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Dynamical systems with low recurrence rate

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Abstract. The question on the recurrence rate of a dynamical system in a metric space of finite Hausdorff measure is considered. For such systems upper bounds for the rate of simple recurrence are due to Boshernitzan and ones for the rate of multiple recurrence to the present author. The paper is concerned with finding lower bounds for the rate of multiple recurrence. More precisely, an example of a dynamical system (an odometer or a von Neumann transformation) with a low rate of multiple recurrence is constructed. Behrend's theorem on sets containing no arithmetic progressions is used in the proof.

Bibliography: 22 titles.

§ 1. Introduction

Let X be a set with a sigma-algebra $\mathscr{B}$ of its subsets and let T be a measurable map of X into itself preserving a measure μ. We shall assume in what follows that $\mu(X) = 1$. We call the quadruple $(X, \mathscr{B}, \mu, T)$ a dynamical system with invariant measure. The well-known *Poincaré's recurrence theorem* says that for each measurable subset E of X, $\mu E > 0$, there exists a positive integer n such that $\mu(E \cap T^{-n} E) > 0$.

Additionally, let X be a metric space with metric $d(\,\cdot\,,\cdot\,)$ and let $\mathscr{B}$ be the Borel sigma-algebra. Then one can state Poincaré's theorem as follows.

Theorem 1.1. *Let X be a metric space with metric $d(\,\cdot\,,\cdot\,)$ and μ a Borel measure on X. Let T be a map of X into itself preserving the measure μ. Then the following equality holds for almost all $x \in X$:*

$$\liminf_{n\to\infty} d(T^n x, x) = 0. \tag{1}$$

In [1] (see also [2] and [3]) Furstenberg generalized Poincaré's theorem to the case of several powers of the map T.

Theorem 1.2. *Let X be a space with sigma-algebra of measurable sets $\mathscr{B}$ and μ a measure on X. Let T be a map of X into itself preserving μ and assume that $k \geqslant 3$. Then for each measurable set E with $\mu E > 0$ there exists a positive integer n such that*

$$\mu(E \cap T^{-n} E \cap T^{-2n} E \cap \dots \cap T^{-(k-1)n} E) > 0.$$

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If X is a metric space, then we can state Theorem 1.2 as follows.

Theorem 1.2′. *Let X be a metric space with metric $d(\cdot,\cdot)$ and μ a Borel measure on X. Let T be a map of X into itself preserving μ and assume that $k \geqslant 3$. Then for almost all $x \in X$,*

$$\liminf_{n\to\infty} \max\bigl\{d(T^n x, x), d(T^{2n} x, x), \dots, d(T^{(k-1)n} x, x)\bigr\} = 0.$$

In [4] Furstenberg and Katznelson extended Theorem 1.2 to several commuting maps. We state their result in the case when X is a metric space.

Theorem 1.3. *Let X be a metric space with metric $d(\cdot,\cdot)$ and μ a Borel measure on X. Assume that $k \geqslant 2$ and let $T_1, T_2, \dots, T_k$ be commuting maps of X into itself preserving the measure μ. Then for almost all $x \in X$,*

$$\liminf_{n\to\infty} \max\bigl\{d(T_1^n x, x), d(T_2^n x, x), \dots, d(T_k^n x, x)\bigr\} = 0.$$

Let A be a subset of the set of positive integers and let $[N]$ be the interval of positive integers $\{1, 2, \dots, N\}$. By the *upper density* of A one means the quantity

$$D^*(A) = \limsup_{N\to\infty} \frac{|A \cap [N]|}{N},$$

where $|A \cap [N]|$ is the cardinal number of the set $A \cap [N]$.

As shown in [2], Theorem 1.2 is equivalent to Szemerédi's famous theorem on arithmetic progressions.

Theorem 1.4. *Let A be an arbitrary set of positive integers and assume that $D^*(A) > 0$. Then for each integer $k \geqslant 3$, A contains an arithmetic progression of length k.*

We must point out that for $k = 3$ Theorem 1.4 had been proved before by Roth (see [5]).

One shows easily that Theorems 1.2 and 1.2′ follow from Theorem 1.4 (see [2]). In actual fact, Theorem 1.2 (1.2′) and Theorem 1.4 are equivalent. For the proof of their equivalence Furstenberg has established the following beautiful result, which is called *Furstenberg's correspondence principle*.

Theorem 1.5. *Let A be an arbitrary set of positive integers such that $D^*(A) > 0$. Then there exist a dynamical system with invariant measure $(X, \mathscr{B}, \mu, T)$ and a measurable set E, $\mu E = D^*(A)$, such that for all integers $k \geqslant 3$ and all positive integers $m_1, m_2, \dots, m_{k-1}$ one has the relation*

$$D^*(A \cap (A + m_1) \cap \dots \cap (A + m_{k-1})) \geqslant \mu(E \cap T^{-m_1} E \cap \dots \cap T^{-m_{k-1}} E). \tag{2}$$

Theorem 1.5 points to close connections between ergodic theory and combinatorial problems on arithmetic progressions.

Assertion 1.6. *Theorems 1.2 and 1.5 yield Theorem 1.4.*

Proof. Let $k \geqslant 3$ be an integer and let $A \subseteq \mathbb{N}$ be a set containing no arithmetic progressions of length k and having a positive upper density. By Theorem 1.5 there exist a dynamical system $(X, \mathscr{B}, \mu, T)$ and a measurable set E of positive measure such that inequality (2) holds for all positive integers $m_1, m_2, \dots, m_{k-1}$. On the other hand, there exists by Theorem 1.2 an integer $n > 0$ such that

$$\mu(E \cap T^{-n}E \cap T^{-2n}E \cap \cdots \cap T^{-(k-1)n}E) > 0. \tag{3}$$

We set $m_1 = n$, $m_2 = 2n$, $\dots$, $m_{k-1} = (k-1)n$. Then (2) and (3) yield the inequality $D^*(A \cap (A+n) \cap \cdots \cap (A+(k-1)n)) > 0$, which contradicts the assumption that A contains no arithmetic progressions of length k. The proof is complete.

The main aim of the present paper is to find lower bounds for the multiple recurrence rate for metric spaces of finite Hausdorff measure. As Boshernitzan proved in [6], if $(X, \mathscr{B}, \mu, T, d)$ is a dynamical system such that the space X has a finite Hausdorff measure, then one can significantly refine Poincaré's Theorem 1.1 (we formulate Boshernitzan's result more accurately below). In the wake of this result one comes in the natural way to the question about similar refined versions of Theorems 1.2 and 1.3 for spaces of finite Hausdorff measure. Such versions of Theorem 1.2 and, partially, also Theorem 1.3 (the case $k = 2$) have recently been obtained in [7]–[11]. In all these versions lower bounds for the multiple recurrence rate are considered. In this paper we present simple lower bounds for this recurrence rate.

Before stating Boshernitzan's theorem and our central result we give several definitions.

Consider the measure $H_h(\,\cdot\,)$ on X defined as follows:

$$H_h(E) = \lim_{\delta \to 0} H_h^\delta(E), \tag{4}$$

where $h(t)$ is a non-negative ($h(0) = 0$) continuous increasing function and $H_h^\delta(E) = \inf\bigl\{\sum h(\delta_j)\bigr\}$, where one takes inf over countable covers of E by open sets $\{B_j\}$, $\operatorname{diam}(B_j) = \delta_j < \delta$. If $h(t) = t^\alpha$, then $H_h(\,\cdot\,)$ is the standard Hausdorff measure, which we denote by $H_\alpha(\,\cdot\,)$.

The outer measure $H_h(\,\cdot\,)$ is sigma-additive on the sigma-algebra of Carathéodory-measurable subsets. As is well known, this sigma-algebra contains all Borel subsets.

We shall say that measures μ and H_h are *compatible* if each μ-measurable set is also H_h-measurable (in the sense of Carathéodory measurability).

Definition 1.7. Let $x \in X$. We call the quantity

$$C(x) = C^h(x) := \liminf_{n \to \infty} \{n \cdot h(d(T^n x, x))\}$$

the *recurrence constant* of the point x.

In [6] Boshernitzan obtained the first quantitative analogue of Theorem 1.1. A similar result has been independently proved by Moshchevitin [12].

Theorem 1.8. *Let X be a metric space with $H_h(X) < \infty$ and let T be a map of X into itself preserving the measure μ. Assume also that the measures μ and H_h are compatible. Then $C(x) < \infty$ for almost all points x in X with respect to μ.*

Boshernitzan's result is a significant improvement over Poincaré's theorem: consider the simplest example when $H_1(X) < \infty$. By Boshernitzan's theorem, for each $\varepsilon > 0$ and almost all $x \in X$ one has

$$\liminf_{n\to\infty}\{n^{1-\varepsilon}\cdot d(T^n x, x)\} = 0,$$

whereas Poincaré's theorem only yields $\liminf_{n\to\infty}\{d(T^n x, x)\} = 0$.

In [6] Boshernitzan obtained several applications of Theorem 1.8 to various dynamical systems.

In [13] we proved a result refining slightly Theorem 1.8.

Theorem 1.9. *Let X be a metric space with $H_h(X) < \infty$ and let T be a map of X into itself preserving the measure μ. Assume that the measures μ and H_h are compatible. Then $C(x)$ is an integrable function (with respect to μ) and for each μ-measurable set A one has*

$$\int_A C(x)\,d\mu \leqslant H_h(A). \tag{5}$$

On the other hand, if $H_h(A) = 0$, then

$$\int_A C(x)\,d\mu = 0$$

without the assumption that μ and H_h are compatible measures.

Remark 1.10. In view of an example in § 7 of [6], the estimate (5) in Theorem 1.9 is best possible.

We now return to Theorems 1.2 and 1.4. We want to discuss several quantitative versions of them.

Let N and $k \geqslant 3$ be positive integers and let

$$a_k(N) = \frac{1}{N}\max\bigl\{|A| : A \subseteq [N],\ A \text{ contains no arithmetic progressions of length } k\bigr\}.$$

It is clear that Theorem 1.4 is equivalent to the convergence $a_k(N) \to 0$ as $N \to \infty$.

The first quantitative result on the rate of approach to zero of the function $a_k(N)$ for $k = 3$ is due to Roth (see [5]), who used the Hardy–Littlewood method to prove the inequality

$$a_3(N) \ll \frac{1}{\log\log N}\,.$$

In other words, Roth proved a quantitative version of Theorem 1.4 and therefore also of Theorem 1.2 for $k = 3$.

After the papers of Roth, Szemerédi, and Furstenberg several authors made considerable improvements on their results. The best result so far on upper bounds for the quantity $a_3(N)$ is Bourgain's [14], who has proved that

$$a_3(N) \ll \sqrt{\frac{\log\log N}{\log N}}\,. \tag{6}$$

Gowers [7] has obtained a quantitative result on the rate of approach to zero of the function $a_k(N)$ for all $k \geqslant 4$.

Theorem 1.11. *For all $k \geqslant 4$,*

$$a_k(N) \ll \frac{1}{(\log\log N)^{c_k}}\,,$$

with constant c_k depending only on k.

Behrend [15] has obtained a lower bound for the quantity $a_3(N)$. Rankin [16] has extended Behrend's result to all $k \geqslant 3$ (see also [17]).

Theorem 1.12. *Let ε be an arbitrary positive number and let k be an integer, $k \geqslant 3$. Then for all sufficiently large N,*

$$a_k(N) \geqslant \exp\bigl(-(1+\varepsilon)C_k(\log N)^{1/(k-1)}\bigr),$$

where C_k is an effective positive constant depending only on k.

A quantitative version of Theorem 1.3 for $k=3$ was obtained in [8], [9] and refined in [11].

Consider the two-dimensional lattice $[N]^2$ with basis $\{(1,0),\,(0,1)\}$. Let

$$L(N) = \frac{1}{N^2}\max\{|A| : A \subseteq [N]^2 \text{ and } A \text{ contains no triples of the form } (k,m),\ (k+d,m),\ (k,m+d),\ d>0\}. \tag{7}$$

Theorem 1.13. *The inequality $L(N) \ll 1/(\log\log N)^{C'}$ holds with effective constant C'.*

This bound yields the following result on the recurrence rate in Theorem 1.3 for $k=2$ (see [10], [11]).

Let S and R be commuting maps of the space X preserving the measure μ.

Definition 1.14. Let $x \in X$. Then one calls the quantity

$$C_{S,R}(x) = C^h_{S,R}(x) := \liminf_{n\to\infty}\bigl\{L^{-1}(n)\cdot\max\{h(d(S^n x,x)),h(d(R^n x,x))\}\bigr\},$$

where $L^{-1}(n) = 1/L(n)$, the *simultaneous* (*or the multiple*) *recurrence constant* of the point x.

Theorem 1.15. *Let X be a metric space with $H_h(X) < \infty$ and let S and R be commuting maps of X into itself preserving a measure μ. Let μ and H_h be compatible measures. Then $C_{S,R}(x)$ is an integrable function* (*with respect to μ*) *and for each μ-measurable set A one has the inequality*

$$\int_A C_{S,R}(x)\,d\mu \leqslant H_h(A). \tag{8}$$

On the other hand, if $H_h(A) = 0$, then

$$\int_A C_{S,R}(x)\,d\mu = 0$$

even without the assumption that μ and H_h are compatible.

Thus, Theorems 1.8, 1.9, 1.15 produce upper bounds for the integrals of the functions $C(x)$ and $C_{S,R}(x)$. As pointed out before (see Remark 1.10), the upper bound (5) for $C(x)$ is best possible. One cannot say the same about inequality (8) because one does not know the actual growth order of $L(n)$. In this paper we obtain lower bounds for the multiple recurrence function in the case of the action on X of several powers of the map T. One can regard such a result as a certain quantitative analogue of Theorem 1.5. Since powers of T commute, this gives us incidentally a lower bound for the multiple recurrence function in the case of several commuting maps acting on X.

We now state our central result in this paper.

Let k be a fixed integer, $k \geqslant 3$. Assume that for each N one fixes a non-empty set $A^{(N)} \subseteq \mathbb{Z}_N$ containing no arithmetic progressions of length k. Let $\rho(N)$ be the density of the set $A^{(N)}$ in $\mathbb{Z}_N$: $\rho(N) = |A^{(N)}|/N$. Then $\rho(1) = 1$. Since $A^{(N)}$ contains no arithmetic progressions in $\mathbb{Z}_N$, it follows that $A^{(N)}$ contains no arithmetic progressions in $\mathbb{Z}$ either. By Theorem 1.4, $\rho(N) \to 0$ as $N \to \infty$.

Theorem 1.16. *Let $\psi\colon \mathbb{N} \to \mathbb{R}^+$ be an arbitrary increasing function, $X = [0,1]$, let μ be Lebesgue measure in X, and let $\{A^{(N)}\}_{N=1}^{\infty}$ be the above-constructed sequence of sets. Then there exists a dynamical system $(X, \mathscr{B}, \mu, T, d)$ such that μ and the Hausdorff measure H_1 are compatible, $H_1(X) = 0$, and the following inequality holds for almost all x in X with respect to μ:*

$$\liminf_{n\to\infty} \biggl\{ \frac{\psi(n)}{\rho(n)} \max\{d(T^n x, x), d(T^{2n}x, x), \dots, d(T^{(k-1)n}x, x)\} \biggr\} \geqslant 1. \tag{9}$$

Recall that we denote by H_1 the Hausdorff measure with function $h(t) = t$.

Remark 1.17. The equality $H_1(X) = 0$ in Theorem 1.16 is very important: without the assumption that $H_1(X) < \infty$ the result is trivial because one can always choose a metric d such that the space X has infinite Hausdorff measure $H_1(X)$ and the lower limit in (9) is $+\infty$. We also point out that we can certainly replace the equality $H_1(X) = 0$ by a stronger equality of the form $H_{tg(t)} = 0$, where $g(t)$ is a non-increasing function such that $g(t) \to +\infty$ as $t \to 0+$.

The method developed for the proof of our central Theorem 1.16 can be applied to the investigation of the ordinary (not multiple) recurrence $C(x)$. In § 3 we prove the following result. Let $\kappa \in [0,1]$ be an arbitrary number. Then there exists a dynamical system $(X, \mathscr{B}, \mu, T, d)$ such that $H_1(X) = 1$ and for almost all points x in X with respect to μ one has $C(x) = \kappa$.

Our constructions develop the approach of [6] and the monograph [2].

§ 2. Proof of Theorem 1.16

We require a simple result proved, in effect, in [18]. The reader can find another proof of a similar result in [19]. Since the statement we require is slightly different from the one in [18], we present a proof of our own.

Lemma 2.1. *Let N be a positive integer, A an arbitrary non-empty subset of $\mathbb{Z}_N$, and $\varphi \geqslant 1$ a real number. Then there exist residues $a_1, \dots, a_l \in \mathbb{Z}_N$ and a partitioning of $\mathbb{Z}_N$ into disjoint subsets $A_1, A_2, \dots, A_l$ and B such that*

(1) $A_i \subseteq A + a_i$ *for* $i = 1, \dots, l$,

(2) $|A_i| \geqslant |A|/\varphi$ *for* $i = 1, \dots, l$,

(3) $|B| \leqslant N/\varphi$.

Proof. We carry out the proof by induction. At the nth step we construct the sets $A_1, \dots, A_n$, the residues $a_1, \dots, a_n$, and auxiliary sets $B_1, \dots, B_n$ such that $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$.

Let $n = 1$. We set $a_1 = 0$, $A_1 = A$, and $B_1 = \mathbb{Z}_N \setminus A_1$. If $|B_1| \leqslant N/\varphi$, then the proof of Lemma 2.1 is complete. In fact, let $B = B_1$. One sees easily that conditions (1)–(3) are fulfilled.

Assume that at the nth step of the inductive procedure we have constructed the sets $A_1, \dots, A_n$ and the residues $a_1, \dots, a_n$. Also let $B_n = \mathbb{Z}_N \setminus \bigl(\bigsqcup_{i=1}^n A_i\bigr)$. If $|B_n| \leqslant N/\varphi$, then the proof of Lemma 2.1 is complete. In fact, we set $B = B_n$. Then conditions (1)–(3) hold for the sets $A_1, \dots, A_n$, B, and the residues $a_1, \dots, a_n$.

Assume that $|B_n| > N/\varphi$. Then we obtain

$$\sum_{t \in \mathbb{Z}_N} |B_n \cap (A + t)| = |A|\,|B_n| \geqslant \frac{N|A|}{\varphi} \tag{10}$$

and there exists $t \in \mathbb{Z}_N$ such that $|B_n \cap (A + t)| \geqslant |A|/\varphi$. We set $a_{n+1} = t$ and $A_{n+1} = B_n \cap (A + a_{n+1})$. Then for all $i = 1, \dots, n$ we have $A_{n+1} \cap A_i = \varnothing$. In addition, $A_{n+1} \subseteq A + a_{n+1}$.

Since $|A_i| \geqslant |A|/\varphi > 0$ for each i, the inductive procedure terminates after fewer than $[N\varphi/|A|] + 1$ steps. The proof is complete.

Proof of Theorem 1.16. Let α_m be an arbitrary non-increasing real number sequence convergent to zero, $\alpha_m \in (0, 1)$. The function $\psi(n)$ is defined only for the positive integer values of the variable n. We extend $\psi(n)$ linearly to the entire real axis obtaining a continuous increasing function, which we denote by $\psi(t)$ again, $t \in \mathbb{R}$. Let $\varphi(t) = \sqrt{\psi(t)}$ and let $\varphi^*(t) = \max\{1, \varphi(t)\}$. Consider the non-decreasing integer sequence

$$N_0 \leqslant N_1 \leqslant \cdots \leqslant N_m \leqslant \cdots,$$

where $N_0 = 1$, and for $m \geqslant 1$ we have $N_m = \lceil \varphi^{-1}(2\alpha_m^{-1}\varphi^*(2)N_0 N_1 \cdots N_{m-1}) \rceil$. Here φ^{-1} is the inverse function of φ. Then we have $N_m \geqslant 2$, $m \geqslant 1$.

Let X be the space of sequences $(x_1, x_2, \dots)$, $0 \leqslant x_i < N_i$, $i \geqslant 1$. We call the set

$$C(a_1, \dots, a_l) = \bigl\{x = (x_1, x_2, \dots) \in X : x_1 = a_1, \ \dots, \ x_l = a_l\bigr\}$$

an *elementary cylinder* of rank l. To each sequence $x = (x_1, x_2, \dots) \in X$ one assigns a point in $[0, 1]$ by the following rule: $x \to \sum_{i=1}^\infty x_i/(N_0 N_1 \cdots N_i)$.

Removing countably many points from X one makes this map bijective. This allows one to regard the space X as the interval $[0,1]$.

Let a be a non-negative integer and $N \in \mathbb{N}$. Let $a^{+}(N)$ be equal to $a+1 \pmod{N}$. Let T be the map of the space X into itself defined by the formula $Tx = y$, $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$, where

$$
\begin{aligned}
&y_1 = x_1^{+}(N_1),\\
&y_2 = \begin{cases} x_2^{+}(N_1) & \text{if } x_1 + 1 = N_1,\\ y_2 & \text{otherwise,}\end{cases}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&y_m = \begin{cases} x_m^{+}(N_1) & \text{if } x_1 + 1 = N_1,\ x_2 + 1 = N_2, \dots, x_{m-1} + 1 = N_{m-1},\\ y_m & \text{otherwise,}\end{cases}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

In the space X we have the natural group operation $+$ and we have $Tx = x + 1$, where $1 = (1, 0, 0, \dots)$. Clearly, T preserves Haar measure μ, which is equal to Lebesgue measure in the present case. An elementary cylinder of rank l has measure $1/(N_0 N_1 \cdots N_l)$.

Consider arbitrary N_s. There exists by assumption a non-empty subset $A^{(N_s)} = A^{(s)}$ of $\mathbb{Z}_{N_s}$ containing no arithmetic progressions of length k. Applying Lemma 2.1 to the set $A^{(s)}$ and $\varphi = \varphi(N_s)$, for each N_s we construct sets $A_1^{(s)}, \dots, A_l^{(s)}$, $l = l(s)$, and $B^{(s)}$ with properties (1)–(3).

Let $x, y \in X$, $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$. Consider the function

$$
\begin{aligned}
d(x,y) = \bigl\{&\psi^{-1}(N_0 \cdots N_m)\rho(N_0 \cdots N_m),\\
&\text{where } m = m(x,y) \text{ is the largest index such}\\
&\text{that } x_1 = y_1, \dots, x_{m-1} = y_{m-1}\\
&\text{and either } x_m, y_m \in A_i^{(m)} \text{ for some } i \in 1, 2, \dots, l(m)\\
&\text{or } x_m, y_m \in B^{(m)}\bigr\};
\end{aligned}
$$

here $\psi^{-1} = 1/\psi$. One sees easily that $d(x,y)$ is a non-Archimedean metric in X. Consider the Hausdorff measure H_1 on X. Since each elementary cylinder is a closed and therefore a Borel subset of the metric space (X, d), the measures μ and H_1 are compatible.

We claim that $H_1(X) = 0$. Consider arbitrary $\delta > 0$. Since $\rho(N) \to 0$ as $N \to \infty$, there exists a positive integer m such that

$$\frac{\rho(N_0 \cdots N_m)}{\psi(N_0 \cdots N_m)} < \delta.$$

Consider a partitioning of the space X into the subsets

$$U_i(\vec{a}) = \bigl\{x = (x_1, x_2, \dots) \in X : x_1 = a_1,\ \dots,\ x_{m-1} = a_{m-1},\ x_m \in A_i^{(m)}\bigr\},$$

where $i = 1, \dots, l(m)$ and

$$B(\vec{a}) = \bigl\{x = (x_1, x_2, \dots) \in X : x_1 = a_1,\ \dots,\ x_{m-1} = a_{m-1},\ x_m \in B^{(m)}\bigr\},$$

where $\vec{a} \in [N_1] \times \cdots \times [N_{m-1}] := F_{m-1}$. Then

$$X = \bigsqcup_{\vec{a} \in F_{m-1}} \left(B(\vec{a}) \sqcup \left(\bigsqcup_{i=1}^{l(m)} U_i(\vec{a}) \right) \right).$$

For each $\vec{a} \in F_{m-1}$ and each $i \in 1, 2, \ldots, l(m)$ we have

$$\operatorname{diam} U_i(\vec{a}) \leqslant \frac{\rho(N_0 \cdots N_m)}{\psi(N_0 \cdots N_m)} < \delta.$$

In a similar way, for each $\vec{a} \in F_{m-1}$ we have

$$\operatorname{diam} B(\vec{a}) \leqslant \frac{\rho(N_0 \cdots N_m)}{\psi(N_0 \cdots N_m)} < \delta.$$

It follows from assumption (2) of Lemma 2.1 that

$$l(m) \leqslant \frac{N_m \varphi(N_m)}{|A^{(m)}|} = \frac{\varphi(N_m)}{\rho(N_m)}\,.$$

In view of the last inequality, we obtain

$$\begin{aligned} H_1^\delta(X) &\leqslant |F_{m-1}| \left(\frac{\varphi(N_m)}{\rho(N_m)} + 1 \right) \frac{\rho(N_0 \cdots N_m)}{\psi(N_0 \cdots N_m)} \\ &\leqslant 2N_0 \cdots N_{m-1} \frac{\varphi(N_m) \rho(N_0 \cdots N_m)}{\rho(N_m) \psi(N_0 \cdots N_m)} \leqslant 2N_0 \cdots N_{m-1} \frac{\varphi(N_m)}{\psi(N_m)}\,. \end{aligned}$$

We have $N_m \geqslant \varphi^{-1}(2\alpha_m^{-1} N_0 \cdots N_{m-1})$. Hence $2N_0 \cdots N_{m-1} \leqslant \alpha_m \varphi(N_m)$. In view of the last inequality, we obtain

$$H_1^\delta(X) \leqslant \alpha_m \frac{\varphi^2(N_m)}{\psi(N_m)} \leqslant \alpha_m.$$

Since $\alpha_m \to 0$ as $m \to \infty$, it follows that $H_1(X) = 0$.

We now prove inequality (9). Let

$$\widetilde{B}^{(s)} = \{x \in X : x_s \in B^{(s)}\} \subseteq X \quad \text{and} \quad \mathbf{B} = \bigcap_{n=1}^{+\infty} \bigcup_{s \geqslant n}^{+\infty} \widetilde{B}^{(s)}.$$

Then we have $\mu(\widetilde{B}^{(s)}) = |B^{(s)}|/N_s \leqslant 1/\varphi(N_s)$. Since $\varphi(N_s) \geqslant N_0 \cdots N_{s-1}$ and $N_s \geqslant 2$ for $s \geqslant 1$, it follows that

$$\sum_{s=1}^{\infty} \mu(\widetilde{B}^{(s)}) \leqslant \sum_{s=1}^{\infty} \frac{1}{\varphi(N_s)} \leqslant \sum_{s=1}^{\infty} \frac{1}{N_0 \cdots N_{s-1}} < \infty. \tag{11}$$

By the Borel–Cantelli lemma we now obtain that $\mu\mathbf{B} = 0$.

We claim that inequality (9) holds for each x outside $\mathbf{B}$.

Let $x = (x_1, x_2, \dots) \in X \setminus \mathbf{B}$. Since $x \notin \mathbf{B}$, there exists $M = M(x) \in \mathbb{N}$ such that $x_n \notin B^{(n)}$ for $n \geqslant M$. Consider a positive integer m_0 such that $N_0 \cdots N_{m_0-1} < M \leqslant N_0 \cdots N_{m_0}$. We shall show that the following inequality holds for each $n \geqslant N_0 \cdots N_{m_0}$:

$$\rho^{-1}(n)\psi(n) \cdot \max\bigl\{d(T^n x, x), d(T^{2n} x, x), \dots, d(T^{(k-1)n} x, x)\bigr\} \geqslant 1. \tag{12}$$

Let $m_1 \geqslant m_0$ be a positive integer and let n be an integer such that

$$N_0 \cdots N_{m_1} \leqslant n < N_0 \cdots N_{m_1+1} \tag{13}$$

for which (12) fails. Then

$$d(T^n x, x),\ d(T^{2n} x, x),\ \dots,\ d(T^{(k-1)n} x, x) < \frac{\rho(N_0 \cdots N_{m_1})}{\psi(N_0 \cdots N_{m_1})}\,.$$

Let $y^{(1)} = T^n x, y^{(2)} = T^{2n} x, \dots, y^{(k-1)} = T^{(k-1)n} x$. Then it follows by the properties of the metric $d(x, y)$ that

$$d(y^{(1)}, x),\ \dots,\ d(y^{(k-1)}, x) \leqslant \frac{\rho(N_0 \cdots N_{m_1+1})}{\psi(N_0 \cdots N_{m_1+1})}\,.$$

Hence

$$x_1 = y_1^{(1)} = \cdots = y_1^{(k-1)}, \dots, x_{m_1} = y_{m_1}^{(1)} = \cdots = y_{m_1}^{(k-1)}$$

and for some i we obtain

$$x_{m_1+1},\ y^{(1)}_{m_1+1},\ \dots,\ y^{(k-1)}_{m_1+1} \in A_i^{(m_1+1)}. \tag{14}$$

We have $n = y^{(1)} - x = y^{(2)} - y^{(1)} = \cdots = y^{(k-1)} - y^{(k-2)}$. Using (14) we now see that

$$y^{(1)} - x = \bigl(\underbrace{0, \dots, 0}_{m_1}, y^{(1)}_{m_1+1} - x_{m_1+1} (\operatorname{mod} N_{m_1+1}), w_1, w_2, \dots\bigr),$$
$$y^{(2)} - y^{(1)} = \bigl(\underbrace{0, \dots, 0}_{m_1}, y^{(2)}_{m_1+1} - y^{(1)}_{m_1+1} (\operatorname{mod} N_{m_1+1}), w'_1, w'_2, \dots\bigr),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y^{(k-1)} - y^{(k-2)} = \bigl(\underbrace{0, \dots, 0}_{m_1}, y^{(k-1)}_{m_1+1} - y^{(k-2)}_{m_1+1} (\operatorname{mod} N_{m_1+1}), w''_1, w''_2, \dots\bigr),$$

where $w_1, w_2, \dots,\ w'_1, w'_2, \dots,$ and $w''_1, w''_2, \dots$ are integers. Hence $x_{m_1+1}, y^{(1)}_{m_1+1}, \dots, y^{(k-1)}_{m_1+1}$ form an arithmetic progression of length k modulo N_{m_1+1}. We have $x_{m_1+1}, y^{(1)}_{m_1+1}, \dots, y^{(k-1)}_{m_1+1} \in A_i^{(m_1+1)}$. Since we have $A_i^{(m_1+1)} \subseteq A^{(m_1+1)} + p$ for some $p \in \mathbb{Z}_{N_{m_1+1}}$, it follows that $A_i^{(m_1+1)}$ contains no non-trivial arithmetic progressions of length k modulo N_{m_1+1}. Hence for all $l = 1, 2, \dots, k-1$ we have

$x_{m_1+1} \equiv y^{(l)}_{m_1+1} (\operatorname{mod} N_{m_1+1})$. Since $0 \leqslant x_{m_1+1}, y^{(1)}_{m_1+1}, \dots, y^{(k-1)}_{m_1+1} < N_{m_1+1}$, it follows that $x_{m_1+1} = y^{(1)}_{m_1+1} = \dots = y^{(k-1)}_{m_1+1}$. Consequently,

$$n = y^{(1)} - x = (\underbrace{0, \dots, 0}_{m_1+1}, n_{m_1+2}, n_{m_1+3}, \dots). \tag{15}$$

It follows from (15) that $n \geqslant N_0 \cdots N_{m_1+1}$, in contradiction with inequality (13). The proof of Theorem 1.16 is complete.

Corollary 2.2. *Let k, $k \geqslant 3$, be an integer and let ε be an arbitrary positive number. Then there exist a dynamical system $(X, \mathscr{B}, \mu, T, d)$ with $X = [0,1]$ and μ equal to Lebesgue measure, μ compatible with Hausdorff measure H_1, $H_1(X) = 1$, and an effective positive constant C_k depending only on k such that for almost all $x \in X$ one has*

$$\liminf_{n\to\infty} \biggl\{ \frac{1}{\rho(n)} \max\{d(T^n x, x), d(T^{2n} x, x), \dots, d(T^{(k-1)n} x, x)\} \biggr\} \geqslant 1, \tag{16}$$

where $\rho(n) = \exp(-(1+\varepsilon) C_k (\log n)^{1/(k-1)})$.

Proof. By Theorem 1.12, for each integer $k \geqslant 3$ and each sufficiently large integer N there exists a set $A^{(N)}_0$ in $[N-1]$ containing no progressions of length k such that $|A^{(N)}_0| \geqslant N \exp(-(1+\varepsilon) C_k (\log N)^{1/(k-1)})$, where C_k is an absolute constant depending only on k. Let $A^{(N)}_1 = A^{(N)}_0 \cap [1, N/k)$, $A^{(N)}_2 = A^{(N)}_0 \cap [N/k, 2N/k)$, …, $A^{(N)}_k = A^{(N)}_0 \cap [N(k-1)/k, N)$. For arbitrary $j \in \{1, \dots, k\}$ the set $A^{(N)}_j$ contains no arithmetic progressions of length k *modulo* N. It is easy to see that there exists $j \in \{1, \dots, k\}$ such that

$$|A^{(N)}_j| \geqslant \frac{N}{2k} \exp\bigl(-(1+\varepsilon) C_k (\log N)^{1/(k-1)}\bigr).$$

We set $A^{(N)} = A^{(N)}_j$. By Theorem 1.16 there exists a dynamical system such that inequality (9) holds with function $\rho(n) = \exp(-C_k(1+\varepsilon')(\log n)^{1/(k-1)})$, where we can take ε', for instance, equal to 2ε. The proof is complete.

Remark 2.3. We can call the dynamical systems $(X, \mathscr{B}, \mu, T, d)$ constructed in Theorem 1.16 and Corollary 2.2 systems with low recurrence rate. We mean here that the multiple recurrence rate in these systems is much lower than the ordinary simple one. Indeed, as follows from Theorems 1.8 and 1.9, for almost all $x \in X$ we have $\liminf_{n\to\infty} \{n \cdot d(T^n x, x)\} = 0$. Hence the simple recurrence rate is higher than the multiple rate evaluated by inequality (16).

§ 3. One-dimensional recurrence

Theorem 3.1. *Let f be a real number, $f \geqslant 1$, let $X = [0,1]$, and let μ be Lebesgue measure on X. Then there exists a dynamical system $(X, \mathscr{B}, \mu, T, d)$ such that μ is compatible with Hausdorff measure H_1, $H_1(X) = 1$, and for almost all points x in X with respect to Lebesgue measure*

$$C_f(x) := \liminf_{n\to\infty} \{n \cdot f \cdot d(T^n x, x)\} = 1. \tag{17}$$

Remark 3.2. Theorem 3.1 has been proved in [6] for $f=1$.

Remark 3.3. Let $X=[0,1]$, $\mathscr{B}$ the Borel sigma-algebra, μ Lebesgue measure, and T_α the map of X into itself defined by the formula

$$T_\alpha x=(x+\alpha)\ (\mathrm{mod}\,1).$$

Also let $d(x,y)=\|x-y\|$, where $\|\cdot\|$ is the distance from the closest integer. Then there exists a quantity λ^* ($\lambda^*=5.68195\dots$) such that for all $f\geqslant\lambda^*$ there exists a map T_α, $\alpha=\alpha(f)$, such that $C_f(x)=1$ for *all* x in $[0,1]$. The ray $[\lambda^*,+\infty)$ is called the *Hall ray* (see [20]). The precise value of λ^* is due to Freĭman [21]. By Theorem 3.1 a dynamical system such that $C_f(x)=1$ for almost all x in $[0,1]$ exists for all $f\geqslant 1$, not only for $f\geqslant\lambda^*$.

Remark 3.4. As in Theorem 1.16, the inequality $H_1(X)\leqslant 1$ in Theorem 3.1 is very important (see Remark 1.17). Moreover, the reverse inequality $H_1(X)\geqslant 1$ is also absolutely necessary. Without it Theorem 3.1 becomes trivial. In fact, assume that $f>1$ and let $(X,\mathscr{B},\mu,T,d)$ be a dynamical system such that $C_1(x)=1$ (see Remark 3.2). We set $\widetilde{d}(x,y)=d(x,y)/f$ and consider the new dynamical system $(X,\mathscr{B},\mu,T,\widetilde{d})$. Then for each $x\in X$ we have $C_f(x)=1$, as required in Theorem 3.1. We observe that the newly constructed dynamical system fails the equality $H_1(X)=1$: in fact, $H_1(X)=1/f<1$.

Proof of Theorem 3.1. Let $N_0=1$, $N_m=\lceil f2^m\rceil^2$, $m=1,2,\dots$, and let X be the space of sequences $(x_1,x_2,\dots)$, $0\leqslant x_i<N_i$, $i\geqslant 1$. The map from X into $[0,1]$ defined by the formula $x\to\sum_{i=1}^{\infty}x_i/(N_0N_1\cdots N_i)$ allows one to regard X as the interval $[0,1]$. Now let $T\colon X\to X$ be the transformation defined by the formula $Tx=x+1$ with addition defined in the proof of Theorem 1.16 and $1=(1,0,0,\dots)$. As pointed out before, the transformation T preserves Lebesgue measure μ.

Let $p_m=\sqrt{N_m}=\lceil f2^m\rceil$, $m\geqslant 1$, and let

$$A_m^{(j)}=\bigl\{x\in\{0,1,\dots,N_m-1\}:x\equiv j\ (\mathrm{mod}\,p_m)\bigr\}.$$

It is clear that the sets $A_m^{(j)}$ partition the interval $\{0,1,\dots,N_m-1\}$. Consider the maps $\varphi_j\colon A_m^{(j)}\to\mathbb{N}_0=\mathbb{N}\cup\{0\}$ defined by the formula $\varphi_j(x)=(x-j)/p_m$. Then we set $\varphi(x):=\varphi_j(x)$ for $x\in A_m^{(j)}$ and obtain a function $\varphi(x)$ defined on the entire interval $\{0,1,\dots,N_m-1\}$.

Let

$$d(x,y)=\biggl\{\frac{r_m(x_m,y_m)}{N_0\cdots N_{m-1}},\quad\text{where } m \text{ is the largest index such that } x_1=y_1,\dots,x_{m-1}=y_{m-1}\text{ and }x_m,y_m\in A_m^{(i)}\text{ for some } i\in 1,2,\dots,l(m)\biggr\},$$

where

$$r_m(x_m,y_m)=\begin{cases}\dfrac{1}{N_m} & \text{for } x_m=y_m,\\[2mm] \dfrac{|\varphi(x_m)-\varphi(y_m)|}{fp_m} & \text{otherwise.}\end{cases}$$

We observe that we always have $1/N_m\leqslant r_m(x_m,y_m)\leqslant 1$.

Assertion. *$d(x,y)$ is a metric in X.*

Proof of the assertion. The symmetry of the function $d(x,y)$ is obvious. It is also clear that $d(x,y)=0$ if and only if $x=y$. We claim that for arbitrary $x,y,z\in X$, $x=(x_1,x_2,\dots)$, $y=(y_1,y_2,\dots)$, $z=(z_1,z_2,\dots)$, we have the inequality

$$d(x,y)\leqslant d(x,z)+d(z,y). \tag{18}$$

If $d(x,y)=0$, then (18) holds. Assume that $d(x,y)>0$. Then there exists $m\in\mathbb{N}$ such that $d(x,y)=r_m(x_m,y_m)/(N_0\cdots N_{m-1})$. If $z_i\neq x_i$ or $z_i\neq y_i$ for some $i\in 1,2,\dots,m-1$, then inequality (18) holds. For this reason we shall assume that $z_i=x_i=y_i$ for all $i=1,2,\dots,m-1$.

Suppose that there exists no j such that x_m, y_m belong to some $A_m^{(j)}$. Then $d(x,y)=1/(N_0\cdots N_{m-1})$. Now, either x_m and z_m or y_m and z_m do not lie in the same set $A_m^{(j)}$, therefore inequality (18) holds again.

It remains to consider the case of x_m and y_m belonging to the same set $A_m^{(j)}$. If $z_m\notin A_m^{(j)}$, then inequality (18) holds. On the other hand, if $z_m\in A_m^{(j)}$, then

$$\begin{aligned} d(x,y)=\frac{|\varphi(x_m)-\varphi(y_m)|}{N_0\cdots N_{m-1}fp_m}&\leqslant\frac{|\varphi(x_m)-\varphi(z_m)|}{N_0\cdots N_{m-1}fp_m}+\frac{|\varphi(z_m)-\varphi(y_m)|}{N_0\cdots N_{m-1}fp_m}\\ &=d(x,z)+d(z,y). \end{aligned}$$

The proof is complete.

We now return to the proof of Theorem 3.1. Consider the Hausdorff measure H_1 in the space X. Since each elementary cylinder is a closed and therefore a Borel subset of the metric set (X,d), the measures μ and H_1 are compatible. We claim that $H_1(X)=1$.

Considering a cover of X by elementary cylinders

$$C(a_1,\dots,a_m)=\bigl\{x=(x_1,x_2,\dots)\in X: x_1=a_1,\ \dots,\ x_m=a_m\bigr\},$$

we see that $H_1(X)\leqslant 1$.

We claim that $H_1(X)\geqslant 1$. Assume that $H_1(X)=a<1$. Since

$$H_1(X)=\lim_{\delta\to 0}H_1^\delta(X)=\sup_{\delta>0}H_1^\delta(X)$$

(see, for instance, [22]), for each $\varepsilon>0$ there exists $\delta>0$ such that

$$a-\varepsilon<H_1^\delta(X)\leqslant a=H_1(X). \tag{19}$$

Assume that $\varepsilon_0=(1-a)/2>0$. Using inequality (19) and the definition of the Hausdorff measure we can find a cover of X by subsets $\{U_i\}$, $r_i=\operatorname{diam}U_i$, $r_i<\delta=\delta(\varepsilon_0)$, such that

$$a-\varepsilon<\sum_i\operatorname{diam}U_i=\sum_i r_i<a+\varepsilon. \tag{20}$$

We observe that if $a=0$, then we can drop the inequality on the left-hand side of (20).

If $r_i = 0$, then the corresponding U_i is a singleton:

$$U_i = \{p_i\}.$$

Let P be the union of all the singletons U_i, that is, $P = \bigcup_{\{i:r_i=0\}} U_i = \bigcup_i \{p_i\}$. It is clear that there exists $U_i \notin P$. We consider only such U_i in what follows. Since the set of distances between elements of the space X has only one limit point at zero, for each U_i there exist $x, y \in U_i$ such that $r_i = \operatorname{diam} U_i = d(x,y)$. Let $d(x,y) = r_m(x_m, y_m)/(N_0 \cdots N_{m-1})$ and assume that there exists $j = j(i)$, such that $x_m, y_m \in A_m^{(j)}$. Let

$$C_i = \bigl\{z = (z_1, z_2, \dots) \in X : z_1 = x_1,\ \dots,\ z_{m-1} = x_{m-1},\ z_m \in A_m^{(j)} \cap [x_m, y_m]\bigr\}, \tag{21}$$

and let

$$C_i = \bigl\{z = (z_1, z_2, \dots) \in X : z_1 = x_1,\ \dots,\ z_{m-1} = x_{m-1}\bigr\}, \tag{22}$$

if there exists no such $A_m^{(j)}$. It is clear that in either case $U_i \subseteq C_i$ and $\operatorname{diam} C_i = \operatorname{diam} U_i$. Hence C_i satisfies inequality (20) and the cylinders C_i together with the set P form a cover of X.

We point out that if C_i is defined by formula (22), then C_i is an elementary cylinder. Let

$$C_i(a) = \bigl\{z = (z_1, z_2, \dots) \in X : z_1 = x_1,\ \dots,\ z_{m-1} = x_{m-1},\ z_m = a\bigr\}.$$

Each set $C_i(a)$, $0 \leqslant a < N_m$, is an elementary cylinder. If C_i is defined by (22), then $C_i = \bigsqcup_{a \in A_m^{(j)} \cap [x_m, y_m]} C_i(a)$. Clearly, $\operatorname{diam} C_i \geqslant \sum_{a \in A_m^{(j)} \cap [x_m, y_m]} \operatorname{diam} C_i(a)$. Hence there exists a countable cover of X by points in P and elementary cylinders C_i', $r_i' = \operatorname{diam} C_i'$, such that

$$\sum_i r_i' \leqslant \sum_i r_i < a + \varepsilon < 1. \tag{23}$$

Two arbitrary elementary cylinders are either disjoint or one of them lies in the other. Hence there exists a subcover C_i'', $r_i'' = \operatorname{diam} C_i''$, of the cover C_i' that together with P partitions the space X into elementary cylinders such that

$$\sum_i r_i'' \leqslant \sum_i r_i' < 1. \tag{24}$$

We have $r_i'' = \mu C_i''$ and $\sum_i r_i'' = \sum_i \mu C_i'' = \mu(X) = 1$. This is a contradiction with inequality (24). Hence $H_1(X) = 1$.

It remains to prove that for almost all $x \in X$ we have

$$\liminf_{n \to \infty} \{n \cdot f \cdot d(T^n x, x)\} = 1. \tag{25}$$

Let $a_m(j)$ be the largest element of $A_m^{(j)}$ and $B_m = \bigsqcup_{j=1}^{p_m} a_m(j)$; $|B_m| = p_m = \sqrt{N_m}$. Let

$$\widetilde{B}_m = \{x \in X : x_m \in B_m\} \subseteq X \quad \text{and} \quad \mathbf{B} = \bigcap_{n=1}^{+\infty} \bigcup_{m \geqslant n}^{+\infty} \widetilde{B}_m.$$

Then $\mu(\widetilde{B}_m) = |B_m|/N_m = 1/p_m$. We have

$$\sum_{m=1}^{\infty} \mu(\widetilde{B}_m) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{N_m}} \leqslant \sum_{m=1}^{\infty} \frac{1}{2^m} < \infty. \tag{26}$$

By the Borel–Cantelli lemma we now obtain that $\mu\mathbf{B} = 0$.

We claim that equality (25) holds for all $x = (x_1, x_2, \dots)$, $x \notin \mathbf{B}$. For each $x \notin \mathbf{B}$ there exists $M = M(x)$ such that $x_n \notin B_n$ for all $n \geqslant M$. Assume that M is sufficiently large. There exists a positive integer m_0 such that $N_0 \cdots N_{m_0} \geqslant M$. Consider now the increasing sequence of positive integers

$$S = \{p_{m+1} N_0 \cdots N_m\}_{m=m_0}^{+\infty} = \{n_m\}_{m=m_0}^{+\infty}.$$

Let $x = (x_1, x_2, \dots, x_m, x_{m+1}, x_{m+2}, \dots)$, where x_{m+1} lies in some $A_{m+1}^{(j)}$. Also let $n_m \in S$. Then $T^{n_m} x = (x_1, \dots, x_m, \widetilde{x}_{m+1}, \widetilde{x}_{m+2}, \dots)$, where $\widetilde{x}_{m+1}, \widetilde{x}_{m+2}, \dots$ are some integers such that $x_{m+1}, \widetilde{x}_{m+1}$ belong to $A_{m+1}^{(j)}$ and $|\varphi(x_{m+1}) - \varphi(\widetilde{x}_{m+1})| = 1$. Hence $d(T^{n_m} x, x) = 1/(N_0 \cdots N_m f p_{m+1})$. Now,

$$n_m f \cdot d(T^{n_m} x, x) = p_{m+1} N_0 \cdots N_m f \frac{1}{N_0 \cdots N_m f p_{m+1}} = 1. \tag{27}$$

Consequently, for all $x \notin \mathbf{B}$ we have $C_f(x) \leqslant 1$.

We claim that for each $x \notin \mathbf{B}$ we have the reverse inequality $C_f(x) \geqslant 1$. Let n be a positive integer such that $n \in [N_0 \cdots N_m, N_0 \cdots N_{m+1}) := J_m$ and $m \geqslant m_0$. We observe that n_m belongs to J_m. If $n = tN_0 \cdots N_m$, $1 \leqslant t < N_{m+1}$, then $T^n x = (x_1, \dots, x_m, \widetilde{x}_{m+1}, \widetilde{x}_{m+2}, \dots)$, where $\widetilde{x}_{m+1}, \widetilde{x}_{m+2}, \dots$ are some integers. If $\widetilde{x}_{m+1} \notin A_{m+1}^{(j)}$, then $d(T^n x, x) = 1/(N_0 \cdots N_m)$, and therefore $nfd(T^n x, x) \geqslant 1$. On the other hand, if $\widetilde{x}_{m+1} \in A_{m+1}^{(j)}$, then

$$1 = n_m f d(T^{n_m} x, x) \leqslant n f d(T^n x, x).$$

It remains to consider the case of $n \neq tN_0 \cdots N_m$, $1 \leqslant t < N_{m+1}$. In this case

$$T^n x = (x_1', \dots, x_m', x_{m+1}', x_{m+2}', \dots),$$

and in addition, there exists $i \in \{1, 2, \dots, m\}$ such that $x_i \neq x_i'$. Hence we obtain $d(T^n x, x) \geqslant 1/(N_0 \cdots N_m)$ and $nfd(T^n x, x) \geqslant 1$ again.

Thus, if $m \geqslant m_0$, then for $n \in [N_0 \cdots N_m, N_0 \cdots N_{m+1})$ we have $1 \leqslant nfd(T^n x, x)$. Hence for $x \notin \mathbf{B}$ we have the inequality $C_f(x) \geqslant 1$. The proof is complete.

Bibliography

[1] H. Furstenberg, "Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions", *J. Anal. Math.* **31** (1977), 204–256.

[2] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton Univ. Press, Princeton, NJ 1981.

[3] H. Furstenberg, Y. Katznelson, and D. Ornstein, "The ergodic theoretical proof of Szemerédi's theorem", *Bull. Amer. Math. Soc. (N.S.)* **7**:3 (1982), 527–552.

[4] H. Furstenberg and Y. Katznelson, “An ergodic Szemerédi theorem for commuting transformations”, *J. Anal. Math.* **34** (1978), 275–291.

[5] K. F. Roth, “On certain sets of integers”, *J. London Math. Soc.* **28** (1953), 104–109.

[6] M. Boshernitzan, “Quantitative recurrence results”, *Invent. Math.* **113** (1993), 617–631.

[7] W. T. Gowers, “A new proof of Szemerédi’s theorem”, *Geom. Funct. Anal.* **11**:3 (2001), 465–588.

[8] I. D. Shkredov, “On a problem of Gowers”, *Dokl. Ross. Akad. Nauk* **400**:2 (2005), 169–172; English transl. in *Dokl. Math.* **71**:1 (2005), 46–48.

[9] I. D. Shkredov, *On one problem of Gowers*, `arXiv: math.NT/0405406`.

[10] I. D. Shkredov, *On multiple recurrence*, `arXiv: math.DS/0406413`.

[11] I. D. Shkredov, *On a generalization of Szemerédi’s theorem*, `arXiv: math.DS/0503639`.

[12] N. G. Moshchevitin, “On a theorem of Poincaré”, *Uspekhi Mat. Nauk* **53**:1 (1998), 223–224; English transl. in *Russian Math. Surveys* **53**:1 (1998), 219–220.

[13] I. D. Shkredov, “Recurrence in mean”, *Mat. Zametki* **72**:4 (2002), 625–632; English transl. in *Math. Notes* **72**:4 (2002), 576–582.

[14] J. Bourgain, “On triples in arithmetic progression”, *Geom. Funct. Anal.* **9**:5 (1999), 968–984.

[15] F. A. Behrend, “On sets of integers which contain no three terms in arithmetic progression”, *Proc. Nat. Acad. Sci.* **32** (1946), 331–332.

[16] R. A. Rankin, “Sets of integers containing not more than a given number of terms in arithmetic progression”, *Proc. Roy. Soc. Edinburgh Sect. A* **65**:4 (1961), 332–344.

[17] I. Laba and M. T. Lacey, *On sets of integers not containing long arithmetic progressions*, `arXiv: math.CO/0108155`.

[18] C. L. Stewart and R. Tijdeman, “On infinite-difference sets”, *Canad. J. Math.* **31**:5 (1979), 897–910.

[19] T. Tao, *Lecture notes* 5, `http://www.math.ucla.edu/~tao/254a.1.03w/notes5.dvi`.

[20] T. W. Cusick and M. E. Flahive, *The Markoff and Lagrange spectra*, Math. Surveys Monogr., vol. 30, Amer. Math. Soc., Providence, RI 1989.

[21] G. A. Freĭman, *Diophantine approximations and the geometry of numbers* (*Markov’s problem*), Kalinin State University, Kalinin 1975.

[22] K. Falconer, *Fractal geometry. Mathematical foundations and applications*, Wiley, New York 1990.

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