^6|\Lambda'|^4|\Lambda_1|^3(|\Lambda_1||\Lambda_2|)^3 \sum_{i\notin\Omega_1, j\notin\Omega_2'(i)} \sum_k \sum_{r,r'} \mu_j(k+r)\mu_j(k+r')\cdot$$

$$\Big|\sum_m \lambda_i(m-k)f(r,m)f(r',m)\Big|^2 + 2^{15}\alpha_0^{2/3}(|\Lambda_1||\Lambda_2|)^4|\Lambda'|^8|\Lambda_1|^4 \leq$$

$$\leq 2^{15}\beta_1^6\beta_2^6|\Lambda'|^4|\Lambda_1|^3(|\Lambda_1||\Lambda_2|)^3 \sum_{i\in\Lambda_1, j\in\Lambda_2} \sum_k \sum_{r,r'} \mu_j(k+r)\mu_j(k+r')\cdot$$

$$\Big|\sum_m \lambda_i(m-k)f(r,m)f(r',m)\Big|^2 + 2^{15}\alpha_0^{2/3}(|\Lambda_1||\Lambda_2|)^4|\Lambda'|^8|\Lambda_1|^4 .$$

By assumption the function f is rectilinearly (α,ε)–uniform. It follows that

$$\Big(\sum_{i\notin\Omega_1, j\notin\Omega_2'(i)} |\sigma_{i,j}|\Big)^4 \leq 2^{15}\alpha\beta_1^8\beta_2^8|\Lambda'|^8|\Lambda_1|^8|\Lambda_2|^4 + 2^{15}\alpha_0^{2/3}|\Lambda'|^8|\Lambda_1|^8|\Lambda_2|^4 \leq$$

$$\leq 2^{16}\alpha\beta_1^8\beta_2^8|\Lambda_1|^8|\Lambda_2|^4 . \tag{3.54}$$

Hence

$$\sum_{i\notin\Omega_1, j\notin\Omega_2'(i)} |\sigma_{i,j}| \leq 2^4\alpha^{1/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2| . \tag{3.55}$$

Using (3.25), (3.36) and (3.23), we have

$$|\sigma_0| \leq 16\kappa|\Lambda_1|^2|\Lambda_2| + 8\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2| + 32\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2| + 2^4\alpha^{1/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2| \leq$$

$$\leq 2^5\alpha^{1/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2| \tag{3.56}$$

as required.

The next result is the main in this section.

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \leq \Lambda_2$, $\Lambda_1 = \Lambda_{\theta,\varepsilon_1,N_1}$, $\theta \in \mathbf{T}^d$, and $E_1 \subseteq \Lambda_1$, $E_2 \subseteq \Lambda_2$, $|E_1| = \beta_1|\Lambda_1|$, $|E_2| = \beta_2|\Lambda_2|$. Let $\mathcal{P}$ be a product set $E_1 \times E_2$.

THEOREM 3.2. *Let A be an arbitrary subset of $E_1 \times E_2$ of cardinality $\delta|E_1||E_2|$. Suppose that the sets E_1, E_2 are $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform, $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$, $\varepsilon = (2^{-100}\alpha_0^2)/(100d)$. Let A be rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform, $\alpha = 2^{-100}\delta^{12}$, $\alpha_1 = 2^{-7}\delta$, and*

$$\log N_1 \geq 2^{10} d \log \frac{1}{\varepsilon_1\varepsilon} . \tag{3.57}$$

Then A contains a triple $\{(k,m), (k+d,m), (k,m+d)\}$, where $d \neq 0$.

Proof. Let Λ' be ε–attendant set of Λ_1, and $\lambda_i = \Lambda' + i$, $i \in \Lambda_1$. Let $G_i = (\lambda_i \times \Lambda_2) \cap A$, $f_i(\vec{s}) = f(s_1 + i, s_2)\Lambda'(s_1, s_2)$, $i \in \Lambda_1$. By G_i denote the characteristic functions of the sets G_i. Let

$$B_1 = \{i \in \Lambda_1 \mid E_1 \cap \lambda_i \text{ is not } (8\alpha_0^{1/4}, \varepsilon)\text{–uniform}\}\,,$$

$$B_2 = \{i \in \Lambda_1 \mid |\delta_{\lambda_i}(E_1) - \beta_1| \ge 4\alpha_0^{1/2}\}\,,$$

$$B_3 = \{i \in \Lambda_1 \mid \|f_i\|^4_{\Lambda' \times \Lambda_2, \varepsilon} > \alpha\beta_1^2\beta_2^2 |\Lambda'(\varepsilon)|^4 |\Lambda'|^2 |\Lambda_2|\}, \text{ and } B = B_1 \cup B_2 \cup B_3\,.$$

By assumption E_1 is (α_0, ε)–uniform. By Lemma 3.3, we get $|B_1| \le 8\alpha_0^{1/4}|\Lambda_1|$. and $|B_2| \le 8\alpha_0^{1/4}|\Lambda_1|$. Since A is rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform, it follows that $|B_3| \le \alpha_1|\Lambda_1|$. Hence $|B| \le 16\alpha_0^{1/4}|\Lambda_1| + \alpha_1|\Lambda_1| \le 2\alpha_1|\Lambda_1|$.

Let $\vec{e} = \vec{e}_1 + \vec{e}_2$, and $\vec{s} = x\vec{e}_1 + y\vec{e}$. Using Lemma 2.3, we obtain

$$A(\vec{s}) = \frac{1}{|\Lambda'|} \cdot \sum_{i \in \Lambda_1} G_i(\vec{s}) + \epsilon(\vec{s})\,, \tag{3.58}$$

where $\kappa = \alpha_0^2$. onsider the sum

$$\sigma = \frac{1}{|\Lambda'|} \sum_{i \in \Lambda_1} \sum_{\vec{s}} G_i(\vec{s})\,. \tag{3.59}$$

We have $|A| = \delta\beta_1\beta_2|\Lambda_1||\Lambda_2|$. Using (3.58), we get

$$\sigma \ge \frac{\delta\beta_1\beta_2}{4}|\Lambda_1||\Lambda_2|\,. \tag{3.60}$$

Split σ as

$$\sigma = \frac{1}{|\Lambda'|} \sum_{i \in B} \sum_{\vec{s}} G_i(\vec{s}) + \frac{1}{|\Lambda'|} \sum_{i \notin B} \sum_{\vec{s}} G_i(\vec{s}) = \sigma_1 + \sigma_2\,. \tag{3.61}$$

Let us estimate σ_1. For any $i \in \Lambda_1$ we have $\sum_{\vec{s}} G_i(\vec{s}) \le \beta_2|\Lambda'||\Lambda_2|$. If $i \notin B_2$, then $\sum_{\vec{s}} G_i(\vec{s}) \le 2\beta_1\beta_2|\Lambda'||\Lambda_2|$. It follows that

$$\sigma_1 \le \frac{1}{|\Lambda'|} \sum_{i \in B \cap B_2} \sum_{\vec{s}} G_i(\vec{s}) + \frac{1}{|\Lambda'|} \sum_{i \in B, i \notin B_2} \sum_{\vec{s}} G_i(\vec{s}) \le$$

$$\le 8\alpha_0^{1/4}|\Lambda_1||\Lambda_2| + 4\alpha_1\beta_1\beta_2|\Lambda_1||\Lambda_2|\,. \tag{3.62}$$

By assumption $\alpha_0^{1/4} \le \alpha_1\beta_1\beta_2$. This implies that

$$\sigma_1 \le 8\alpha_1\beta_1\beta_2|\Lambda_1||\Lambda_2| + 4\alpha_1\beta_1\beta_2|\Lambda_1||\Lambda_2| < 16\alpha_1\beta_1\beta_2|\Lambda_1||\Lambda_2|\,. \tag{3.63}$$

We have $\alpha_1 = 2^{-7}\delta$. Using this and (3.60), (3.63), we obtain

$$\frac{1}{|\Lambda'|}\sum_{i\notin B}\sum_{\vec{s}} G_i(\vec{s}) \ge \frac{\delta\beta_1\beta_2}{8}|\Lambda_1||\Lambda_2|\,. \tag{3.64}$$

The formula (3.64) implies that there exists $i_0 \notin B$ such that

$$\sum_{\vec{s}} G_{i_0}(\vec{s}) \ge \frac{\delta\beta_1\beta_2}{8}|\Lambda'||\Lambda_2| = 2^{-3}\delta\beta_1\beta_2|\Lambda'||\Lambda_2|\,. \tag{3.65}$$

Let $G(\vec{s}) = G_{i_0}(\vec{s})$. We have

$$\sum_k\sum_m G(k,m) \ge 2^{-3}\delta\beta_1\beta_2|\Lambda'||\Lambda_2|\,. \tag{3.66}$$

We have $m \in \Lambda_2$ and $k+m \in \lambda_i$. It follows that $k \in \lambda_i - \Lambda_2$. Using Lemma 2.3 we obtain that k belongs to a set of cardinality at most $2|\Lambda_2|$. By the Cauchy–Schwarz inequality, we get

$$2^{-6}\delta^2\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|^2 \le \sum_k\Big(\sum_m G(k,m)\Big)^2 \cdot 2|\Lambda_2|\,. \tag{3.67}$$

It follows that

$$\sum_k\Big(\sum_m G(k,m)\Big)^2 = \sum_k\sum_{m,p} G(k,m)G(k,p) \ge 2^{-7}\delta^2\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|\,. \tag{3.68}$$

Let $\vec{s} = x\vec{e}_1 + y\vec{e}_2$. Consider the sum

$$\sigma_0 = \sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e})A(\vec{s}+r\vec{e}_2)\,. \tag{3.69}$$

Then

$$\sigma_0 = \delta\sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e})\mathcal{P}(\vec{s}+r\vec{e}_2) + \sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e})f(\vec{s}+r\vec{e}_2) \tag{3.70}$$

$$= \delta\sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e}) + \sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e})f(\vec{s}+r\vec{e}_2) \tag{3.71}$$

Let us estimate the second term in (3.71). Let $\overline{f}(\vec{s}) = f(\vec{s})$ if $\vec{s} \in \lambda_i \times \Lambda_2$ and $\overline{f}(\vec{s}) = 0$ otherwise. We have

$$G(\vec{s})G(\vec{s}+r\vec{e})f(\vec{s}+r\vec{e}_2) = G(\vec{s})G(\vec{s}+r\vec{e})\overline{f}(\vec{s}+r\vec{e}_2)\,. \tag{3.72}$$

It follows that

$$\sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e})f(\vec{s}+r\vec{e}_2) = \sum_{\vec{s}}\sum_r G(\vec{s})G(\vec{s}+r\vec{e})\overline{f}(\vec{s}+r\vec{e}_2) =$$

$$= \sum_{\vec{s}}\sum_r G(\vec{s}+i_0\vec{e}_1)G(\vec{s}+r\vec{e}+i_0\vec{e}_1)f(\vec{s}+r\vec{e}_2+i_0\vec{e}_1) =$$

$$= \sum_{\vec{s}} \sum_{r} G(\vec{s} + i_0\vec{e}_1) G(\vec{s} + r\vec{e} + i_0\vec{e}_1) f_{i_0}(\vec{s} + r\vec{e}_2) \tag{3.73}$$

Since $i_0 \notin B$, it follows that $\|f_{i_0}\|^4 \le \alpha\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$ and $\delta_{\lambda_{i_0}}(E_1) \le 2\beta_1$. By assumption $\alpha = 2^{-100}\delta^{12}$. By Theorem 3.1 the second term in (3.71) does not exceed $2^{10}\alpha^{1/4}\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2| \le 2^{-8}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$. The inequality (3.68) implies that the first term in (3.71) is greater then $2^{-7}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$. It follows that $\sigma_0 \ge 2^{-8}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$.

The sum (3.69) is the number of triples $\{(k,m),(k+d,m),(k,m+d)\}$, where $k \in \Lambda_{i_0}$, $m \in \Lambda_2$, $d \in \mathbf{Z}$. The number of triples with $d = 0$ does not exceed $|\Lambda'||\Lambda_2|$. By assumption $\log N_1 \ge 2^{10} d \log \frac{1}{\varepsilon_1\varepsilon}$. Using Lemma 2.1, we get $|\Lambda'| > 2^8(\delta^3\beta_1^2\beta_2^2)^{-1}$. Hence, $2^{-8}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2| > |\Lambda'||\Lambda_2|$. It follows that A contains a triple $\{(k,m),(k+d,m),(k,m+d)\}$ with $d \ne 0$. This completes the proof.

4. *Non–uniform case.*

LEMMA 4.1. *Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, and Λ' be ε–attendant set of Λ_1, $\varepsilon = \kappa/(100d)$. Let set A be a subset of $C \subseteq \Lambda_1 \times \Lambda_2$ of cardinality $\delta|C|$. By B define the set of $s \in \Lambda_1$ such that $|A \cap ((\Lambda' + s) \times \Lambda_2)| < (\delta - \eta)|C \cap ((\Lambda' + s) \times \Lambda_2)|$, where $\eta > 0$. Then*

$$\sum_{s \in (\Lambda_1 \setminus B)} |A \cap ((\Lambda' + s) \times \Lambda_2)| \ge \delta \sum_{s \in (\Lambda_1 \setminus B)} |C \cap ((\Lambda' + s) \times \Lambda_2)| +$$

$$+ \eta \sum_{s \in B} |C \cap ((\Lambda' + s) \times \Lambda_2)| - 4\kappa|\Lambda'||\Lambda_1||\Lambda_2|\,.$$

Proof. Let $\vec{s} = k\vec{e}_1 + m\vec{e}_2$. Using Lemma 2.3, we get

$$\delta|C| = \sum_{\vec{s}} A(\vec{s})\Lambda_1(k)\Lambda_2(m) = \frac{1}{|\Lambda'|} \sum_{n \in \Lambda_1} \sum_{\vec{s}} A(\vec{s})((\Lambda' + n) \times \Lambda_2)(\vec{s}) + 2\vartheta\kappa|\Lambda_1||\Lambda_2|\,, \tag{4.1}$$

where $|\vartheta| \le 1$. Split the sum (4.1) into a sum over $n \in B$ and a sum over $n \in \Lambda_1 \setminus B$. We have

$$\delta|C| < \frac{1}{|\Lambda'|}(\delta - \eta) \sum_{n \in B} |C \cap ((\Lambda' + n) \times \Lambda_2)| +$$

$$+\frac{1}{|\Lambda'|}\sum_{n\in(\Lambda_1\setminus B)}|A\cap((\Lambda'+n)\times\Lambda_2)|+2\kappa|\Lambda_1||\Lambda_2|\,. \qquad (4.2)$$

In the same way

$$|C|=\frac{1}{|\Lambda'|}\sum_{n\in B}|C\cap((\Lambda'+n)\times\Lambda_2)|+\frac{1}{|\Lambda'|}\sum_{n\in(\Lambda_1\setminus B)}|C\cap((\Lambda'+n)\times\Lambda_2)|+2\vartheta_1\kappa|\Lambda_1||\Lambda_2|\,, \qquad (4.3)$$

where $|\vartheta_1|\le 1$. Combining (4.2) and (4.3), we obtain the required result.

Proposition 4.1 (B. Green). *Let A be a subset of $E_1\times E_2$ of cardinality $|A|=\delta|E_1||E_2|$. Suppose that $\alpha>0$ is a real number, and A is not rectilinearly α–uniform. Then there are two sets $F_1\subseteq E_1$ and $F_2\subseteq E_2$ such that*

$$|A\bigcap(F_1\times F_2)|>(\delta+2^{-14}\alpha^2)|F_1||F_2| \quad \textit{and} \qquad (4.4)$$

$$|F_1|\ge 2^{-8}\alpha|E_1|\,,\quad |F_2|\ge 2^{-8}\alpha|E_2|\,. \qquad (4.5)$$

In [**16**] the author used spectral methods to prove Proposition 4.1. His proof gives worse constants than (4.4), (4.5). B. Green [**11**] took a more simple approach, which provided better bounds.

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1\le\Lambda_2$, $\Lambda_1=\Lambda_{\theta,\varepsilon_0,N}$, $\theta\in\mathbf{T}^d$, and $E_1\subseteq\Lambda_1$, $E_2\subseteq\Lambda_2$, $|E_1|=\beta_1|\Lambda_1|$, $|E_2|=\beta_2|E_2|$. Let $\mathcal{P}$ be a product set $E_1\times E_2$.

Theorem 4.1. *Let A be a subset of $\mathcal{P}$ of cardinality $|A|=\delta|E_1||E_2|$. Suppose that A has no triples $\{(k,m),(k+d,m),(k,m+d)\}$ with $d\ne 0$, E_1,E_2 are $(\alpha_0,2^{-10}\varepsilon^2)$–uniform, $\alpha_0=2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$, $\varepsilon=(2^{-100}\alpha_0^2)/(100d)$, $\varepsilon'=2^{-10}\varepsilon^2$, and*

$$\log N\ge 2^{10}d\log\frac{1}{\varepsilon_0\varepsilon}\,.$$

Then there exists a Bohr set $\tilde\Lambda$, two sets F_1, F_2 and a vector $\vec{y}=(y_1,y_2)\in\mathbf{Z}^2$, $F_1\subseteq E_1\cap(\tilde\Lambda+y_1)$, $F_2\subseteq E_2\cap(\tilde\Lambda+y_2)$ such that

$$|F_1|\ge 2^{-125}\delta^{12}\beta_1|\tilde\Lambda|,\quad |F_2|\ge 2^{-125}\delta^{12}\beta_2|\tilde\Lambda| \quad \textit{and} \qquad (4.6)$$

$$\delta_{F_1\times F_2}(A)\ge\delta+2^{-500}\delta^{37}\,. \qquad (4.7)$$

Besides that for $\tilde{\Lambda} = \Lambda_{\tilde{\theta},\tilde{\varepsilon},\tilde{N}}$ *we have* $\tilde{\theta} = \theta$, $\tilde{\varepsilon} \ge 2^{-5}\varepsilon'\varepsilon_0$ *and* $\tilde{N} \ge 2^{-5}\varepsilon' N$.

Proof. Let Λ' be ε–attendant of Λ_1, and Λ'' be ε–attendant of Λ'. Suppose that A is rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform, $\alpha = 2^{-100}\delta^{12}$, $\alpha_1 = 2^{-7}\delta$. Using Theorem 3.2, we obtain that A contains a triple $\{(k,m),(k+d,m),(k,m+d)\}$ with $d \ne 0$. Hence, the set A is not rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform.

Let

$$B_1 = \{ s \in \Lambda_1 \mid |\delta_{\Lambda'+s}(E_1) - \beta_1| \ge 4\alpha_0^{1/2} \},$$

$$B_2 = \{ s \in \Lambda_1 \mid \Lambda' \cap (E_1 - s) \text{ is not } (8\alpha_0^{1/4}, \varepsilon)\text{–uniform} \},$$

and

$$B = \{ i \in \Lambda_1 \mid \|f_i\|^4_{\Lambda' \times \Lambda_2, \varepsilon} > \alpha\beta_1^2\beta_2^2|\Lambda'(\varepsilon)|^4|\Lambda'|^2|\Lambda_2| \}.$$

Since A is not rectilinearly $(\alpha, \varepsilon, \varepsilon')$–uniform, it follows that $|B| > \alpha_1|\Lambda_1|$. By assumption E_1, E_2 are (α_0, ε')–uniform. Using Lemma 3.3, we obtain $|B_1| \le 4\alpha_0^{1/2}|\Lambda_1|$, $|B_2| \le 8\alpha_0^{1/2}|\Lambda_1|$. Let $B_3 = B_1 \cup B_2$. Then $|B_3| \le 12\alpha_0^{1/2}|\Lambda_1|$. Let $B' = B \setminus B_3$. Since $32\alpha_0^{1/2} < \alpha_1$, it follows that $|B'| \ge \alpha_1|\Lambda_1|/2$. Note that for all $l \in B'$ we have

$$|\delta_{\Lambda'+s}(E_1) - \beta_1| < 4\alpha_0^{1/2}. \tag{4.8}$$

Let $\eta = 2^{-100}\alpha^3$. Let $\lambda_l = \Lambda' + l$, $l \in \Lambda_1$. Suppose that for any $l \in B'$ we have

$$|A \cap (\lambda_l \times \Lambda_2)| \le (\delta - \eta)|\lambda_l \cap E_1||\Lambda_2 \cap E_2|. \tag{4.9}$$

Let $B'^c = \Lambda_1 \setminus B'$. Using Lemma 4.1 and (4.8), we get

$$\sum_{l \in B'^c} |A \cap (\lambda_l \times \Lambda_2)| \ge \delta|\Lambda_2 \cap E_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + \eta|\Lambda_2 \cap E_2| \sum_{l \in B'} |\lambda_l \cap E_1| - \alpha_0^2|\Lambda'||\Lambda_1||\Lambda_2|$$

$$\ge \delta\beta_2|\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + \eta\frac{\alpha_1|\Lambda_1|}{2}\frac{\beta_1|\Lambda'|}{4}\beta_2|\Lambda_2| \ge$$

$$\ge \delta\beta_2|\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + 2^{-3}\alpha_1\eta\beta_1\beta_2|\Lambda'||\Lambda_1||\Lambda_2|. \tag{4.10}$$

We have

$$\sum_{l \in B_1} |A \cap (\lambda_l \times \Lambda_2)| \le 4\alpha_0^{1/2}|\Lambda_1||\Lambda'||\Lambda_2| \le 2^{-4}\alpha_1\eta\beta_1\beta_2|\Lambda'||\Lambda_1||\Lambda_2|. \tag{4.11}$$

Combining (4.10) and (4.11), we obtain

$$\sum_{l\in(B'^c\setminus B_1)} |A\cap(\lambda_l\times\Lambda_2)| \ge \delta\beta_1|\Lambda_2| \sum_{l\in B'^c} |\lambda_l\cap E_1| + 2^{-4}\alpha_1\eta\beta_1\beta_2|\Lambda'||\Lambda_1||\Lambda_2| \,. \quad (4.12)$$

This implies that, there exists a number $l \in B'^c \setminus B_1$ such that

$$|A\cap(\lambda_l\times\Lambda_2)| > (\delta + 2^{-5}\alpha_1\eta)|\lambda_l\cap E_1||\Lambda_2\cap E_2| \,. \quad (4.13)$$

Put $\tilde{\Lambda} = \Lambda'$, $y_1 = l_0$ and $F_1 = (\tilde{\Lambda} + l_0)\cap E_1$. Since $l_0 \notin B_1$, it follows that $|F_1| \ge \beta_1|\tilde{\Lambda}|/2$. The set E_2 is $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform. This yields that there exists a number a such that $F_2 = (\tilde{\Lambda} + a)\cap E_2$ has the cardinality at least $\beta_2|\tilde{\Lambda}_1|/2$ and for $\vec{y} = (l_0, a)$ we have

$$|A\cap(\tilde{\Lambda} + \vec{y})| > (\delta + 2^{-6}\alpha_1\eta)|F_1||F_2| \,.$$

and the theorem is proven.

Let $\vec{x} = r\vec{e}_1 + m\vec{e}_2$, and $f(\vec{x})$ be a balanced function of A. There exists $l_0 \in B'$ such that

$$|A\cap(\lambda_{l_0}\times\Lambda_2)| > (\delta - \eta)|\lambda_{l_0}\cap E_1||\Lambda_2\cap E_2| \,.$$

If

$$|A\cap(\lambda_{l_0}\times\Lambda_2)| \ge (\delta + \eta)|\lambda_{l_0}\cap E_1||\Lambda_2\cap E_2| \,, \quad (4.14)$$

then the theorem is proven.

Hence there exists $l_0 \in B'$ such that

$$|\sum_{r,m} f(r,m)\lambda_{l_0}(r)\Lambda_2(m)| < \eta|\lambda_{l_0}\cap E_1||\Lambda_2\cap E_2| \,. \quad (4.15)$$

Let $\Lambda_0 = \Lambda' + l_0$. Put $\nu_i = \Lambda'' + i$, $i\in\Lambda_0$ and $\mu_j = \Lambda'' + j$, $j \in \Lambda_2$. Consider the sum

$$\sigma^* = \sum_{i\in\Lambda_0}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \,. \quad (4.16)$$

Suppose that i and j are fixed in the sum (4.16). Using Lemma 2.3, we obtain that k runs a set of cardinality at most $2|\Lambda_0|$. Besides that if i, j, k are fixed, then m, r run sets of size at most $|\Lambda''|$. Using Lemma 2.3 once again, we obtain

$$\sigma^* = |\Lambda''|^2 \sum_{k}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k)\Lambda_2(k+r) + \vartheta\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| \,, \quad (4.17)$$

where $|\vartheta| \le 1$. Let $\Lambda_3 = \Lambda_2 - \Lambda' - l_0$. Using Lemma 2.3, we get $|\Lambda_2| \le |\Lambda_3| \le (1+\alpha_0^2)|\Lambda_2|$. Note that k belongs to the set Λ_3 in (4.17). If $k \in \Lambda_2^- - l_0$, then $\Lambda_2(k+r) = 1$, for all $r \in \Lambda_0$. If k is fixed in (4.17), then r and m run sets of cardinality at most $|\Lambda_0|$. It follows that

$$\frac{\sigma^*}{|\Lambda''|^2} = \sum_{k\in(\Lambda_2^- - l_0)} \sum_m \sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k) + \sum_{k\in(\Lambda_3\setminus(\Lambda_2^- - l_0))} \sum_m \sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k)\Lambda_2(k+r)$$

$$= \sum_{k\in(\Lambda_2^- - l_0)} \sum_m \sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k) + \alpha_0^2\vartheta_1|\Lambda_0|^2|\Lambda_2| =$$

$$\sum_k \sum_m \sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k) + 2\alpha_0^2\vartheta_2|\Lambda_0|^2|\Lambda_2| = |\Lambda_0| \sum_m \sum_{r\in\Lambda_0} f(r,m) + 2\alpha_0^2\vartheta_2|\Lambda_0|^2|\Lambda_2|\,, \tag{4.18}$$

where $|\vartheta_1|, |\vartheta_2| \le 1$. Using (4.15), we get

$$|\sigma^*| < \eta|\Lambda''|^2|\Lambda_0||\Lambda_0\cap E_1||\Lambda_2\cap E_2| + 4\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| \tag{4.19}$$

If j is fixed, then k runs a set $-\Lambda_0 + j + \Lambda''$ in (4.16). Clearly, the cardinality of this set does not exceed $(1+\alpha_0^2)|\Lambda'|$. Hence, replacing $4\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|$ in (4.19) by $8\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|$, we can assume that k runs $-\Lambda_0 + j$ in (4.16).

Since $l \in B'$, it follows that $\beta_1|\Lambda_0|/2 \le |\Lambda_0 \cap E_1| \le 2\beta_1|\Lambda_0|$. Besides that $16\alpha_0^2 < \eta\beta_1\beta_2$. This implies that

$$|\sum_{i\in\Lambda_0}\sum_{j\in\Lambda_2}\sum_{k\in-\Lambda_0+j}\sum_m\sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r)| <$$

$$< 2\eta|\Lambda''|^2|\Lambda_0|\cdot|\Lambda_0\cap E_1|\cdot|\Lambda_2\cap E_2| \le 4\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|\,. \tag{4.20}$$

Let

$$\Omega = \{j \in \Lambda_2 \mid \frac{1}{|\Lambda'|}\sum_{k\in\Lambda'+j} |\delta_{\Lambda''+k}(E_2) - \beta_2|^2 \ge 4\alpha_0^{1/2}\},\ \text{and}\ G = \Lambda_2 \setminus \Omega\,.$$

Since E_2 is (α_0, ε')–uniform, it follows that $|\Omega| \le 8\alpha_0^{1/2}|\Lambda_2|$. Let $i \in \Lambda_0$ be fixed. Let

$$\Omega(i) = \{j \in \Lambda_2 \mid \frac{1}{|\Lambda'|}\sum_{k\in-\Lambda_0+j} |\delta_{\Lambda''+i+k}(E_2) - \beta_2|^2 \ge 4\alpha_0^{1/2}\},\ \text{and}\ G(i) = \Lambda_2 \setminus \Omega(i)\,.$$

Since

$$\sum_{k\in -\Lambda_0+j} |\delta_{\Lambda''+i+k}(E_2)-\beta_2|^2 = \sum_{k\in \Lambda'+j+(i-l_0)} |\delta_{\Lambda''+k}(E_2)-\beta_2|^2\,,$$

it follows that $\Lambda_2\cap(G+l_0-i)\subseteq G(i)$. Hence, $|\Omega(i)|\le|\Lambda_2|-|\Lambda_2\cap(G+l_0-i)|$. Since i belongs to Λ_0, this implies that a number $a=l_0-i$ belongs to Λ'. Using Lemma 2.3 for Λ_2 and its ε–attendant Λ', we get $(G\cap\Lambda_2^-)+a\subseteq\Lambda_2$ and

$$|\Lambda_2\cap(G+a)|\ge|\Lambda_2\cap((G\cap\Lambda_2^-)+a)|\ge|(G\cap\Lambda_2^-)+a|=|G\cap\Lambda_2^-|\ge|G|-8\alpha_0^2|\Lambda_2|\,.$$

Hence $|\Omega(i)|\le 8\alpha_0^{1/2}|\Lambda_2|$.

Since $l_0\in B'$, it follows that

$$\frac{1}{|\Lambda'|}\sum_{k\in\Lambda'} |\delta_{\Lambda''+k}(E_1-l_0\cap\Lambda')-\beta_1|^2\le 2^6\alpha_0^{1/2} \tag{4.21}$$

It is clear that for any j the sum (4.21) equals

$$\frac{1}{|\Lambda'|}\sum_{k\in-\Lambda_0+j} |\delta_{\Lambda''+j-k}(E_1\cap\Lambda_0)-\beta_1|^2\,.$$

Indeed

$$\sum_{k\in-\Lambda_0+j} |\delta_{\Lambda''+j-k}(E_1\cap\Lambda_0)-\beta_1|^2=\sum_{k\in\Lambda'+l_0} |\delta_{\Lambda''+k}(E_1\cap\Lambda'+l_0)-\beta_1|^2=$$

$$=\sum_{k\in\Lambda'} |\delta_{\Lambda''+k}(E_1-l_0\cap\Lambda')-\beta_1|^2$$

Let

$$\Omega_1(i,j)=\{k\in-\Lambda_0+j\ :\ |\delta_{\Lambda''+i+k}(E_2)-\beta_2|\ge 4\alpha_0^{1/8}\}\,,$$

$$\Omega_2(i,j)=\{k\in-\Lambda_0+j\ :\ |\delta_{\Lambda''+j-k}(E_1\cap\Lambda_0)-\beta_1|\ge 4\alpha_0^{1/8}\},\ \text{and}$$

$$\Omega_3(i,j)=\Omega_1(i,j)\cup\Omega_2(i,j)\,.$$

For all $j\notin\Omega(i)$ we have $|\Omega_1(i,j)|\le 2\alpha_0^{1/4}|\Lambda'|$. The inequality (4.21) implies that $|\Omega_2(i,j)|\le 4\alpha_0^{1/4}|\Lambda'|$. Hence $|\Omega_3(i,j)|\le 8\alpha_0^{1/4}|\Lambda'|$ if $j\notin\Omega(i)$.

Since $l_0\in B'$, it follows that

$$\sigma=\sum_{i\in\Lambda_0}\sum_{j\in\Lambda_2}\sum_k\sum_{m,u}\nu_i(m-k)\nu_i(u-k)\Big|\sum_r\mu_j(k+r)\tilde f_{l_0}(r,m)\tilde f_{l_0}(r,u)\Big|^2\ge$$

$$\ge\alpha\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,, \tag{4.22}$$

where $\tilde{f}_{l_0}$ is a restriction of f to $\lambda_{l_0} \times \Lambda_2$. If j is fixed, then k runs $-\Lambda_0 + j + \Lambda''$ in (4.22). Clearly, the cardinality of this set does not exceed $(1 + \alpha_0^2)|\Lambda'|$. Hence, replacing α by $\alpha/2$ in (4.22), we can assume that k runs $-\Lambda_0 + j$ in (4.22). Using $|\Omega(i)| \le 8\alpha_0^{1/2}|\Lambda_2|$, we get

$$\sigma = \sum_{i\in\Lambda_0} \sum_{j\notin\Omega(i)} \sum_k \sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big| \sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge$$

$$\ge \frac{\alpha}{4}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.23}$$

Now we can prove the theorem.

Let

$$J = \{(i,j,k) \mid i \in \Lambda_0,\ j \notin \Omega(i),\ k \notin \Omega_3(i,j) \text{ such that}$$

$$\sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big| \sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge \frac{\alpha}{64}\beta_1^2\beta_2^2|\Lambda''|^4\}\,.$$

Using (4.23), we get

$$\sum_{i\in\Lambda_0} \sum_{j\notin\Omega(i)} \sum_{k\notin\Omega_3(i,j)} \sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big| \sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge$$

$$\ge \frac{\alpha}{8}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.24}$$

It follows that

$$\sum_{(i,j,k)\in J} \sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big| \sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge$$

$$\ge \frac{\alpha}{16}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.25}$$

Let us estimate the cardinality of J. For any triple (i,j,k) belongs to J we have $||E_2 \cap (\nu_i + k)| - \beta_2|\Lambda''|| \le 4\alpha_0^{1/8}|\Lambda''|$ and $||(E_1 \cap \Lambda_0) \cap (\mu_j - k)| - \beta_1|\Lambda''|| \le 4\alpha_0^{1/8}|\Lambda''|$. Using (4.25), we get

$$32|J| \cdot |\Lambda''|^4\beta_1^2\beta_2^2 \ge \frac{\alpha}{16}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.26}$$

This yields that $|J| \ge 2^{-12}\alpha|\Lambda_0|^2|\Lambda_2|$.

Let us assume that for all $(i,j,k) \in J$ we have

$$\sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) < -2^{15}\frac{\eta}{\alpha}\beta_1\beta_2|\Lambda''|^2 \,. \qquad (4.27)$$

Using (4.20), we get

$$\sum_{(i,j,k)\in \overline{J}} \sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge 4\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| \,, \qquad (4.28)$$

where $\overline{J} = \{(i,j,k) \ : \ (i,j,k) \in (\Lambda_0 \times \Lambda_2 \times (-\Lambda_0 + j)) \setminus J\}$. Since $|\Omega(i)| \le 8\alpha_0^{1/2}|\Lambda_2|$, $i \in \Lambda_0$, it follows that

$$\sum_{(i,j,k)\in \overline{J}, j \notin \Omega(i)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge 2\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| \,. \quad (4.29)$$

Hence, there exist i and j, $j \notin \Omega(i)$ such that

$$\sum_{k \in Q(i,j)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge \frac{\eta}{2}\beta_1\beta_2|\Lambda'|^2|\Lambda_0| \,, \qquad (4.30)$$

where $Q(i,j)$ is a subset of $-\Lambda_0 + j$. Since $j \notin \Omega(i)$, it follows that $|\Omega_3(i,j)| \le 8\alpha_0^{1/4}|\Lambda'|$. Hence

$$\sum_{k \in Q(i,j)\setminus \Omega_3(i,j)} \sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge \frac{\eta}{4}\beta_1\beta_2|\Lambda''|^2|\Lambda_0| \,. \qquad (4.31)$$

This implies that there exists $k \notin \Omega_3(i,j)$ such that

$$\sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge \frac{\eta}{8}\beta_1\beta_2|\Lambda''|^2 \,. \qquad (4.32)$$

Put $\tilde{\Lambda} = \Lambda''$, $\vec{y} = (j-k, k+i)$ and $F_1 = (\tilde{\Lambda}+y_1)\cap(E_1\cap\Lambda_0)$, $F_2 = (\tilde{\Lambda}+y_2)\cap E_2$. Since $k \notin \Omega_3(i,j)$, it follows that $\beta_1|\Lambda''|/2 \le |F_1| \le 2\beta_1|\Lambda''|$, $\beta_2|\Lambda''|/2 \le |F_2| \le 2\beta_2|\Lambda''|$. Using this and (4.32), we get

$$|A \cap (F_1 \times F_2)| = |A \cap (((\mu_j - k) \cap \Lambda_0) \times ((\nu_i + k) \cap \Lambda_2))| \ge$$

$$\ge \delta|(\mu_j - k) \cap E_1 \cap \Lambda_0||(\nu_i + k) \cap E_2| + \frac{\eta}{8}\beta_1\beta_2|\Lambda''|^2 \ge$$

$$\ge (\delta + \frac{\eta}{32})|F_1||F_2| \,.$$

Hence, if for all $(i,j,k) \in J$ we have (4.27), then the theorem is proven.

Now assume that there exists a triple $(i,j,k) \in J$ such that

$$\sum_m \sum_{r \in \Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge -2^{15}\frac{\eta}{\alpha}\beta_1\beta_2|\Lambda''|^2 \,. \qquad (4.33)$$

We can assume that for all $(i,j,k) \in J$ we have

$$|\sum_m \sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r)| \le 2^{15}\frac{\eta}{\alpha}\beta_1\beta_2|\Lambda''|^2 \,. \tag{4.34}$$

Indeed, if

$$\sum_m \sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) > 2^{15}\frac{\eta}{\alpha}\beta_1\beta_2|\Lambda''|^2 \,,$$

then we might apply the same reasoning as above. For sets $\tilde{\Lambda}_1 = \Lambda''$, $\tilde{\Lambda}_2 = \Lambda''$, a vector $\vec{y} = (j-k, k+i)$ and $F_1 = (\tilde{\Lambda}_1 + y_1) \cap (E_1 \cap \Lambda_0)$, $F_2 = (\tilde{\Lambda}_2 + y_2) \cap E_2$ we have $|F_1| \ge \beta_1|\tilde{\Lambda}_1|/2$, $|F_2| \ge \beta_2|\tilde{\Lambda}_2|/2$ and

$$|A \cap (F_1 \times F_2)| \ge (\delta + 2^6\frac{\eta}{\alpha})|F_1||F_2|.$$

Since $(i,j,k) \in J$, it follows that

$$\sum_{m,u\in\nu_i+k} \Big| \sum_{r\in\mu_j-k} \tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge 2^{-6}\alpha\beta_1^2\beta_2^2|\Lambda''|^4 \,. \tag{4.35}$$

Note that m,u belong to $\nu_i + k \cap \Lambda_2$ in (4.35) and r belongs to a set $\mu_j - k \cap \Lambda_0$. Put $\mathcal{L}_1 = \mu_j - k \cap \Lambda_0$, $\mathcal{L}_2 = \nu_i + k \cap \Lambda_2$, $E'_1 = E_1 \cap \mathcal{L}_1$ and $E'_2 = E_2 \cap \mathcal{L}_2$. We can assume that $\tilde{f}_{l_0}$ is zero outside $\mathcal{L}_1 \times \mathcal{L}_2$ in (4.35). Let $A_1 = A \cap (\mathcal{L}_1 \times \mathcal{L}_2)$, $\delta_1 = \delta_{E'_1\times E'_2}(A)$, and f_1 be a balanced function of A_1. Using (4.34), we get $|\delta_1 - \delta| \le 2^{20}\frac{\eta}{\alpha}$. We have $k \notin \Omega_3(i,j)$. Using this, we obtain

$$\|\tilde{f}_{l_0} - f_1\|^4 = |E'_1|^2|E'_2|^2(\delta_1-\delta)^2 \le 2^{44}\beta_1^2\beta_2^2\frac{\eta^2}{\alpha^2}|\Lambda'|^4 \,. \tag{4.36}$$

We have $\eta \le 2^{-50}\alpha^2$. Using this and Lemma 3.2, we get

$$\sum_{m,u\in\nu_i+k} \Big| \sum_{r\in\mu_j-k} f_1(r,m)f_1(r,u)\Big|^2 \ge 2^{-7}\alpha\beta_1^2\beta_2^2|\Lambda'|^4 \,. \tag{4.37}$$

Since $k \notin \Omega_3(i,j)$, it follows that $2^{-1}\beta_1|\Lambda'| \le |E'_1| \le 2\beta_1|\Lambda'|$, $2^{-1}\beta_2|\Lambda'| \le |E'_2| \le 2\beta_2|\Lambda'|$. Hence

$$\sum_{m,u\in\nu_i+k} \Big| \sum_{r\in\mu_j-k} f_1(r,m)f_1(r,u)\Big|^2 \ge 2^{-11}\alpha|E'_1|^2|E'_2|^2 \,. \tag{4.38}$$

Using Proposition 4.1, we obtain sets $F_1 \subseteq E'_1 \subseteq \mu_j - k$, $F_2 \subseteq E'_2 \subseteq \nu_i + k$ such that

$$|A \cap (F_1 \times F_2)| \ge |A_1 \cap (F_1 \times F_2)| \ge (\delta_1 + 2^{-36}\alpha^2)|F_1||F_2| \ge (\delta + 2^{-40}\alpha^2)|F_1||F_2|$$

and

$$|F_i| \ge 2^{-19}\alpha |E'_i| \ge 2^{-25}\alpha\beta_i |\Lambda'|, \quad i = 1, 2\,.$$

Put $\tilde{\Lambda} = \Lambda''$, $\vec{y} = (j-k, k+i)$ and $F_1 = (\tilde{\Lambda}_1 + y_1) \cap (E_1 \cap \Lambda_0)$, $F_2 = (\tilde{\Lambda}_2 + y_2) \cap E_2$. The sets $\tilde{\Lambda}$ and F_1, F_2 satisfy (4.6), (4.7). This concludes the proof.

5. *On dense subsets of Borh sets.*

We need a simple lemma.

LEMMA 5.1. *Let Λ be a Bohr set, Λ' be ε–attendant of Λ, $\varepsilon = \kappa/(100d)$, and Q be a subset of Λ. Let $g : 2^{\mathbf{Z}} \times \mathbf{Z}^2 \to \mathbf{D}$ be the function such that $g(\Lambda, \vec{x}) = \delta^2_{\Lambda+\vec{x}}(Q)$. Then*

$$\frac{1}{|\Lambda|^2} \sum_{\vec{x} \in \Lambda} g(\Lambda', \vec{x}) \ge g(\Lambda, 0) - 8\kappa\,. \tag{5.1}$$

Proof. Using the Cauchy–Schwarz inequality and Lemma 2.4, we get

$$|\Lambda|^2 \sum_{\vec{x} \in \Lambda} g(\Lambda', \vec{x}) \ge \Big(\sum_{\vec{x} \in \Lambda} \delta_{\Lambda'+\vec{x}}(Q) \Big)^2 = |\Lambda|^2 (\delta_\Lambda(Q) + 4\vartheta\kappa)^2\,,$$

where $|\vartheta| \le 1$. This implies that

$$\frac{1}{|\Lambda|^2} \sum_{\vec{x} \in \Lambda} g(\Lambda', \vec{x}) \ge \delta^2_\Lambda(Q) - 8\kappa = g(\Lambda, 0) - 8\kappa$$

as required.

NOTE. Clearly, the one–dimension analog of Lemma 5.1 takes place.

LEMMA 5.2. *Let Λ be a Bohr set, Λ' be ε–attendant of Λ, $\varepsilon = \kappa/(100d)$, $\alpha > 0$ be a real number, and Q be a subset of Λ, $|Q| = \delta|\Lambda|$. Suppose that*

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} |\delta_{\Lambda'+\vec{n}}(Q) - \delta|^2 \ge \alpha\,. \tag{5.2}$$

Then

$$\sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(Q) \ge \delta^2 + \alpha - 4\kappa\,. \tag{5.3}$$

Proof. Using (5.2), we have

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(Q) \ge \frac{2\delta}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta_{\Lambda'+\vec{n}}(Q) - \delta^2 + \alpha\,. \tag{5.4}$$

The first term in (5.4) equals

$$\frac{2\delta}{|\Lambda'|^2|\Lambda|^2} \sum_{\vec{s}} Q(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n})\Lambda'(\vec{s}-\vec{n}) = \frac{2\delta}{|\Lambda'|^2|\Lambda|^2} \sum_{\vec{s}} Q(\vec{s})(\Lambda * \Lambda')(\vec{s})\,.$$

Using Lemma 2.3, we obtain

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(Q) \ge \frac{2\delta}{|\Lambda|^2} \sum_{\vec{s}} Q(\vec{s})\Lambda(\vec{s}) - \delta^2 + \alpha - 4\kappa \ge \delta^2 + \alpha - 4\kappa\,. \tag{5.5}$$

This completes the proof.

Note. Clearly, the one–dimension analog of Lemma 5.2 takes place.

Corollary 5.1. *Let Λ be a Bohr set, $\alpha > 0$ be a real number, and E_1, E_2 be sets, $|E_1 \cap \Lambda| = \beta_1|\Lambda|$, $|E_2 \cap \Lambda| = \beta_2|\Lambda|$. Suppose that either E_1 or E_2 does not satisfy (3.13). Let Λ' be an arbitrary $(2^{-10}\alpha^2\beta_1^2\beta_2^2)/(100d)$–attendant set of Λ. Then*

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1 \times E_2) \ge \beta_1^2\beta_2^2(1 + \frac{\alpha^2}{2})\,. \tag{5.6}$$

Proof. Let $\vec{n} = (x, y)$ and $\kappa = 2^{-10}\alpha^2\beta_1^2\beta_2^2$. We have

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1 \times E_2) = \Big(\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta^2_{\Lambda'+x}(E_1)\Big)\Big(\frac{1}{|\Lambda|} \sum_{y\in\Lambda} \delta^2_{\Lambda'+y}(E_2)\Big) \tag{5.7}$$

Without loss of generality it can be assumed that E_1 does not satisfy (3.13). Using Lemma 5.2, we get

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta^2_{\Lambda'+x}(E_1) \ge \beta_1^2 + \alpha^2 - 4\kappa\,. \tag{5.8}$$

Let us estimate the second factor in (5.7). Using Lemma 5.1, we obtain

$$\frac{1}{|\Lambda|} \sum_{y \in \Lambda} \delta^2_{\Lambda'+y}(E_2) \ge \beta_2^2 - 8\kappa \,. \tag{5.9}$$

Combining (5.8) and (5.9), we have

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1 \times E_2) \ge (\beta_1^2 + \alpha^2 - 4\kappa)(\beta_2^2 - 8\kappa) \ge \beta_1^2 \beta_2^2 (1 + \frac{\alpha^2}{2}) \,.$$

This concludes the proof.

The following lemma was proven by J. Bourgain in [**3**]. We give his proof for the sake of completeness.

Lemma 5.3. *Let $\Lambda = \Lambda_{\theta,\varepsilon,M}$ be a Bohr set, $\alpha > 0$ be a real number, and Q be a set, $|Q \cap \Lambda| = \delta |\Lambda|$. Suppose that*

$$\| (Q \cap \Lambda - \delta \Lambda)\widehat{\ } \|_\infty \ge \alpha |\Lambda| \,. \tag{5.10}$$

Then there exists a Bohr set $\Lambda' = \Lambda_{\theta',\varepsilon',N'}$ such that Λ' is ε_1–attendant of Λ, $\varepsilon_1 = \frac{\kappa}{100d}$, $\kappa \le \alpha/32$ and

$$\frac{1}{|\Lambda|} \sum_{n \in \Lambda} |\delta_{\Lambda'+n}(Q) - \delta|^2 \ge \frac{\alpha^2}{4} \,, \tag{5.11}$$

$\theta' \in \mathbf{T}^{d+1}$.

Proof. Let $Q_1 = Q \cap \Lambda$. Using (5.10), we obtain

$$|\widehat{Q}_1(x_0) - \delta \widehat{\Lambda}(x_0)| \ge \alpha |\Lambda| \,, \tag{5.12}$$

where $x_0 \in \mathbf{T}$. We have $\Lambda = \Lambda_{\theta,\varepsilon,M}$. Put $\theta' = \theta \cup \{x_0\} \in \mathbf{T}^{d+1}$ and

$$\Lambda' = \Lambda_{\theta', \frac{\kappa}{100d}\varepsilon, \frac{\kappa}{100d} M} \,.$$

Using Lemma 2.3, we get

$$\widehat{Q}_1(x_0) = \sum_n Q(n)\Lambda(n) e^{2\pi i n x_0} = \frac{1}{|\Lambda'|} \sum_n (\Lambda * \Lambda')(n) Q(n) e^{2\pi i n x_0} + 2\kappa\vartheta|\Lambda| \,,$$

where $|\vartheta| \le 1$. We have

$$\widehat{Q}_1(x_0) = \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m) Q(n) e^{2\pi i n x_0} + 2\kappa\vartheta|\Lambda| =$$

$$= \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m)Q(n)e^{2\pi i m x_0} +$$

$$+ \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m)Q(n)[e^{2\pi i n x_0} - e^{2\pi i m x_0}] + 2\kappa\vartheta|\Lambda| =$$

$$= \sum_{m\in\Lambda} \delta_{\Lambda'+m}(Q)e^{2\pi i m x_0} + O(\frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m)Q(n)|e^{2\pi i(n-m)x_0} - 1|) +$$

$$+2\kappa\vartheta|\Lambda| = \sum_{m\in\Lambda} \delta_{\Lambda'+m}(Q)e^{2\pi i m x_0} + (14\kappa + 2\kappa)\vartheta|\Lambda|\,. \tag{5.13}$$

Using (5.10) and (5.13), we obtain

$$\Big| \sum_{m\in\Lambda} \delta_{\Lambda'+m}(Q)e^{2\pi i m x_0} - \delta \sum_{m\in\Lambda} e^{2\pi i m x_0} \Big| \geq \frac{\alpha}{2}|\Lambda|\,. \tag{5.14}$$

Hence

$$\sum_{m\in\Lambda} |\delta_{\Lambda'+m}(Q) - \delta| \geq \frac{\alpha}{2}|\Lambda|\,. \tag{5.15}$$

Using the Cauchy–Schwarz inequality, we get

$$\frac{1}{|\Lambda|} \sum_{\vec{n}\in\Lambda} |\delta_{\Lambda'+\vec{n}}(Q) - \delta|^2 \geq \frac{\alpha^2}{4}\,. \tag{5.16}$$

This completes the proof.

COROLLARY 5.2. *Let Λ be a Bohr set, $\alpha > 0$ be a real number, and E_1, E_2 be sets, $|E_1 \cap \Lambda| = \beta_1|\Lambda|$, $|E_2 \cap \Lambda| = \beta_2|\Lambda|$ Suppose that either E_1 or E_2 satisfies (5.10). Then there exists $(2^{-10}\alpha^2\beta_1^2\beta_2^2)/(100d)$–attendant set $\Lambda' = \Lambda_{\theta',\varepsilon',N'}$ of Bohr set Λ such that*

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1 \times E_2) \geq \beta_1^2\beta_2^2(1 + \frac{\alpha^2}{8}) \tag{5.17}$$

and

$$\theta' \in \mathbf{T}^{d+1}\,. \tag{5.18}$$

Proof. Let $\vec{n} = (x, y)$, and $\kappa = 2^{-10}\alpha^2\beta_1^2\beta_2^2$. We have

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1 \times E_2) = \Big(\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta^2_{\Lambda'+x}(E_1) \Big) \Big(\frac{1}{|\Lambda|} \sum_{y\in\Lambda} \delta^2_{\Lambda'+y}(E_2) \Big) \tag{5.19}$$

We can assume without loss of generality that E_1 satisfies (5.10). Using Lemma 5.3

and Lemma 5.2, we obtain

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\delta^2_{\Lambda'+x}(E_1)\ge\beta_1^2+\frac{\alpha^2}{4}-4\kappa\,. \tag{5.20}$$

Let us estimate the second term in (5.19). Using Lemma 5.1, we get

$$\frac{1}{|\Lambda|}\sum_{y\in\Lambda}\delta^2_{\Lambda'+y}(E_2)\ge\beta_2^2-8\kappa\,. \tag{5.21}$$

Combining (5.20) and (5.21), we obtain

$$\frac{1}{|\Lambda|^2}\sum_{\vec{n}\in\Lambda}\delta^2_{\Lambda'+\vec{n}}(E_1\times E_2)\ge(\beta_1^2+\frac{\alpha^2}{4}-4\kappa)(\beta_2^2-8\kappa)\ge\beta_1^2\beta_2^2(1+\frac{\alpha^2}{8})\,.$$

This concludes the proof.

We shall say that the vector θ' from (5.18) is constructed by Corollary 5.2.

Clearly, all lemmas of this section apply to translations of Bohr sets.

Let $\mathbf{\Lambda}$ be a union of a family of Bohr sets $\Lambda_0^*,\Lambda_1^*(\vec{x}_0),\dots,\Lambda_n^*(\vec{x}_0,\dots,\vec{x}_{n-1})$ and a sequence of some translations of Bohr sets $\Lambda_0,\Lambda_1(\vec{x}_0),\dots,\Lambda_n(\vec{x}_0,\dots,\vec{x}_{n-1})$ such that

$$\Lambda_1(\vec{x}_0) \text{ and } \Lambda_1^*(\vec{x}_0) \text{ are defined iff } \vec{x}_0\in\Lambda_0$$

$$\Lambda_2(\vec{x}_0,\vec{x}_1) \text{ and } \Lambda_2^*(\vec{x}_0,\vec{x}_1) \text{ are defined iff } \vec{x}_1\in\Lambda_1(\vec{x}_0),\vec{x}_0\in\Lambda_0$$

$$\dots$$

$$\Lambda_n(\vec{x}_0,\dots,\vec{x}_{n-1}) \text{ and } \Lambda_n^*(\vec{x}_0,\dots,\vec{x}_{n-1}) \text{ are defined iff}$$

$$\vec{x}_{n-1}\in\Lambda_{n-1}(\vec{x}_0,\dots,\vec{x}_{n-2}),\vec{x}_{n-2}\in\Lambda_{n-2}(\vec{x}_0,\dots,\vec{x}_{n-3}),\dots,\vec{x}_0\in\Lambda_0\,. \tag{5.22}$$

Let $m\ge 0$ be an integer number and $\mathbf{\Lambda}$ be a family of Bohr sets satisfies (5.22). Let $g: 2^{\mathbf{Z}}\times\mathbf{Z^2}\to\mathbf{D}$ be a function. Let us define the *index* of g, respect $\mathbf{\Lambda}$, for all $k=0,\dots,m$ by

$$\mathrm{ind}_k(\mathbf{\Lambda})(g)=\frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\frac{1}{|\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})|^2}\sum_{\vec{y}\in\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})}g(\Lambda_k^*(\vec{x}_0,\dots,\vec{x}_{k-1}),\vec{y})\,. \tag{5.23}$$

Let $M_k = M_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ be the family of sets such that $M_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \subseteq \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ for all $(\vec{x}_0, \dots, \vec{x}_{k-1})$. For any $k = 0, \dots, m$ by $\mathrm{ind}_k(\mathbf{\Lambda}, M)(g)$ define the following expression

$$\mathrm{ind}_k(\mathbf{\Lambda}, M)(g) = \frac{1}{|\Lambda_0|^2} \sum_{\vec{x}_0 \in \Lambda_0} \frac{1}{|\Lambda_1(\vec{x}_0)|^2} \sum_{\vec{x}_1 \in \Lambda_1(\vec{x}_0)} \dots$$

$$\frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in M_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) \,. \qquad (5.24)$$

Clearly, we have $|\,\mathrm{ind}_k(\mathbf{\Lambda}, M)(g)| \le 1$, for any natural $k \ge 0$, a family M_k and a function $g : 2^{\mathbf{Z}} \times \mathbf{Z}^{\mathbf{2}} \to \mathbf{D}$.

LEMMA 5.4. *Let Q be a subset of $\Lambda_0 \times \Lambda_0$, and $|Q| = \delta |\Lambda_0|^2$. Suppose that $\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1})$ is an arbitrary ε–attendant of $\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})$, $\varepsilon = \kappa/(100d)$. Let $g(M, \vec{x}) = \delta_{M+\vec{x}}(Q)$. Then for all $k = 0, \dots, n$ we have*

$$\Big| \mathrm{ind}_k(\mathbf{\Lambda})(g) - \delta \Big| \le 4\kappa(k+1) \,. \qquad (5.25)$$

Proof. If $k = 0$, then Lemma 2.4 implies the result. Let $k > 0$. Using Lemma 2.4 once again, we get

$$\mathrm{ind}_k(\mathbf{\Lambda})(g) \ge \mathrm{ind}_{k-1}(\mathbf{\Lambda})(g) - 4\kappa \ge \dots \ge \mathrm{ind}_0(\mathbf{\Lambda})(g) - 4\kappa k \ge \delta - 4\kappa(k+1) \,.$$

In the same way we obtain the reverse inequality. This completes the proof.

The next result is the main in this section.

PROPOSITION 5.1. *Let $\Lambda = \Lambda(\theta, \varepsilon_0, N)$ be a Bohr set, $\theta \in \mathbf{T}^d$, and $\vec{s} = (s_1, s_2)$ be an integer vector. Let $\varepsilon, \sigma, \tau, \delta \in (0,1)$ be real numbers, E_1, E_2 be sets, $E_i = \beta_i |\Lambda|$, $i = 1, 2$. Suppose that $\mathbf{E} = E_1 \times E_2$ is a subset of $(\Lambda + s_1) \times (\Lambda + s_2)$, $A \subset \mathbf{E}$, $\delta_{\mathbf{E}}(A) = \delta + \tau$, and $\varepsilon \le \kappa/(100d)$, $\kappa = 2^{-100} (\tau \beta_1 \beta_2)^5 \sigma^3$. Let*

$$N \ge (2^{-100} \varepsilon_0 \varepsilon)^{-2^{100}((\tau\beta_1\beta_2)^{-5}\sigma^{-3}+d)^2} \,, \qquad (5.26)$$

and $\sigma \le 2^{-100} \tau \beta_1 \beta_2$. Then there exists a Bohr set $\Lambda' = \Lambda(\theta', \varepsilon', N')$, $\theta' \in \mathbf{T}^D$, $D \le 2^{30} (\tau\beta_1\beta_2)^{-5} \sigma^{-3} + d$, $\varepsilon' \ge (2^{-10}\varepsilon)^D \varepsilon_0$, $N' \ge (2^{-10}\varepsilon)^D N$ and an integer vector

$\vec{t} = (t_1, t_2)$ *such that if* $E'_1 = (E_1 - t_1) \cap \Lambda'$, $E'_2 = (E_2 - t_2) \cap \Lambda'$, $\mathbf{E}' = E'_1 \times E'_2$, *then*

1) $|\mathbf{E}'| \ge \beta_1 \beta_2 \tau |\Lambda'|/16$;

2) E'_1, E'_2 *are* (σ, ε)*–uniform subsets of* Λ';

3) $\delta_{\mathbf{E}'}(A - \vec{t}) \ge \delta + \tau/16$.

Proof. Let $\beta = \beta_1\beta_2$, and $\tilde{E}_1 = E_1 - s_1$, $\tilde{E}_2 = E_2 - s_2$, $\tilde{E} = \tilde{E}_1 \times \tilde{E}_2$. If the sets $\tilde{E}_1$, $\tilde{E}_2$ are (σ, ε)–uniform subsets of Λ, then Proposition 5.1 is proven.

Suppose that $\tilde{E}_1$, $\tilde{E}_2$ are not (σ, ε)–uniform subsets of Λ. We shall construct a family of Bohr sets $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}$ satisfies the conditions (5.22). The proof of Proposition 5.1 is a sort of an algorithm. At the first step of our algorithm we put $\Lambda_0 = \Lambda = \Lambda_{\theta, \varepsilon_0, N}$. If either $\tilde{E}_1$ or $\tilde{E}_2$ does not satisfy (3.14) with $\alpha = \sigma/2$, then let Λ_0^* be ε–attendant of Λ_0 such that Λ_0^* is constructed by Corollary 5.2. In the other cases let Λ_0^* be ε–attendant of Λ_0 with the same θ. Define

$$R_0 = \{\vec{p} = (p_1, p_2) \in \Lambda_0 \mid \tilde{E}_1 - p_1, \tilde{E}_2 - p_2 \text{ are } (\sigma, \varepsilon)\text{–uniform in } \Lambda_0^*$$

$$\text{or } \delta_{\Lambda_0^* + \vec{p}}(\tilde{E}_1 \times \tilde{E}_2) < \beta\tau/16\}$$

and $\overline{R}_0 = (\Lambda_0 \times \Lambda_0) \setminus R_0$.

Let $\tilde{\Lambda}$ be an arbitrary Bohr set, and $\vec{n} \in \mathbf{Z}^2$ be an arbitrary integer vector. Put $g(\tilde{\Lambda}, \vec{n}) = \delta^2_{\tilde{\Lambda}+\vec{n}}(\tilde{E})$, $g_1(\tilde{\Lambda}, \vec{x}) = \delta_{\tilde{\Lambda}+\vec{n}}(A)$, $g_2(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{E} \cap \tilde{\Lambda}+\vec{n}}(A)$ and $g_3(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{\Lambda}+\vec{n}}(\tilde{E})$. Clearly, $g(\tilde{\Lambda}, \vec{n}) = g_3^2(\tilde{\Lambda}, \vec{n})$ and $g_1(\tilde{\Lambda}, \vec{x}) \le g_3(\tilde{\Lambda}, \vec{n})$. Besides that, we have

$$g_1(\tilde{\Lambda}, \vec{n}) = g_2(\tilde{\Lambda}, \vec{n}) g_3(\tilde{\Lambda}, \vec{n})\,.$$

Let $\mathbf{\Lambda}_0 = \{\Lambda_0\}$. If $\mathrm{ind}_0(\mathbf{\Lambda}_0, \overline{R}_0)(g_3) < \tau\beta/4$, then we stop the algorithm at step 0.

Using Lemma 2.4 and the Cauchy–Schwarz inequality, we get

$$\mathrm{ind}_0(\mathbf{\Lambda}_0)(g) \ge \Big(\frac{1}{|\Lambda_0|^2} \sum_{\vec{y} \in \Lambda_0} \delta_{\Lambda_0^* + \vec{y}}(\tilde{E}) \Big)^2 \ge \beta/2\,. \tag{5.27}$$

Let after the kth step of the algorithm the family of Bohr sets $\mathbf{\Lambda}_k$ has been constructed, $k \ge 0$.

Let

$$\Lambda_{k+1}(\vec{x}_0,\dots,\vec{x}_k)=\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})+\vec{x}_k\,,\ \vec{x}_k\in\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})$$

Let $\vec{x}_k=(a,b)$, and $\Lambda^*_k=\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})$. If either $(\tilde{E}_1-a)\cap\Lambda^*_k$ or $(\tilde{E}_2-b)\cap\Lambda^*_k$ does not satisfy (3.14) with $\alpha=\sigma/2$, then let $\Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$ be ε–attendant of $\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_k)$ such that $\Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$ is constructed by Corollary 5.2. In the other cases let $\Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$ be ε–attendant of $\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_k)$ with the same generative vector.

By $R_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$, $\overline{R}_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$ denote the sets

$$R_{k+1}(\vec{x}_0,\dots,\vec{x}_k)=\{\vec{p}=(p_1,p_2)\in\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})+\vec{x}_k\mid\tilde{E}_1-p_1,\tilde{E}_2-p_2$$

$$\text{are } (\sigma,\varepsilon)\text{–uniform in } \Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$$

$$\text{or } \delta_{\Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)+\vec{p}}(\tilde{E}_1\times\tilde{E}_2)<\tau\beta/16\}$$

and $\overline{R}_{k+1}(\vec{x}_0,\dots,\vec{x}_k)=(\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})+\vec{x}_k)\setminus R_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$.

By $E_k(\vec{x}_0,\dots,\vec{x}_{k-1})$ denote the sets

$$E_k(\vec{x}_0,\dots,\vec{x}_{k-1})=$$

$$\{\vec{p}=(p_1,p_2)\in\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})+\vec{x}_{k-1}\mid\delta_{\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})+\vec{p}}(\tilde{E}_1\times\tilde{E}_2)<\tau\beta/16\}.$$

Obviously, $E_k(\vec{x}_0,\dots,\vec{x}_{k-1})\subseteq R_k(\vec{x}_0,\dots,\vec{x}_{k-1})$, $k=0,1,\dots$

Let $\mathbf{\Lambda'}_{k+1}=\{\Lambda_{k+1}(\vec{x}_0,\dots,\vec{x}_k)\}$, $\vec{x}_k\in\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})$, and $\mathbf{\Lambda}_{k+1}=\{\mathbf{\Lambda}_k,\mathbf{\Lambda'}_{k+1}\}$. If $\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1},\overline{R}_{k+1})(g_3)<\tau\beta/4$, then we stop the algorithm at step $k+1$.

Let $\Lambda^*_{k-1}=\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$, and $\beta'_k=\delta_{\Lambda^*_{k-1}}(\tilde{E}_1)$, $\beta''_k=\delta_{\Lambda^*_{k-1}}(\tilde{E}_2)$. Suppose $\vec{x}_{k-1}=(a',b')$ belongs to $\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$. Note that $\vec{x}_{k-1}$ does not belong to $E_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$. Let us consider three cases.

Case 1 : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.12).

Case 2 : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.13).

Case 3 : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.14).

Note that α equals σ in all these cases.

Let us consider the following situation : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$

does not satisfy (3.14) with $\alpha = 2^{-4}\sigma^{3/2}$. Let

$$S_0 = \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) \,, \tag{5.28}$$

where $\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ is ε–attendant of $\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ such that $\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ is constructed by Corollary 5.2. Using Corollary 5.2, we get

$$S_0 \ge g(\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1}), 0)(1 + 2^{-11}\sigma^3) =$$

$$= g(\Lambda^*_{k-1}(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1})(1 + 2^{-11}\sigma^3) \,. \tag{5.29}$$

Note that in this case, we have $\dim \Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}) = \dim \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1}) + 1$.

Suppose that either $(\tilde{E}_1 - a') \cap \Lambda^*_{k-1}$ or $(\tilde{E}_2 - b') \cap \Lambda^*_{k-1}$ does not satisfy (3.13) with $\alpha = 2^{-4}\sigma^{3/2}$. Using Corollary 5.1, we obtain

$$S_0 \ge g(\Lambda^*_{k-1}(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1})(1 + 2^{-11}\sigma^3) \,. \tag{5.30}$$

In this case, we have $\dim \Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}) = \dim \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})$.

Finally, suppose that either $(\tilde{E}_1 - a') \cap \Lambda^*_{k-1}$ or $(\tilde{E}_2 - b') \cap \Lambda^*_{k-1}$ does not satisfy (3.12) with $\alpha = \sigma$. Note that $(\tilde{E}_1 - a') \cap \Lambda^*_{k-1}$ *and* $(\tilde{E}_2 - b') \cap \Lambda^*_{k-1}$ satisfy (3.13) with $\alpha = 2^{-4}\sigma^{3/2}$. Let $\Lambda^*_k = \Lambda^*_k(\vec{x}_0, \dots, \vec{x}_k)$. Define

$$B_k(\vec{x}_0, \dots, \vec{x}_{k-1}) = \{\vec{p} = (p_1, p_2) \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \;:$$

$$\|((\tilde{E}_1 - p_1) - \beta'_k \Lambda^*_k)\widehat{\ }\|_\infty \ge \sigma|\Lambda^*_k| \text{ or } \|((\tilde{E}_2 - p_2) - \beta''_k \Lambda^*_k)\widehat{\ }\|_\infty \ge \sigma|\Lambda^*_k|\} \,.$$

We have

$$|B_k(\vec{x}_0, \dots, \vec{x}_{k-1})| \ge \sigma |\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2 \,. \tag{5.31}$$

Let

$$\tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1}) = \{\vec{p} = (p_1, p_2) \in B_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \;:$$

$$|\delta_{\Lambda^*_k}(\tilde{E}_1 - p_1) - \beta'_k| \le \sigma/8 \quad \text{and} \quad |\delta_{\Lambda^*_k}(\tilde{E}_2 - p_2) - \beta''_k| \le \sigma/8\} \,.$$

For all $\vec{p} \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$, we have either $(\tilde{E}_1 - p_1) \cap \Lambda^*_k$ or $(\tilde{E}_2 - p_2) \cap \Lambda^*_k$ does not $\sigma/2$–uniform. The sets $(\tilde{E}_1 - a') \cap \Lambda^*_{k-1}$ and $(\tilde{E}_2 - b') \cap \Lambda^*_{k-1}$ satisfy (3.13) with α equals $2^{-4}\sigma^{3/2}$. This implies that

$$|\tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})| \ge \frac{\sigma}{2} |\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2 \tag{5.32}$$

Suppose that

$$g_3(\Lambda^*_{k-1}, \vec{x}_{k-1}) = \beta'_k \beta''_k \ge \tau\beta/8\,. \tag{5.33}$$

It follows from (5.33) that

$$g_3(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{p}) \ge \beta'_k \beta''_k - \sigma/2 \ge \tau\beta/16\,, \tag{5.34}$$

for all $\vec{p} \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$.

Let us consider the sum

$$S = S(\vec{x}_0, \dots, \vec{x}_{k-1}) = \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{x}_k \in \Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})} \frac{1}{|\Lambda_{k+1}(\vec{x}_0, \dots, \vec{x}_k)|^2}$$
$$\cdot \sum_{\vec{y} \in \Lambda_{k+1}(\vec{x}_0,\dots,\vec{x}_k)} g(\Lambda^*_{k+1}(\vec{x}_0, \dots, \vec{x}_k), \vec{y})\,.$$

Write the sum S as $S' + S''$, where the summation in S' is taken over $\vec{x}_k \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ and the summation in S'' is taken over $\vec{x}_k \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \setminus \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$. Note that if $\vec{x}_k \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$, then the Bohr set $\Lambda^*_{k+1}(\vec{x}_0, \dots, \vec{x}_k)$ is constructed by Corollary 5.2. Using this corollary, we obtain

$$S' \ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \tilde{B}_k(\vec{x}_0,\dots,\vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y})(1 + \frac{\sigma^2}{32}) \tag{5.35}$$

Let us estimate the sum S''. Using Lemma 5.1, we get

$$S'' \ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1}) \setminus \tilde{B}_k(\vec{x}_0,\dots,\vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) - 8\kappa \tag{5.36}$$

Combining (5.34), (5.35), (5.36) and (5.32), we have

$$S \ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) +$$
$$+ \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \tilde{B}_k(\vec{x}_0,\dots,\vec{x}_{k-1})} 2^{-13}\tau^2\beta^2\sigma^2 - 2^4\kappa \ge$$
$$\ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) + 2^{-14}\tau^2\beta^2\sigma^3 - 2^4\kappa$$

Using Lemma 5.1, we obtain

$$S \ge g(\Lambda^*_{k-1}(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1}) + 2^{-14}\tau^2\beta^2\sigma^3 - 2^5\kappa \ge$$
$$\ge g(\Lambda^*_{k-1}(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1}) + 2^{-15}\tau^2\beta^2\sigma^3 \ge$$

$$\geq g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-15}\tau^2\beta^2\sigma^3)\,. \qquad (5.37)$$

On the other hand, S_0 is an estimate for S. Using Lemma 5.1, we get

$$S \geq S_0 - 8\kappa\,.$$

Thus if $\vec{x}_{k-1}$ belongs to $\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$ and $\vec{x}_{k-1}$ satisfies (5.33), then we have

$$S \geq g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-15}\tau^2\beta^2\sigma^3) - 8\kappa\,. \qquad (5.38)$$

Now suppose that $\vec{x}_{k-1}$ is an arbitrary vector, $\vec{x}_{k-1}\in\Lambda_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$. Using Lemma 5.1 twice, we have

$$S \geq g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1}) - 16\kappa\,. \qquad (5.39)$$

Let us consider $\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g)$. We have

$$\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g) =$$

$$\frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots\sum_{\vec{x}_{k-1}\in\Lambda_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})} S(\vec{x}_0,\dots,\vec{x}_{k-1})$$

By assumption $\mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1},\overline{R}_{k-1})(g_3)\geq\tau\beta/4$. In other words

$$\frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\sum_{\vec{x}_{k-1}\in\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})} g_3(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1}) \geq \tau\beta/4\,. \qquad (5.40)$$

By $M_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$ denote the set of $\vec{x}_{k-1}\in\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$ such that $\vec{x}_{k-1}$ satisfies (5.33). Using (5.40), we obtain

$$S_M = \frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\sum_{\vec{x}_{k-1}\in M_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})} g_3(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1}) \geq \tau\beta/8\,. \qquad (5.41)$$

Using (5.33), (5.38), (5.39) and (5.41), we get

$$\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g) \geq \frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\Big\{ \sum_{\vec{x}_{k-1}\in M_{k-1}(\vec{x}_0,\ldots,\vec{x}_{k-2})} (g(\Lambda^*_{k-1}(\vec{x}_0,\ldots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-15}\tau^2\beta^2\sigma^3)-8\kappa)+$$

$$+\sum_{\vec{x}_{k-1}\in \Lambda_{k-1}(\vec{x}_0,\ldots,\vec{x}_{k-2})\setminus M_{k-1}(\vec{x}_0,\ldots,\vec{x}_{k-2})} (g(\Lambda^*_{k-1}(\vec{x}_0,\ldots,\vec{x}_{k-2}),\vec{x}_{k-1})-16\kappa)\Big\} \geq$$

$$\geq \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g)+2^{-15}\tau^2\beta^2\sigma^3\Big(\frac{\tau\beta}{8}\Big)S_M-24\kappa \geq$$

$$\geq \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g)+2^{-24}\tau^4\beta^4\sigma^3-24\kappa \geq$$

$$\geq \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g)+2^{-25}\tau^4\beta^4\sigma^3\,.$$

In other words, for all $k \geq 1$, we have

$$\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g) \geq \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g)+2^{-25}\tau^4\beta^4\sigma^3\,. \tag{5.42}$$

Since for any k we have $\mathrm{ind}_k(\mathbf{\Lambda}_k)(g) \leq 1$, it follows that the total number of steps of the algorithm does not exceed $K_0 = 2^{30}\tau^{-4}\beta^{-4}\sigma^{-3}$.

Suppose that the algorithm stops at step K, $K \geq 1$, $K \leq 2^{30}\tau^{-4}\beta^{-4}\sigma^{-3}$. We have

$$\mathrm{ind}_K(\mathbf{\Lambda}_K,\overline{R}_K)(g_3) < \frac{\tau\beta}{4}\,. \tag{5.43}$$

Using Lemma 5.4, we get

$$\mathrm{ind}_K(\mathbf{\Lambda}_K)(g_1) \geq (\delta+\tau)\beta - 8\kappa K \geq (\delta+\frac{7\tau}{8})\beta\,.$$

Using (5.43), we obtain

$$\mathrm{ind}_K(\mathbf{\Lambda}_K,R_K)(g_1) \geq (\delta+\frac{3\tau}{8})\beta\,. \tag{5.44}$$

The summation in (5.44) is taken over the sets $\Lambda^*_K(\vec{x}_0,\ldots,\vec{x}_{K-1})+\vec{y}$, where $\vec{y}\in R_K(\vec{x}_0,\ldots,\vec{x}_{K-1})$.

Let E_K be the family of vectors $\vec{y}$ such that $\vec{y}\in E_K(\vec{x}_0,\ldots,\vec{x}_{K-1})$, and R^*_K be the family of vectors $\vec{y}$ such that $\vec{y}\in R_K(\vec{x}_0,\ldots,\vec{x}_{K-1})$, but $\vec{y}$ does not belong to $E_K(\vec{x}_0,\ldots,\vec{x}_{K-1})$. We have

$$\mathrm{ind}_K(\mathbf{\Lambda}_K,E_K)(g_1) < \frac{\tau\beta}{16}\,\mathrm{ind}_K(\mathbf{\Lambda}_K)(1) \leq \frac{\tau\beta}{16}\,. \tag{5.45}$$

Combining (5.44), (5.45), we get

$$\mathrm{ind}_K(\mathbf{\Lambda}_K,R^*_K)(g_1) > (\delta+\frac{\tau}{4})\beta\,. \tag{5.46}$$

Suppose that for all $\vec{y} \in R^*_K(\vec{x}_0, \dots, \vec{x}_{K-1})$, we have $g_2(\Lambda^*_K(\vec{x}_0, \dots, \vec{x}_{K-1}), \vec{y}) < (\delta + \tau/16)$. Then

$$(\delta + \frac{\tau}{4})\beta < \mathrm{ind}_K(\mathbf{\Lambda}_K, R^*_K)(g_1) \le (\delta + \frac{\tau}{16}) \,\mathrm{ind}_K(\mathbf{\Lambda}_K, R^*_K)(g_3) \le$$

$$\le (\delta + \frac{\tau}{16}) \,\mathrm{ind}_K(\mathbf{\Lambda}_K)(g_3)\,. \tag{5.47}$$

Using Lemma 5.4 once again, we obtain

$$(\delta + \frac{\tau}{4})\beta < (\delta + \frac{\tau}{16}) \,\mathrm{ind}_K(\mathbf{\Lambda}_K)(g_3) \le (\delta + \frac{\tau}{16})(\beta + 8\kappa K) \le (\delta + \frac{\tau}{4})\beta$$

with contradiction. Whence there exist vectors $\vec{x}_0, \dots, \vec{x}_{K-1}$, $\vec{y}$ such that $g_2(\Lambda^*_K(\vec{x}_0, \dots, \vec{x}_{K-1}), \vec{y}) \ge (\delta + \tau/16)$ and $\vec{y} \in R_K(\vec{x}_0, \dots, \vec{x}_{K-1}) \setminus E_K(\vec{x}_0, \dots, \vec{x}_{K-1})$. Put $\vec{t} = \vec{y} + \vec{s}$ and $\Lambda' = \Lambda^*_K(\vec{x}_0, \dots, \vec{x}_{K-1})$. We obtain the vector $\vec{t}$, the sets $E'_1 = (\tilde{E}_1 - y_1) \cap \Lambda'$, $E'_2 = (\tilde{E}_2 - y_2) \cap \Lambda'$ and the Bohr set Λ' which satisfy the conditions 1)—3).

Let us estimate D, ε' and N'. At the each step of the algorithm the dimension of Bohr sets increases at most 1. Since the total number of steps does not exceed K_0, it follows that $D \le d + 2^{30}\tau^{-5}\beta^{-5}\sigma^{-3}$, $\varepsilon' \ge (2^{-10}\varepsilon)^D \varepsilon_0$, $N' \ge (2^{-10}\varepsilon)^D N$. Using Lemma 2.1 and (5.26), we obtain that the set Λ' is not empty. This completes the proof.

6. *Proof of main result.*

Let us put Theorem 4.1 and Proposition 5.1 together in a single proposition.

PROPOSITION 6.1. *Let $\Lambda = \Lambda(\theta, \varepsilon_0, N)$ be a Bohr set, $\theta \in \mathbf{T}^d$, and $\vec{s} = (s_1, s_2) \in \mathbf{Z}^2$. Let E_1, E_2 be sets, $E_i = \beta_i |\Lambda|$, $i = 1, 2$, $\beta = \beta_1\beta_2$. Suppose $\mathbf{E} = E_1 \times E_2$ is a subset of $(\Lambda + s_1) \times (\Lambda + s_2)$, E_1, E_2 are $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform subsets of $\Lambda + s_1$, $\Lambda + s_2$, respectively, $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$, $\varepsilon = (2^{-100}\alpha_0^2)/(100d)$. Suppose that A is a subset of $\mathbf{E}$, $\delta_{\mathbf{E}}(A) = \delta$, and A has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \ne 0$. Let*

$$\log N \ge 2^{1000000} (2^{250000} \delta^{-20000} \beta^{-200} + d)^3 \log \frac{1}{\delta\beta\varepsilon_0}\,. \tag{6.1}$$

Then there is a Bohr set $\tilde{\Lambda}$ and a vector $\vec{y} = (y_1, y_2) \in \mathbf{Z}^2$ with the following properties : there exist sets $E_1' \subseteq (E_1 - y_1 \cap \tilde{\Lambda})$, $E_2' \subseteq (E_2 - y_2 \cap \tilde{\Lambda})$ such that

1) *Let* $|E_1'| = \beta_1'|\tilde{\Lambda}|, |E_2'| = \beta_2'|\tilde{\Lambda}|$ *and* $\beta' = \beta_1'\beta_2'$. *Then* $\beta' \geq 2^{-1500}\delta^{100}\beta$.

2) E_1', E_2' *are* $(\alpha_0', 2^{-10}\varepsilon'^2)$*–uniform, where* $\alpha_0' = 2^{-2000}\delta^{96}\beta'^{48}$,
$\varepsilon' = \frac{2^{-100}\alpha_0'^2}{100D'}$, $D \leq D' = 2^{250000}\delta^{-20000}\beta^{-200} + d$.

3) *For* $\tilde{\Lambda} = \Lambda_{\tilde{\theta}, \tilde{\varepsilon}, \tilde{N}}$ *we have* $\tilde{\theta} \in \mathbf{T}^D$, $\tilde{\varepsilon} \geq (2^{-100}\varepsilon'^2)^D\varepsilon_0$ *and* $\tilde{N} \geq (2^{-100}\varepsilon'^2)^D N$.

4) $\delta_{E_1' \times E_2'}(A) \geq \delta + 2^{-600}\delta^{37}$.

The following lemma is due to B. Green.

Lemma 6.1. *Let N be a natural number. Suppose A is a subset of $[-N, N]^2$, $|A| = \delta(2N+1)^2$, and A has no triples $\{(k,m), (k+d,m), (k,m+d)\}$ with $d > 0$. Then there exists a set $A_1 \subseteq A$ such that*

1) $|A_1| \geq \delta^2(2N+1)^2/4$ *and*

2) A_1 *has no triples* $\{(k,m), (k+d,m), (k,m+d)\}$ *with* $d \neq 0$.

Proof. Since $|A| = \delta(2N+1)^2$, it follows that

$$\sum_{\vec{v}}\sum_{\vec{s}} A(\vec{s})A(\vec{v}-\vec{s}) = \sum_{\vec{s}}\sum_{\vec{v}} A(\vec{s})A(\vec{v}-\vec{s}) = \delta^2(2N+1)^4 \,. \tag{6.2}$$

Clearly, the summation in (6.2) is taken over $\vec{v} \in [-2N, -2N+1, \dots, 2N-1, 2N]^2$. Hence there exists a vector $\vec{v}$ such that $|A \cap (\vec{v} - A)| \geq \delta^2(2N+1)^4/(4N+1)^2 \geq \delta^2(2N+1)^2/4$. Put $A_1 = A \cap (\vec{v} - A)$. We have $A_1 \subseteq A$. It follows that A_1 does not contain any triple $\{(k,m), (k+d,m), (k,m+d)\}$ with $d > 0$. Since $A_1 \subseteq \vec{v} - A$, it follows that A_1 has no triples $\{(k,m), (k+d,m), (k,m+d)\}$ with $d < 0$. This completes the proof.

Proof of Theorem 1.3.

Suppose $A \subseteq [-N, N]$ and A has no triples $\{(k,m), (k+d,m), (k,m+d)\}$ with $d > 0$. Using Lemma 6.1, we get the set A', $A' \subseteq A$, $|A'| \geq \delta^2/4(2N+1)^2$ such that A' has no triples $(k,m), (k+d,m), (k,m+d)$ with $d \neq 0$. Let $\delta' = \delta^2/4$.

The proof of Theorem 1.3 is a sort of an algorithm.

After the ith step of the algorithm an integer vector $\vec{s}_i = (s_i^{(1)}, s_i^{(2)})$ and sets : a regular Bohr set $\Lambda_i = \Lambda_{\theta_i,\varepsilon_i,N_i}$, sets $E_i^{(1)} - s_i^{(1)} \subseteq \Lambda_i$, $E_i^{(2)} - s_i^{(2)} \subseteq \Lambda_i$, will be constructed. Let $|E_i^{(1)}| = \beta_i^{(1)}|\Lambda_i|$, $|E_i^{(2)}| = \beta_i^{(2)}|\Lambda_i|$, $\beta_i = \beta_i^{(1)}\beta_i^{(2)}$, $\mathbf{E_i} = E_i^{(1)} \times E_i^{(2)}$.

The sets Λ_i, $E_i^{(1)}$, $E_i^{(2)}$ satisfy the following conditions

1) $\beta_i \ge 2^{-1500}\delta'^{100}\beta_{i-1}$.

2) $E_i^{(1)}, E_i^{(2)}$ are $(\alpha_0^{(i)}, 2^{-10}(\varepsilon_i')^2)$–uniform, $\alpha_0^{(i)} = 2^{-2000}\delta'^{96}\beta_i^{48}$, $\varepsilon_i' = 2^{-100}(\alpha_0^{(i)})^2/(100d_i)$.

3) $\Lambda_i = \Lambda_{\theta_i,\varepsilon_i,N_i}, \tilde{\theta} \in \mathbf{T}^{d_i}, d_i \le 2^{250000}\delta'^{-20000}\beta_{i-1}^{-200} + d_{i-1}, \varepsilon_i \ge (2^{-100}(\varepsilon_i')^2)^{d_i}\varepsilon_{i-1}$, $N_i \ge (2^{-100}(\varepsilon_i')^2)^{d_i}N_{i-1}$.

4) $\delta_{\mathbf{E_i}}(A') \ge \delta_{\mathbf{E_{i-1}}}(A') + 2^{-600}\delta'^{37}$.

Proposition 6.1 allows us to carry the $(i+1)$th step of the algorithm. By this Proposition there exists a new vector $\vec{s}_{i+1} = (s_{i+1}^{(1)}, s_{i+1}^{(2)}) \in \mathbf{Z}^2$ and sets : a regular Bohr set $\Lambda_{i+1} = \Lambda_{\theta_{i+1},\varepsilon_{i+1},N_{i+1}}$, sets $E_{i+1}^{(1)} - s_i^{(1)} \subseteq \Lambda_{i+1}$, $E_{i+1}^{(2)} - s_{i+1}^{(2)} \subseteq \Lambda_{i+1}$, $\mathbf{E_{i+1}} = E_{i+1}^{(1)} \times E_{i+1}^{(2)}$, which satisfy 1) — 4).

Put $\theta_0 = \{0\}$, $\Lambda_0 = \Lambda_{\theta_0,1,N}$ and $E_1 = E_2 = [-N, N]$, $\beta_0 = 1$. Clearly, E_1, E_2 are $(2^{-2000}\delta'^{96}, 2^{-10000}\delta'^{400})$–uniform. Hence we have constructed zeroth step of the algorithm.

Let us estimate the total number of steps of our procedure. For an arbitrary i we have $\delta_{\mathbf{E_i}}(A') \le 1$. Using this and condition 4), we obtain that the total number of steps cannot be more then $2^{700}\delta'^{-36} = K$.

Condition 3) implies $\beta_i \ge (2^{-1500}\delta'^{100})^i$. Hence $d_i \le (C_1\delta)^{-C_1' i}$, where $C_1, C_1' > 0$ are absolute constants.

To prove Theorem 1.3, we need to verify condition (6.1) at the last step of the algorithm. Using 3), we get

$$N_K \ge (C_2\delta)^{C_3\delta^{-C_4 K}} N\,,$$

where $C_2, C_3, C_4 > 0$ are absolute constants. Condition (6.1) can be rewrite as

$$N_K \ge (C_2'\delta)^{-C_3'\delta^{-C_4' K}}\,,$$

where $C_2', C_3', C_4' > 0$ are absolute constants. Whence we need to check up the following inequality

$$N \ge (C_2''\delta)^{-C_3''\delta^{-C_4''K}} = \exp(\delta^{-C'\delta'^{-36}})\,, \tag{6.3}$$

where $C_2'', C_3'', C_4'', C' > 0$ are absolute constants. By assumption

$$\delta \gg \frac{1}{(\log\log N)^{1/73}}\,.$$

It follows that

$$\delta' \gg \frac{1}{(\log\log N)^{2/73}}$$

and we get (6.3). Hence A' has a triple $\{(k,m),(k+d,m),(k,m+d)\}$, where $d \ne 0$. This contradiction concludes the proof.

NOTE. Certainly, the constant 73 in Theorem 1.3 can be slightly decreased. Nevertheless, it is the author's opinion that this constant cannot be lowered to anything like 1 without a new idea.

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