On sets with small doubling *

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Annotation.

Let G be an arbitrary Abelian group and let A be a finite subset of G. A has small additive doubling if $|A+A| \leq K|A|$ for some $K > 0$. These sets were studied in papers of G.A. Freiman, Y. Bilu, I. Ruzsa, M.C.–Chang, B. Green and T.Tao. In the article we prove that if we have some minor restrictions on K then for any set with small doubling there exists a set Ł, $Ł \ll_\varepsilon K \log|A|$ such that $|A \cap Ł| \gg |A|/K^{1/2+\varepsilon}$, where $\varepsilon > 0$. In contrast to the previous results our theorem is nontrivial for large K. For example one can take K equals $|A|^\eta$, where $\eta > 0$. We use an elementary method in our proof.

1. Introduction.

Let G be an arbitrary Abelian group with additive group operation $+$. Suppose that A, B are two finite subsets of G and define their sumset $A+B$ to be the set of all pairwise sums $a+b$ with $a \in A$, $b \in B$. Let log stand for the logarithm to base 2.

Suppose that A is a set such that $|A+A| \leq K|A|$, where $K \geq 1$ is small (for example $K = \log\log|A|$ or $K = 2$). These sets are called sets with *small doubling*. The properties of such sets were studied in papers [4, 5, 6, 7, 9, 10, 13, 14, 15, 16]. G.A. Freiman (see [4]) proved the following wonderful result on the structure of these sets.

Recall that a set $Q \subseteq G$ is called a *d–dimensional arithmetic progression* if

$$Q = \{n_0 + n_1 l_1 + \cdots + n_d l_d \ : \ 0 \leq l_i < m_i\},$$

where $m_i, n_i \in \mathbb{Z}$ and $m_i \geq 0$.

Let $G = \mathbb{Z}$.

Theorem 1.1 (Freiman) *Let $K \geq 1$ be a real number, and $A \subseteq \mathbb{Z}$ be a finite set. Let also $|A+A| \leq K|A|$. Then there exist numbers $d = d(K)$ and $C = C(K)$ depend on K only and d–dimensional arithmetic progression Q such that $|Q| \leq C|A|$ and $A \subseteq Q$.*

The functions $d = d(K)$ and $C = C(K)$ were studied in [6, 7]. In paper [7] M.–C. Chang proved that $d = O(K^2 \log^2 K)$ and $C = \exp(O(K^2 \log^2 K))$ (as usual we use $X = O(Y)$ or $X \ll Y$ to denote an estimate of the form $X \leq MY$ for some absolute constant M).

Let n be a positive integer. Sets with small doubling in groups $G = (\mathbb{Z}/2\mathbb{Z})^n$ were considered in [9, 17, 13, 14, 16]. For example we formulate a theorem from [9]. Note that $(\mathbb{Z}/2\mathbb{Z})^n$ is a vector space.

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Theorem 1.2 *Let $K \geq 1$ be a real number. Let $A \subseteq (\mathbb{Z}/2\mathbb{Z})^n$ be a set such that $|A+A| \leq K|A|$. Then A is contained in a subspace H with $|H| \leq K^2 2^{K^4}|A|$.*

There are another structural results on sets with small doubling. Let A be a set with small doubling and d is a small positive integer. Is it true that A has large *intersection* with some d—dimensional arithmetic progression? It is known that there is a positive answer at the question and we give two examples of such results. In [14] the following theorem was proved.

Theorem 1.3 *Let $K \geq 1$ be a real number. Suppose that $A \subseteq (\mathbb{Z}/2\mathbb{Z})^n$ is a set such that $|A+A| \leq K|A|$. Then there exists a subspace H such that $|H| \ll K^{O(1)}|A|$ and $|A \cap H| \gg \exp(-K^{O(1)})|A|$.*

Finally, in recent paper [16] B. Green and T. Tao proved the following theorem.

Theorem 1.4 *Let $K \geq 1$ be a real number. Let $A \subseteq (\mathbb{Z}/2\mathbb{Z})^n$ be a set such that $|A+A| \leq K|A|$. Then there exists a subspace H and $x \in (\mathbb{Z}/2\mathbb{Z})^n$ such that $|H| \gg K^{-O(\sqrt{K})}|A|$ and $|A \cap (x+H)| \geq \frac{1}{2K}|H|$.*

Let us formulate our main result.

Let $E = \{e_1, \dots, e_{|E|}\} \subseteq G$ be a finite set. By $\operatorname{Span} E$ denote the set $\operatorname{Span} E = \{ \sum_{i=1}^{|E|} \varepsilon_i e_i \ : \ \varepsilon_i \in \{-1,0,1\} \}$.

Theorem 1.5 *Let G be an Abelian group. Let K, ε be real numbers, $\varepsilon \in (0, 1/2]$, $A \subseteq G$ be a finite set, $|A| \geq 2^{32/\varepsilon}$, $1 \leq K \leq \min\{ (2^{-58}\varepsilon^{-4}\frac{|A|}{\log|A|})^{(3/2+\varepsilon)^{-1}}, |A|^{\varepsilon} \}$. Let also A contains at least $|A|^3/K$ quadruples with $a_1 + a_2 = a_3 + a_4$. Then there exists a set Ł such that $|\operatorname{Span} Ł \cap A| \geq \frac{1}{2} \cdot \frac{|A|}{K^{1/2+\varepsilon}}$ and $|Ł| \leq 2^{30}\varepsilon^{-2}K \log|A|$.*

It is easy to see that the number K in Theorems 1.1,1.2,1.3,1.4 cannot be too large. For example Theorem 1.4 is trivial if $K \gg \left(\frac{\log|A|}{\log\log|A|}\right)^2$. In contrast to these results our Theorem 1.5 is nontrivial for large K (for example one can take $K = |A|^{\eta}$, where $\eta > 0$ is a sufficiently small number). On the other hand the cardinality of the set Ł depends on $|A|$. This fact differences our main result from Theorems 1.1—1.4.

This paper is organized as follows. In §2 we study so–called "connected sets" in Abelian groups. We prove that any such set has large intersection with $\operatorname{Span} Ł$ for some small set Ł (in more detail see Propositions 2.7, 2.9). Besides in the section we show that any set contains large connected subset. These two facts imply Theorem 1.5. We give its proof in §3. In the last section we discuss some relations between our definition of connectedness and a graph–theoretical definition of connectedness from paper [8].

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2. On connected sets in Abelian groups.

Let G be an arbitrary Abelian group with additive group operation $+$. Let $A \subseteq G$ be a finite set, and $k \geq 2$ be a positive integer. By $T_k(A)$ denote the following number

$$T_k(A) := |\{a_1 + \dots + a_k = a'_1 + \dots + a'_k \ : \ a_1, \dots, a_k, a'_1, \dots, a'_k \in A\}| \, .$$

Denote by the same letter A the characteristic function of A. Thus $A(x) = 1$ if $x \in A$ and $A(x) = 0$ otherwise. We shall write $\sum_x$ instead of $\sum_{x \in G}$.

Definition 2.1 Let $k \geq 2$ be a positive integer, and $\beta \in [0,1]$ be a real number. Suppose that A is a finite nonempty set $A \subseteq G$. A is called *β–connected of degree k* if there is an absolute constant $C \in (0,1]$ such that for any $B \subseteq A$, $|B| \geq \beta|A|$ we have

$$T_k(B) \geq C^{2k} \left(\frac{|B|}{|A|}\right)^{2k} T_k(A) \, . \qquad (1)$$

If $\beta = 0$ then A is *connected of degree* k.

The class of connected sets is wide enough. On the one hand very structured sets like arithmetic progressions, multidimensional arithmetic progressions, subspaces are connected sets (see Corollary 2.4 below). On the other hand any so–called *dissociated* set (see Definition 2.5) belongs to this class. Other examples of connected of degree k sets will be considered in section 4.

Definition 2.2 Let $f, g : G \to \mathbb{R}$ be arbitrary functions. Denote by $(f * g)(x)$ the function

$$(f * g)(x) = \sum_s f(s)g(x-s)\,. \qquad (2)$$

Clearly, $(f * g)(x) = (g * f)(x)$, $x \in G$. By $(f \circ g)(x)$ denote the function

$$(f \circ g)(x) = \sum_s f(s)g(s-x)\,. \qquad (3)$$

Obviously, $(f \circ g)(x) = (g \circ f)(-x)$, $x \in G$.

Suppose that $A, B \subseteq G$ are any sets. Then $(A*B)(x) \neq 0$ iff $x \in A+B$ and $(A \circ B)(x) \neq 0$ iff $x \in A - B$. Hence $T_2(A) = \sum_x (A * A)^2(x)$. Further denote by $*_k$ the composition of k operations $*$, $k \geq 1$. Then $T_k(A) = \sum_x (A *_{k-1} A)^2(x)$, $k \geq 2$. Since

$$T_2(A) := |\{a_1 + a_2 = a_1' + a_2' \ : \ a_1, a_2, a_1', a_2' \in A\}| = |\{a_1 - a_1' = a_2' - a_2 \ : \ a_1, a_2, a_1', a_2' \in A\}|$$

it follows that $T_2(A) = \sum_x (A \circ A)^2(x)$.

Let $f : G \to \mathbb{R}$ be a function. By $T_k(f)$ denote the quantity $T_k(f) = \sum_x (f *_{k-1} f)^2(x)$. Let us prove the following simple lemma.

Lemma 2.3 *Let p_1,p_2 be positive integers, and $k_1 = 2^{p_1}$, $k_2 = 2^{p_2}$. Let also $f_1, \dots, f_{k_1}, g_1, \dots, g_{k_2} : G \to \mathbb{R}$ be functions. Then*

$$\left| \sum_x (f_1 * \dots * f_{k_1})(x) \cdot (g_1 * \dots * g_{k_2})(x) \right| \leq$$

$$\leq (T_{k_1}(f_1))^{1/2k_1} \dots (T_{k_1}(f_{k_1}))^{1/2k_1} (T_{k_2}(g_1))^{1/2k_2} \dots (T_{k_2}(g_{k_2}))^{1/2k_2}\,. \qquad (4)$$

Proof. First of all let us suppose that $k_1 = k_2 = k = 2^p$, where p is a positive integer. We prove the lemma by induction. Put $\sigma = \sum_x (f_1 * \dots * f_k)(x) \cdot (g_1 * \dots g_k)(x)$. Using the Cauchy–Schwartz inequality, we get

$$\sigma^2 \leq \sum_x (f_1 * \dots * f_k)^2(x) \cdot \sum_x (g_1 * \dots * g_k)^2(x) = \sigma_1 \sigma_2\,. \qquad (5)$$

Consider the sum σ_1. By definitions of $*, \circ$, we obtain

$$\sigma_1 = \sum_x ((f_1 \circ f_1) * \dots * (f_{2^{p-1}} \circ f_{2^{p-1}}))(x) \cdot ((f_{2^{p-1}+1} \circ f_{2^{p-1}+1}) * \dots * (f_k \circ f_k))(x)$$

By the induction hypothesis, we have

$$\sigma_1 \leq (T_{2^{p-1}}(f_1 \circ f_1))^{1/k} \dots (T_{2^{p-1}}(f_k \circ f_k))^{1/k}\,. \qquad (6)$$

Besides, $T_{2^{p-1}}(f_1 \circ f_1) = T_k(f_1)$. Hence

$$\sigma_1 \leq (T_k(f_1))^{1/k} \dots (T_k(f_k))^{1/k}\,. \qquad (7)$$

In the same way

$$\sigma_2 \le (T_k(g_1))^{1/k} \dots (T_k(g_k))^{1/k} \,. \qquad (8)$$

Combining (7), (8) and (5), we obtain that (4) holds.

Let now $k_1 = 2^{p_1}$, $k_2 = 2^{p_2}$, and $p_1 \ne p_2$. Put $\sigma' = \sum_x (f_1 * \dots * f_{k_1})(x) \cdot (g_1 * \dots g_{k_2})(x)$. Using the Cauchy–Schwartz inequality, we get

$$\sigma'^2 \le \sum_x (f_1 * \dots * f_{k_1})^2(x) \cdot \sum_x (g_1 * \dots * g_{k_2})^2(x) = \sigma'_1 \sigma'_2 \,. \qquad (9)$$

Using (4) for σ'_1, σ'_2, we have

$$|\sigma'| \le (T_{k_1}(f_1))^{1/2k_1} \dots (T_{k_1}(f_{k_1}))^{1/2k_1} (T_{k_2}(g_1))^{1/2k_2} \dots (T_{k_2}(g_{k_2}))^{1/2k_2} \,. \qquad (10)$$

This completes the proof.

Let us derive a corollary from Lemma 2.3. Let n be positive integer, q be a prime and let G be $(\mathbb{Z}/q\mathbb{Z})^n$. As was noted above G is a vector space.

Corollary 2.4 *Let n, p be positive integers, $k = 2^p$, q be a prime, and $G = (\mathbb{Z}/q\mathbb{Z})^n$. Let also P be a subspace of G. Then P is a connected of degree k set and (1) is true for $C = 1$.*

Proof. Let $B \subseteq P$ be a set and let $\sigma(B) := \sum_x (B * P *_{k-2} P)(x) \cdot (P *_{k-1} P)(x)$. The sum $\sigma(B)$ equals the number of solutions of the equation $b + p_2 + \dots + p_k = p'_1 + \dots + p'_k$, where $b \in B$ and $p_2, \dots, p_k, p'_1, \dots, p'_k \in P$. Since P is a subspace of $(\mathbb{Z}/q\mathbb{Z})^n$ it follows that $b + p_2 + \dots + p_k - p'_1 - \dots - p'_k \in P$. Hence $\sigma(B) \ge |B||P|^{2k-2}$. In particular $\sigma(P) = T_k(B) \ge |P|^{2k-1}$. Since $T_k(P) \le |P|^{2k-1}$ it follows that $T_k(P) = |P|^{2k-1}$. Using Lemma 2.3 with $f_1 = B$, $f_2 = \dots = f_k = g_1 = \dots = g_k = A$, we get

$$\sigma^{2k}(B) \le T_k(B) \cdot T_k^{2k-1}(A) \,. \qquad (11)$$

Combining the last inequality and the lower bound for $\sigma(B)$, we obtain

$$T_k(B) \ge \frac{\sigma^{2k}(B)}{T_k^{2k-1}(A)} \ge \frac{|B|^{2k}|P|^{(2k-2)2k}}{|P|^{(2k-1)^2}} = \left(\frac{|B|}{|P|}\right)^{2k} |P|^{2k-1} = \left(\frac{|B|}{|P|}\right)^{2k} T_k(P) \,.$$

This completes the proof.

Thus very structured sets like subspaces are connected sets. Consider another examples of connected sets.

We need in the following definition (see [18] or [7]).

Definition 2.5 We say that $Ł = \{l_1, \dots, l_{|Ł|}\} \subseteq G$ is *dissociated* if the equality

$$\sum_{i=1}^{|Ł|} \varepsilon_i l_i = 0 \,, \qquad (12)$$

where $\varepsilon_i \in \{-1, 0, 1\}$ implies that all ε_i are equal to zero.

If Ł is a dissociated set then there exists a good upper bound for $T_k(Ł)$ (see [18] and also [11, 19]).

Statement 2.6 *There is an absolute constant $M > 0$ such that for any dissociated set $Ł \subseteq G$ and any positive integer $k \ge 2$, we have*

$$T_k(Ł) \le M^k k^k |Ł|^k \,, \qquad (13)$$

where $M \le 288$.

Any connected of degree k set has the following property.

Proposition 2.7 *Let $k \ge 2$ be a positive integer. Suppose that $A \subseteq G$ is a connected of degree k set and for $C > 0$ inequality (1) holds. Then there exists a set $Ł \subseteq A$, $|Ł| \le 288C^{-2}k\frac{|A|^2}{T_k^{1/k}(A)}$ such that any $a \in A$ can be expressed in the form*

$$a = \sum_{i=1}^{|Ł|} \varepsilon_i l_i \,, \qquad (14)$$

where $\varepsilon_i \in \{-1, 0, 1\}$.

Proof. Let Ł be a maximal dissociated subset of A. Prove that any $a \in A$ can be expressed in the form

$$a = \sum_{i=1}^{|Ł|} \varepsilon_i l_i \,, \qquad (15)$$

where $\varepsilon_i \in \{-1, 0, 1\}$. If $a = 0$ then (15) holds. Let a be an arbitrary element of $A \setminus Ł$, $a \ne 0$. Consider all equations $\sum_{i=1}^{|Ł|+1} \varepsilon_i \tilde{l}_i = 0$, where $\tilde{l}_i \in Ł \bigsqcup \{a\}$ and $\varepsilon_i \in \{-1, 0, 1\}$, $i \in \{1, 2, \dots, |Ł|+1\}$. If all these equations are trivial, i.e. we have $\varepsilon_i = 0$, $i \in \{1, 2, \dots, |Ł|+1\}$ then we obtain a contradiction with the maximality of Ł. It follows that there exists non–trivial equation $\varepsilon a + \sum_{i=1}^{|Ł|} \varepsilon_i l_i = 0$, $\varepsilon, \varepsilon_i \in \{-1, 0, 1\}$ such that not all $\varepsilon, \varepsilon_i$ are equal to zero. Note that $\varepsilon \ne 0$. Whence any $a \in A$ is involved in some equation (14).

Let us prove that $|Ł| \le 288C^{-2}k\frac{|A|^2}{T_k^{1/k}(A)}$. Using Statement 2.6, we have $T_k(Ł) \le (288)^k k^k |Ł|^k$. On the other hand, the set A is connected of degree k. Hence $T_k(Ł) \ge C^{2k}(|Ł|/|A|)^{2k} \cdot T_k(A)$. It follows that $|Ł| \le 288C^{-2}k\frac{|A|^2}{T_k^{1/k}(A)}$. This completes the proof.

We need in a more delicate definition of connectedness.

Definition 2.8 Let $k \ge 2$ be a positive integer, and $\beta_1, \beta_2 \in [0, 1]$ be real numbers, $\beta_1 \le \beta_2$. Nonempty set $A \subseteq G$ is called *(β_1, β_2)–connected of degree k* if there exists an absolute constant $C \in (0, 1]$ such that for any $B \subseteq A$, $\beta_1|A| \le |B| \le \beta_2|A|$ we have

$$T_k(B) \ge C^{2k} \left(\frac{|B|}{|A|}\right)^{2k} T_k(A) \,. \qquad (16)$$

Clearly, any β–connected of degree k set is a (β, β_2)–connected of degree k, where $\beta_2 \in [\beta, 1]$ is an arbitrary number. Nevertheless we have the following weak analog of Proposition 2.7 for (β_1, β_2)–connected of degree k sets.

Proposition 2.9 *Let $k \ge 2$ be a positive integer, $0 < \beta_1 \le \beta_2$ be real numbers. Let also $A \subseteq G$ be a (β_1, β_2)–connected of degree k set and for $C > 0$ inequality (1) holds. Suppose that $\beta_2 \ge \beta_1 + 1/|A|$, $T_k(A) \ge 2^{14k}C^{-2k}k^k|A|^k$ and $|A| \ge 1/\beta_1$. Then there exists a set $Ł \subseteq A$ such that*

$$|Ł| \le 2^{13}C^{-2}k\frac{|A|^2}{T_k^{1/k}(A)} \,, \qquad (17)$$

and $|\mathrm{Span}\, Ł \cap A| \ge (1 - \beta_1)|A|$.

Proof. The proof of the proposition is a sort of inductive process. Let $Ł_1$ be a dissociated subset of A such that $|\mathrm{Span}\, Ł_1 \cap A| \ge (1 - \beta_1)|A|$. Clearly, there exists such set $Ł_1$, for example one put $Ł_1$ to be a maximal dissociated subset of A. Let $l = 2^{13}C^{-2}k\frac{|A|^2}{T_k^{1/k}(A)}$. If

$|Ł_1| \le l$ then the proposition is proved. Suppose that $|Ł_1| > l$. Let $Ł'_1 \subseteq Ł_1$ be an arbitrary set of the cardinality l. Obviously, that $Ł'_1$ is a dissociated set. Consider the set $A_1 = A \setminus Ł'_1$. If $|A_1| < (1-\beta_1)|A|$ then we stop our algorithm. If $|A_1| \ge (1-\beta_1)|A|$ then let $Ł_2$ be a dissociated subset of A_1 such that $|\operatorname{Span} Ł_2 \cap A_1| \ge (1-\beta_1)|A|$. Suppose that $|Ł_2| \le l$. Then $|\operatorname{Span} Ł_2 \cap A| \ge |\operatorname{Span} Ł_2 \cap A_1| \ge (1-\beta_1)|A|$ and we are done. It follows that $|Ł_2| > l$. Let $Ł'_2 \subseteq Ł_2$ be an arbitrary set of the cardinality l and consider the set $A_2 = A_1 \setminus Ł'_2$. An so on. We get the sets $A_0 = A, A_1, A_2, \dots, A_s$ and disjoint dissociated sets $Ł'_1, \dots, Ł'_s$ from A. We have $|A_s| < (1-\beta_1)|A|$. Since for all $l = 1, 2, \dots, s$ the following holds $A_l = A \setminus \bigsqcup_{i=1}^{l} Ł'_i$ it follows that $\sum_{i=1}^{s} |Ł'_i| = |A| - |A_s| > \beta_1|A|$. Let $B = \bigsqcup_{i=1}^{s} Ł'_i$. Then $|B| > \beta_1|A|$. We can remove some elements from $Ł'_s$ and assume that the cardinality of $\bigsqcup_{i=1}^{s} Ł'_i$ equals $[\beta_1|A|] + 1$. Denote by the same letter B our modified set. We have $B \subseteq A$ and $|B| \ge \beta_1|A|$. Since $\beta_2 \ge \beta_1 + 1/|A|$ it follows that $|B| \le \beta_2|A|$. By assumption the set A is (β_1, β_2) connected of degree k. Hence

$$T_k(B) \ge C^{2k}\beta_1^{2k} T_k(A)\,. \tag{18}$$

On the other hand

$$T_k(B) \le T_k\left(\bigsqcup_{i=1}^{s} Ł'_i\right) = \sum_{i_1,\dots,i_k=1}^{s} \sum_{j_1,\dots,j_k=1}^{s} \sum_{x} (Ł'_{i_1} * \dots * Ł'_{i_k})(x) \cdot (Ł'_{j_1} * \dots * Ł'_{j_k})(x)\,. \tag{19}$$

Using Lemma 2.3, Statement 2.6 and (19), we get

$$T_k(B) \le s^{2k} \max_{i=1,\dots,s} T_k(Ł'_i) \le s^{2k}(288)^k k^k l^k\,. \tag{20}$$

By assumption $T_k(A) \ge 2^{14k} C^{-2k} k^k |A|^k$. Whence $|A| \ge 2^{14} C^{-2} k \frac{|A|^2}{T_k^{1/k}(A)} = 2l$ and $s \ge 2$. Since $\bigsqcup_{i=1}^{s-1} Ł'_i \subseteq B$ and $|A| \ge 1/\beta_1$ it follows that $sl/2 \le (s-1)l \le |B| \le 2\beta_1|A|$. Hence $s \le 4\beta_1|A|/l$. Combining the last inequality and (20), we have

$$T_k(B) \le 2^{4k}\beta_1^{2k}(288)^k k^k \frac{|A|^{2k}}{l^k}\,.$$

This contradicts with (18) and we obtain the required result.

Let us prove now that any $A \subseteq G$ contains some large (β_1, β_2)–connected of degree k set. We begin with some notation.

Definition 2.10 Let $A \subseteq G$ be an arbitrary finite set, $|A| \ge 2$, and $k \ge 2$ be a positive integer. By $\zeta_k(A)$ denote the quantity

$$\zeta_k = \zeta_k(A) := \frac{\log T_k(A)}{\log |A|}\,.$$

In other words $T_k(A) = |A|^{\zeta_k}$. Clearly, for any set A, we have $k \le \zeta_k(A) \le 2k-1$.

Let $A \subseteq G$ be a finite set, $|A| = m \ge 2$, p be a positive integer, and $k = 2^p$. Write ζ for $\zeta_k(A)$.

Theorem 2.11 *Let $\beta_1, \beta_2 \in (0,1)$ be real numbers, $\beta_1 \le \beta_2$. Then there exists a set $A' \subseteq A$ such that*

1) *A' is (β_1, β_2)–connected of degree k set such that (16) holds for any $C \le 1/32$.*
2) *$|A'| \ge m \cdot 2^{\frac{\log((2k-1)/\zeta)}{\log(1+\kappa)} \log(1-\beta_2)}$, where $\kappa = \frac{\log((1-\beta_1)^{-1})}{\log m}(1-16C)$.*
3) *$\zeta_k(A') \ge \zeta_k(A)$.*

Proof. Let $C \le 1/32$ be a real number. The proof of Theorem 2.11 is a sort algorithm. If A is (β_1, β_2)–connected of degree k and (16) is true with the constant C then there is nothing to prove. Suppose that A is not (β_1, β_2)–connected of degree k set (with the constant C). Then there exists a set $B \subseteq A$, $\beta_1|A| \le |B| \le \beta_2|A|$ such that (16) does not hold. Note that $|A| > 2$. Let $\overline{B} = A \setminus B$. We have

$$T_k(A) = \sum_x (A *_{k-1} A)^2(x) =$$

$$= \sum_x (B * A *_{k-2} A)(x)(A *_{k-1} A)(x) + \sum_x (\overline{B} * A *_{k-2} A)(x)(A *_{k-1} A)(x) = \sigma_1 + \sigma_2 \,. \quad (21)$$

Using Lemma 2.3 with $f_1 = B$, $f_2 = \cdots = f_k = g_1 = \cdots = g_k = A$, we obtain

$$\sigma_1^{2k} \le T_k(B) \cdot T_k^{2k-1}(A) \,. \quad (22)$$

In the same way

$$\sigma_2^{2k} \le T_k(\overline{B}) \cdot T_k^{2k-1}(A) \,. \quad (23)$$

Let $c_B = |B|/|A|$. Combining $T_k(B) < C^{2k} c_B^{2k} T_k(A)$ and (22), we have $\sigma_1 < C c_B T_k(A)$. Using the last inequality, (21) and (23), we get

$$T_k(\overline{B}) > T_k(A)(1 - C c_B)^{2k} \,. \quad (24)$$

Let $\overline{\zeta} = \zeta_k(\overline{B})$, $b = |B|$ and $\overline{b} = |\overline{B}| = m - b$. Using (24), we obtain

$$\overline{\zeta} \log \overline{b} > \zeta \log m + 2k \log(1 - C c_B) \,.$$

Hence

$$\overline{\zeta} > \frac{\zeta \log m + 2k \log(1 - C c_B)}{\log \overline{b}} = \frac{\zeta \log m + 2k \log(1 - C c_B)}{\log m + \log(1 - b/m)} = \frac{\zeta + 2k \frac{\log(1 - C c_B)}{\log m}}{1 + \frac{\log(1 - c_B)}{\log m}} \ge$$

$$\ge \left(\zeta + 2k \frac{\log(1 - C c_B)}{\log m}\right) \left(1 - \frac{\log(1 - c_B)}{\log m}\right) \ge \zeta + \zeta \frac{\log((1 - c_B)^{-1})}{\log m} (1 - 16C) \ge$$

$$\ge \zeta (1 + \frac{\log((1 - \beta_1)^{-1})}{\log m} (1 - 16C)) = \zeta (1 + \kappa) \,, \quad (25)$$

where $\kappa = \frac{\log((1-\beta_1)^{-1})}{\log m}(1 - 16C) > 0$. Besides, by the definition of (β_1, β_2)–connectedness of degree k, we have

$$|\overline{B}| \ge (1 - \beta_2) m = (1 - \beta_2)|A| \,. \quad (26)$$

Thus if the set A is not (β_1, β_2)–connected of degree k then there is a set $\overline{B} \subseteq A$ such that (25), (26) hold. Put $A_1 = \overline{B}$ and apply the arguments above to A_1. And so on. We get the sets $A_0 = A, A_1, A_2, \ldots, A_s$. Clearly, for any A_i, we have $\zeta(A_i) \le 2k - 1$. Using this and (25), we obtain that the total number of steps of our algorithm does not exceed $\frac{\log((2k-1)/\zeta)}{\log(1+\kappa)}$. At the last step of the algorithm, we find the set $A' = A_s \subseteq A$ such that A' is (β_1, β_2)–connected of degree k and $\zeta_k(A') \ge \zeta = \zeta_k(A)$. Thus A' has the properties 1) and 3) of the theorem. Let us prove 2). Using (26), we obtain

$$|A'| \ge (1 - \beta_2)^s m \ge m \cdot 2^{\frac{\log((2k-1)/\zeta)}{\log(1+\kappa)} \log(1-\beta_2)} \,.$$

This completes the proof.

Corollary 2.12 *Let G be an Abelian group, ε, δ be real numbers, $\varepsilon \in (0, 1/8]$, $\delta \in (0, 1]$, $\delta \geq |G|^{-\varepsilon}$ and let $A \subseteq G$ be a set, $|A| \geq \delta |G| \geq 2$. Let also p be a positive integer, $k = 2^p$, and $\beta_1, \beta_2 \in (0, 1)$ be real numbers, $\beta_1 \leq \beta_2$, $\beta_1 \leq 1 - |A|^{-2\varepsilon}$. Then there exists a set $A' \subseteq A$ such that*

1) *A' is (β_1, β_2)–connected of degree k and (16) is true for any $C \leq 1/32$.*
2) *$|A'| \geq |G| \cdot \delta^{\left(\frac{2}{2k-1} + 32\varepsilon\right) \cdot \frac{\log(1-\beta_2)}{\log(1-\beta_1)} + 1}$.*
3) *$\zeta_k(A') \geq \zeta_k(A)$.*

In particular, if $\beta_2 = \beta_1$, $k = 2$ and $\varepsilon = 1/8$ then the cardinality of $|A'|$ is at least $\delta^6 |G|$.

Proof. Using Theorem 2.11 with $C = 1/32$, we find a set $A' \subseteq A$ with properties 1)—3) guaranteed by the theorem. Let us prove that $|A'| \geq |G| \cdot \delta^{\left(\frac{2}{2k-1} + 32\varepsilon\right) \cdot \frac{\log(1-\beta_2)}{\log(1-\beta_1)} + 1}$. Let $N = |G|$, $m = |A|$, and $\zeta = \zeta_k(A)$. Clearly, $T_k(A) \geq \delta^{2k} N^{2k-1}$. Hence

$$\zeta \geq 2k - 1 + (2k-1)\frac{\log(1/\delta)}{\log N} - \frac{2k}{1-\varepsilon}\frac{\log(1/\delta)}{\log N}\,. \tag{27}$$

Since $\delta \geq N^{-\varepsilon}$ it follows that

$$2k - 1 - \zeta \leq \frac{\log(1/\delta)}{\log N}(1 + 4k\varepsilon) \quad \text{and} \quad \zeta \geq (2k-1)(1 - 5\varepsilon)\,. \tag{28}$$

By Theorem 2.11, we have $|A'| \geq m \cdot 2^{\frac{\log((2k-1)/\zeta)}{\log(1+\kappa)} \log(1-\beta_2)}$, where $\kappa = \frac{\log((1-\beta_1)^{-1})}{2\log m}$. Using the last inequality, $\beta_1 \leq 1 - |A|^{-2\varepsilon}$ and (27), (28), we get

$$|A'| \geq m \cdot 2^{\left(\frac{2}{2k-1} + 32\varepsilon\right) \cdot \frac{\log(1-\beta_2)}{\log(1-\beta_1)} \cdot \frac{\log\delta}{\log N} \log m} \geq N \cdot \delta^{\left(\frac{2}{2k-1} + 32\varepsilon\right) \cdot \frac{\log(1-\beta_2)}{\log(1-\beta_1)} + 1}\,.$$

This completes the proof.

Note 2.13 Certainly, the constant 32 at the second point of Corollary 2.12 can be decreased. The constant 2 in the numerator of $\frac{2}{2k-1}$ depends on an upper bound for C. If C is less than $1/32$ then the number 2 is also decreases.

3. The proof of main result.

Lemma 3.1 *Let A be a finite nonempty set, and k be a positive integer, $k \geq 2$. Then $T_k(A) \geq T_2^{k-1}(A)/|A|^{k-2}$.*

Proof. The proof is trivial.

The proof of Theorem 1.5 Let $m = |A|$, $\beta_1 = 1/2$, $\beta_2 = \beta_1 + 1/\log m$, $C = \varepsilon 2^{-7}$, $k = 2^p$, $p = [\log\ln m] + 1$. Clearly, $C \leq 1/32$. Using Theorem 2.11, we find $A' \subseteq A$ such that 1) — 3) hold. By assumption $T_2(A) \geq |A|^3/K$. Using Lemma 3.1, we get $T_k(A) \geq T_2^{k-1}(A)/|A|^{k-2} \geq |A|^{2k-1}/K^{k-1}$. Whence

$$\zeta = \zeta_k(A) \geq 2k - 1 - (k-1)\frac{\log K}{\log m}\,. \tag{29}$$

Using $K \leq m^{\varepsilon}$ and (29), we obtain

$$\zeta \geq (2k-1)\left(1 - \frac{k-1}{2k-1}\frac{\log K}{\log m}\right) \geq (2k-1)\left(1 - \frac{\varepsilon}{2}\right)\,. \tag{30}$$

By 2) of Theorem 2.11, we have

$$|A'| \geq m \cdot 2^{-\frac{\log((2k-1)/\zeta)}{\log(1+\kappa)} \log((1-\beta_2)^{-1})} = m 2^{-\sigma}\,, \tag{31}$$

where $\kappa = \frac{\log((1-\beta_1)^{-1})}{\log m}(1-16C)$. Let us obtain an upper bound on σ. Using simple inequalities $\log(1+x) \le \frac{x}{\ln 2}$, $\log(1+x) \ge \frac{1}{\ln 2}(x - x^2/2)$, $x \ge 0$, and inequalities $m \ge 2^{32/\varepsilon}$, (29), (30), we get

$$\sigma \le \log\left(1 + \frac{2k-1-\zeta}{\zeta}\right)\frac{\ln 2}{\kappa}(1+\kappa)\log((1-\beta_2)^{-1}) \le$$

$$\le \frac{k-1}{2k-1}\cdot\frac{\log K}{\log m}\,(1+\varepsilon)\,\frac{\log m}{1-16C}\left(1+\frac{1}{\log m}\right)\frac{\log((1-\beta_2)^{-1})}{\log((1-\beta_1)^{-1})} \le$$

$$\le \log K^{1/2}\,(1+\varepsilon)\,(1+32C)\left(1+\frac{8}{\log m}\right) \le \log K^{1/2+\varepsilon}\,.$$

Hence $|A'| \ge \frac{m}{K^{1/2+\varepsilon}}$. Since $\zeta_k(A') \ge \zeta_k(A)$ it follows that

$$T_k(A') \ge \frac{|A'|^{2k-1}}{K^{k-1}} \ge \frac{|A'|^{2k-1}}{K^k}\,. \tag{32}$$

Using $|A'| \ge \frac{m}{K^{1/2+\varepsilon}}$, (32) and $K \le (2^{-58}\varepsilon^{-4}\frac{|A|}{\log|A|})^{(3/2+\varepsilon)^{-1}}$ it is easy to see that $T_k(A') \ge 2^{14k}C^{-2k}k^k|A'|^k$. Using Proposition 2.9, we find a set Ł such that $|\mathrm{Span}\,Ł \cap A'| \ge |A'|/2$ and

$$|Ł| \le 2^{27}\varepsilon^{-2}k\frac{|A'|^2}{T_k^{1/k}(A')}\,. \tag{33}$$

We have

$$|\mathrm{Span}\,Ł \cap A| \ge |\mathrm{Span}\,Ł \cap A'| \ge \frac{|A'|}{2} \ge \frac{1}{2}\cdot\frac{m}{K^{1/2+\varepsilon}}\,. \tag{34}$$

Let us prove that $|Ł| \le 2^{30}\varepsilon^{-2}K\log m$. Combining (33) and (32), we get

$$|Ł| \le 2^{27}\varepsilon^{-2}Kk|A'|^{1/k} \le 2^{27}\varepsilon^{-2}Kkm^{1/k}\,.$$

Recall that $k = 2^p$, $p = [\log\ln m] + 1$, we finally obtain $|Ł| \le 2^{30}\varepsilon^{-2}K\log m$. This completes the proof.

Corollary 3.2 *Let G be an Abelian group . Let K, ε be real numbers, $\varepsilon \in (0, 1/2]$, $A \subseteq G$ be an arbitrary set, $|A| \ge 2^{32/\varepsilon}$, $1 \le K \le \min\{\,(2^{-58}\varepsilon^{-4}\frac{|A|}{\log|A|})^{(3/2+\varepsilon)^{-1}}, |A|^{\varepsilon}\,\}$. Let also $|A+A| \le K|A|$. Then there exists a set Ł such that $|\mathrm{Span}\,Ł \cap A| \ge \frac{1}{2}\cdot\frac{|A|}{K^{1/2+\varepsilon}}$ and $|Ł| \le 2^{30}\varepsilon^{-2}K\log|A|$.*

Proof. We have $|A+A| \le K|A|$. By the Cauchy–Schwartz inequality

$$|A|^4 = \left(\sum_x (A*A)(x)\right)^2 \le \sum_x (A*A)^2(x)\cdot|A+A| \le T_2(A)\cdot K|A|\,.$$

Hence $T_2(A) \ge |A|^3/K$. Using Theorem 1.5, we obtain the required result.

4. Another definitions of connectedness.

In the section we discuss some relations between our definition of connectedness and a graph–theoretical definition of connectedness.

Suppose that $\Gamma = (V, f)$ is a graph, where V is the set of vertices of Γ and f is the characteristic function of a symmetric subset of $V \times V$. Let $X, Y \subseteq V$ be arbitrary sets. By $e(X,Y)$ denote the number of vertices between X and Y. In other words $e(X,Y) = \sum_{x\in X}\sum_{y\in Y} f(x,y)$. Recall that Γ is *connected* if for any vertex x, we have $e(x, V\setminus\{x\}) > 0$. In [8] I. Ruzsa and G. Elekes gave the following definition.

Definition 4.1 Let $\alpha \in (0,1]$ be a real number. A graph $\Gamma = (V, f)$ is called *α–dense–connected* if for any partition of the set of vertices into two disjoint parts, say $E \bigsqcup F = V$, we have

$$e(E,F) \geq \alpha |E||F| \,.$$

We give an analog of the definition above for subsets of Abelian groups. Let G be an Abelian group, and $A \subseteq G$ be a finite set. In papers [2, 3, 7] the graph of "popular differences" of A was considered. This graph $\Gamma_A = (V_A, f_A)$ played a significant role in various problems of combinatorial number theory (see articles [2, 3, 7] and book [21]). The vertex set V_A of the graph Γ_A is A, and the function f_A is the characteristic function of the symmetric set of "popular differences"

$$f(x,y) = \begin{cases} 1, & \text{if} \quad |\{x-y = a_1 - a_2 \ : \ a_1, a_2 \in A\}| \geq h \,, \\ 0, & \text{otherwise.} \end{cases}$$

Here h is a number, $0 \leq h \leq |A|$. In many problems of combinatorial number theory h was taken approximately $T_2(A)/|A|^2$. Thus the function $f(x,y)$ equals 1 if $(A \circ A)(x-y) \geq h$ and equals 0 otherwise. Ruzsa and Elekes applied α–dense–connected subgraphs of Γ_A to prove some results on sumsets (see details in [8]).

In the article we define a new (generalized) graph $\Gamma'_A = (V'_A, f'_A)$, where f'_A is a symmetric function but not the characteristic function of some subset of $V'_A \times V'_A$. Put $V'_A := A$ and $f'_A(x,y) := (A \circ A)(x-y)$. The constructed graph Γ'_A is an "approximation" of the graph Γ_A in the sense that the function f_A is a normalized and truncated version of the function f'_A : $f_A(x,y) = \theta(f'_A(x,y)/h)$, where θ is a shifted Heaviside's function : $\theta(x) = 1$ if $x \geq 1$ and $\theta(x) = 0$ if $x < 1$. Then the graph Γ'_A is (generalized) α–dense–connected if for any partition of the set of vertices into two disjoint parts E and V, $E \bigsqcup F = A$, we have

$$e(E,F) = \sum_{x \in E} \sum_{y \in F} (A \circ A)(x-y) = \sum_z (E \circ F)(z) \cdot (A \circ A)(z) \geq \alpha |E||F| \,. \qquad (35)$$

We shall call a set A is *strongly connected* if inequality (35) holds. As was noted above in many problems of combinatorial number theory the order of the number h was $T_2(A)/|A|^2$. We also put $\alpha = C \cdot T_2(A)/|A|^2$, where $C > 0$ is a constant.

Definition 4.2 Let $k \geq 2$ be a positive integer. An arbitrary nonempty finite set $A \subseteq G$ is called *strongly connected of degree k* if there is an absolute constant $C \in (0,1]$ such that for any disjoint sets $E, F \subseteq A$, $E \bigsqcup F = A$, we have

$$\sum_x (E \circ F)(x) \cdot ((A *_{k-2} A) \circ (A *_{k-2} A))(x) \geq C c_E c_F T_k(A) \,, \qquad (36)$$

where $c_E = |E|/|A|$, $c_F = |F|/|A|$.

First of all let us show that any strongly connected set is a connected set.

Statement 4.3 *Let p be a positive integer, and $k = 2^p$. Suppose that A is a strongly connected of degree k set such that (36) holds with some constant C. Then A is connected of degree k and inequality (1) holds with $C/8$.*

Proof. If the cardinality of A is less than two then there is nothing to prove. Let $|A| \geq 3$, B be an arbitrary subset of A, and $\overline{B} = A \setminus B$. Let also

$$\sigma = \sum_x (B \circ \overline{B})(x) \cdot ((A *_{k-2} A) \circ (A *_{k-2} A))(x) \,. \qquad (37)$$

Since A is a strongly connected of degree k it follows that

$$\sigma \geq C\frac{|B|}{|A|}\frac{|\overline{B}|}{|A|}\cdot T_k(A)\,. \tag{38}$$

We have

$$\sigma = \sum_x (B * A *_{k-2} A)(x)\cdot(\overline{B} * A *_{k-2} A)(x)\,. \tag{39}$$

Using Lemma 2.3, we get $\sigma^{2k} \leq T_k(B)T_k(\overline{B})T_k^{2k-2}(A)$. Combining the last inequality and (38), we obtain

$$T_k(B)T_k(\overline{B}) \geq C^{2k}\frac{|B|^{2k}}{|A|^{2k}}\cdot\frac{|\overline{B}|^{2k}}{|A|^{2k}}T_k^2(A)\,. \tag{40}$$

If $|B| \leq |A|/2$ then $|\overline{B}| \geq |A|/2$. Using this lower bound for $|\overline{B}|$, we get

$$T_k(B) \geq \left(\frac{C}{2}\right)^{2k}\left(\frac{|B|}{|A|}\right)^{2k}T_k(A) \tag{41}$$

and the statement is proved. Suppose that $|B| > |A|/2$. Then let B_1 be an arbitrary subset of B of the cardinality $[|A|/2]$. Clearly, $|B| \leq 4|B_1|$. By (41), we have

$$T_k(B) \geq T_k(B_1) \geq \left(\frac{C}{2}\right)^{2k}\left(\frac{|B_1|}{|A|}\right)^{2k}T_k(A) \geq \left(\frac{C}{8}\right)^{2k}\left(\frac{|B|}{|A|}\right)^{2k}T_k(A)\,.$$

This completes the proof.

Thus any strongly connected set is connected. In particular, Proposition 2.7 is true for an arbitrary strongly connected set and therefore any strongly connected set is contained in Span Ł for some Ł with small cardinality. Apparently, it was S.V. Konyagin (see [20]) who first proved that an arbitrary strongly connected set is economically contained in some special subgroup (see also another variant of his statement in book [21] p. 114, ex. 2.6.10). We formulate his result in our terms and give the proof for the sake of completeness.

Statement 4.4 *Let $k \geq 2$ be a positive integer. Let also $A \subseteq G$ be a strongly connected of degree k set such that (36) holds with some constant C. Let*

$$S = \left\{h \in G \ :\ ((A *_{k-2} A)\circ(A *_{k-2} A))(x) \geq C\frac{T_k(A)}{|A|^2}\right\}\,.$$

Then there is an element $a \in G$ such that $A \subseteq \langle S\rangle + a$, where $\langle S\rangle$ is the subgroup of G generating by S.

Proof. Assume the converse. Let $H = \langle S\rangle$ and let $A_1, \ldots, A_r \subseteq A$ be intersections of A with cosets of H. If there are two nonempty intersections of cosets of H with A, say, A_i and A_j, $i < j$, $i, j \in \{1, \ldots, r\}$ then put $E = \bigsqcup_{l=1}^{i} A_l$ and $F = A \setminus E$. Clearly, E and F are nonempty sets. Since for any $e \in E$ and $f \in F$, we have $e - f \notin H$, and, consequently, $e - f \notin S$ it follows that

$$\sum_x (E\circ F)(x)\cdot((A *_{k-2} A)\circ(A *_{k-2} A))(x) \leq$$

$$\leq \sum_{x\notin S}(E\circ F)(x)\cdot((A *_{k-2} A)\circ(A *_{k-2} A))(x) < C|E||F|\cdot\frac{T_k(A)}{|A|^2}$$

with contradiction. This completes the proof.

We prove an analog of Theorem 2.11 for strongly connected sets.

Let $E, F \subseteq A$ be sets. Denote by $e(E,F)$ the quantity $\sum_x (E \circ F)(x) \cdot ((A *_{k-2} A) \circ (A *_{k-2} A))(x)$. Clearly, $e(E_1 \bigsqcup E_2, F) = e(E_1, F) + e(E_2, F)$. Suppose that $E \subseteq A$ is an arbitrary set. By c_E denote the ratio $|E|/|A|$.

We need in the following technical definition of strongly connected of degree k sets.

Definition 4.5 Let $k \geq 2$ be a positive integer. An arbitrary nonempty finite set $A \subseteq G$ is called *β—strongly connected of degree k* if there is an absolute constant $C \in (0,1]$ and a set $B \subseteq A$, $|B| \geq \beta |A|$ such that for any disjoint sets $E, F \subseteq B$, $E \bigsqcup F = B$, we have

$$\sum_x (E \circ F)(x) \cdot ((A *_{k-2} A) \circ (A *_{k-2} A))(x) \geq C c_E c_F T_k(A) \,. \qquad (42)$$

Our next statement can be proved likewise Statement 4.3.

Statement 4.6 *Let p be a positive integer, $k = 2^p$, and $\beta \in [0,1]$ be a real number. Let A be a strongly β—connected of degree k set and (42) is true with some C and some $B \subseteq A$, $|B| \geq \beta |A|$. Then B is connected of degree k set and inequality (1) holds with $C\beta^2/8$.*

A graph–theoretical variant of Lemma 4.7 below was proved in [8].

Lemma 4.7 *Let $k \geq 2$ be a positive integer, $\varepsilon_1 \in [0,1]$ be a real number, and let $A \subseteq G$ be a finite set. Then there exists a partition of A into disjoint sets $A_1, \dots, A_l$ such that*

1) *For all $i, j \in \{1, \dots, l\}$, $i \neq j$, we have $e(A_i, A_j) \leq \varepsilon_1 c_{A_i} c_{A_j} T_k(A)$.*

2) *For any $i \in \{1, \dots, l\}$ the set A_i has the following property : for any disjoint sets $E, F \subseteq A_i$, $E \bigsqcup F = A_i$, we have $e(E,F) \geq \varepsilon_1 c_E c_F T_k(A)$.*

Besides, the following inequality holds

3) $\sum_{i=1}^l T_k(A_i) \geq T_k(A) \cdot (1 - (2k-1)\varepsilon_1)$.

Proof. Consider all partitions of A into disjoint subsets $A_1, \dots, A_s$, where s is an arbitrary positive integer. Select one for which the sum

$$\sigma(A_1, \dots, A_s) = \sum_{1 \geq i < j \leq k} (\ e(A_i, A_j) - \varepsilon_1 c_{A_i} c_{A_j} T_k(A)\ ) \qquad (43)$$

is minimal. By minimality of this partition, say $\{A_1, \dots, A_l\}$, we have 2).

Let us prove 1). Suppose that for some $i, j \in \{1, \dots, l\}$, $i \neq j$ the following holds $e(A_i, A_j) > \varepsilon_1 c_{A_i} c_{A_j} T_k(A)$. Constructing the new partition $\mathcal{P}$ of the set A, $\mathcal{P} = \{A_r\}_{r \neq i,j} \bigsqcup (A_i \bigsqcup A_j)$ and using the last inequality, we get

$$\sigma(\mathcal{P}) = \sigma(A_1, \dots, A_s) - (e(A_i, A_j) - \varepsilon_1 c_{A_i} c_{A_j} T_k(A)) < \sigma(A_1, \dots, A_s) \,.$$

with contradiction.

Prove that 1) implies 3). Indeed,

$$T_k(A) = \sum_x (A *_{k-1} A)^2(x) = \sum_{i,j=1}^l \sum_x (A_i * A *_{k-2} A)(x) \cdot (A_j * A *_{k-2} A)(x) =$$

$$= \sum_{i=1}^l \sum_x (A_i * A *_{k-2} A)(x) \cdot (A_i * A *_{k-2} A)(x) + \sum_{i,j=1,\, j \neq i}^l \sum_x (A_i * A *_{k-2} A)(x) \cdot (A_j * A *_{k-2} A)(x)$$

$$= \sum_{i=1}^l \sum_x (A_i * A *_{k-2} A)(x) \cdot (A_i * A *_{k-2} A)(x) + \sum_{i,j=1,\, j \neq i}^l \sum_x (A_i \circ A_j)(x) \cdot ((A *_{k-2} A) \circ (A *_{k-2} A))(x)$$

$$\le \sum_{i=1}^{l}\sum_{x}(A_i * A *_{k-2} A)(x)\cdot(A_i * A *_{k-2} A)(x) + \varepsilon_1 \sum_{i,j=1}^{l} c_{A_i}c_{A_j}T_k(A) \le$$

$$\le \sum_{i=1}^{l}\sum_{x}(A_i * A *_{k-2} A)(x)\cdot(A_i * A *_{k-2} A)(x) + \varepsilon_1 T_k(A)\,.$$

Hence $\sum_{i=1}^{l}\sum_{x}(A_i * A *_{k-2} A)(x)\cdot(A_i * A *_{k-2} A)(x) \ge (1-\varepsilon_1)T_k(A)$. Similarly,

$$\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{x}(A_i * A_j * A *_{k-3} A)(x)\cdot(A_i * A *_{k-2} A)(x) \le$$

$$\le \sum_{i=1}^{l}\sum_{x}(A_i * A_i * A *_{k-3} A)(x)\cdot(A_i * A *_{k-2} A)(x) +$$

$$+\sum_{i=1}^{l}\sum_{j=1,\,j\ne i}^{l}\sum_{x}(A * A_j * A *_{k-3} A)(x)\cdot(A_i * A *_{k-2} A)(x)$$

$$\le \sum_{i=1}^{l}\sum_{x}(A_i * A_i * A *_{k-3} A)(x)\cdot(A_i * A *_{k-2} A)(x) +$$

$$+\sum_{i=1}^{l}\sum_{j=1,\,j\ne i}^{l}\sum_{x}(A_i \circ A_j)(x)\cdot((A *_{k-2} A)\circ(A *_{k-2} A))(x) \le$$

$$\le \sum_{i=1}^{l}\sum_{x}(A_i * A_i * A *_{k-3} A)(x)\cdot(A_i * A *_{k-2} A)(x) + \varepsilon_1 T_k(A)\,.$$

And so on. Finally, we obtain

$$\sum_{i=1}^{l} T_k(A_i) = \sum_{i=1}^{l}\sum_{x}(A_i *_{k-1} A_i)(x)\cdot(A_i *_{k-1} A_i)(x) \ge (1-(2k-1)\varepsilon_1)\cdot T_k(A)\,.$$

This concludes the proof.

Note 4.8 It is easy to see that the third property of the constructed partition implies that there is $i_0 \in \{1,\dots,l\}$ such that $|A_{i_0}| \ge (1-(2k-1)\varepsilon_1)\cdot m^{\frac{\zeta_k(A)-1}{2k-2}} \ge (1-(2k-1)\varepsilon_1)\cdot m^{1/2}$. In fact, we have

$$m^{\zeta_k(A)}(1-(2k-1)\varepsilon_1) \le \sum_{i=1}^{l} T_k(A_i) \le (\max_{i=1,\dots,l}|A_i|)^{2k-2}\sum_{i=1}^{l}|A_i| \le (\max_{i=1,\dots,l}|A_i|)^{2k-2} m\,.$$

This yields that if we put $\beta = (1-(2k-1)\varepsilon_1)m^{-1/2}$ then *any* set $A \subseteq G$, $|A| = m$ is strongly β—connected of degree k and inequality (42) holds with any $C \le 1/(2k-1)$. Thus to obtain nontrivial results on the structure of A one should prove that A is strongly β—connected for *large* β.

Theorem 4.9 *Let $A \subseteq G$ be a set. Let also $\varepsilon, \beta \in (0,1)$ be real numbers, and $|A| \ge \varepsilon/(2\beta^2)$. Then there exists a partition of A into disjoint sets $A_1,\dots,A_t$, Ω such that*
1) *Any set A_i, $i = 1,\dots,t$ is strongly β—connected of degree 2 and inequality (42) holds with*

any $C \le \varepsilon \log(1/\beta)/(3\log(2|A|/\varepsilon))$.
2) $\sum_{i=1}^{t} T_2(A_i) \ge (1-\varepsilon)\cdot T_2(A)$.

Proof. Let $m = |A|$, $s_0 = \log(2m/\varepsilon)/(2\log(1/\beta)) \ge 1$, and $\varepsilon' = \varepsilon/(6s_0)$. Let $C \le \varepsilon'$ be a real number. The proof of Theorem 4.9 is a sort algorithm. If A is strongly β—connected of degree 2 and (42) is true with the constant C then there is nothing to prove. Suppose that A is not strongly β—connected of degree 2 set (with the constant C). Using Lemma 4.7 with $\varepsilon_1 = \varepsilon'$, we get the partition $\mathcal{P}^{(1)}$ of A into $A_1, \dots, A_l$ satisfy properties 1) — 3) of the lemma. Since A is not strongly β—connected of degree 2 it follows that for any $l \in \{1, \dots, l\}$, we have $|A_i| < \beta|A|$. Using the third property of the partition $\mathcal{P}^{(1)}$, we obtain

$$\sum_{\mathcal{A}\in\mathcal{P}^{(1)}} T_2(\mathcal{A}) = \sum_{i=1}^{l} T_2(A_i) \ge (1-3\varepsilon')T_2(A)\,.$$

Let $B^{(1)} = \{A_i$ — is not strongly $\beta-$connected of degree $2\}$, and $G^{(1)}$ be the collection of all other sets of the partition $\mathcal{P}^{(1)}$. Let us construct a new partition of A. We do not change the sets A_i from $G^{(1)}$. Further, for any A_i belongs to $B^{(1)}$, we use Lemma 4.7 with $\varepsilon_1 = \varepsilon'$. We get a new partition of A_i into subsets A_{ij}, $j \in \{1, \dots, l(i)\}$. So we construct a new partition $\mathcal{P}^{(2)}$ of the set A. For any $A_i \in B^{(1)}$ the following holds $\sum_{j=1}^{l(i)} T_2(A_{ij}) \ge (1-3\varepsilon')T_2(A_i)$. Hence

$$\sum_{\mathcal{A}\in\mathcal{P}^{(2)}} T_2(\mathcal{A}) \ge (1-3\varepsilon')^2\cdot T_2(A)\,. \tag{44}$$

Let $B^{(2)} = \{A_{ij}$ — is not strongly $\beta-$ connected of degree $2\}$. For an arbitrary $A_{ij} \in B^{(2)}$, we use Lemma 4.7 with $\varepsilon_1 = \varepsilon'$. We get a new partitions of the sets A_{ij} into disjoint subsets A_{ijr}. And so on. At s–th step of the algorithm, we construct the partition $\mathcal{P}^{(s)}$ such that

$$\sum_{\mathcal{A}\in\mathcal{P}^{(s)}} T_2(\mathcal{A}) \ge (1-3\varepsilon')^s\cdot T_2(A) \ge (1-3\varepsilon' s)\cdot T_2(A)\,. \tag{45}$$

It is easy to see that if for some $s \le s_0$ the following holds

$$\sum_{\mathcal{A}\in\mathcal{P}^{(s)}\setminus B^{(s)}} T_2(\mathcal{A}) \ge (1-\varepsilon)\cdot T_2(A)\,, \tag{46}$$

then we are done. Indeed, just put $\Omega = \bigsqcup_{\mathcal{A}\in B^{(s)}} \mathcal{A}$. Suppose that for all $s \le s_0$ inequality (46) does not hold. Using inequality $s \le s_0$ and (45), we get $\sum_{\mathcal{A}\in\mathcal{P}^{(s)}} T_2(\mathcal{A}) \ge (1-\varepsilon/2)\cdot T_2(A)$. Hence

$$\sum_{\mathcal{A}\in B^{(s)}} T_2(\mathcal{A}) \ge \frac{\varepsilon}{2}\cdot T_2(A) \ge \frac{\varepsilon m^2}{2}\,. \tag{47}$$

For any $\mathcal{A} \in B^{(s)}$, we have $|\mathcal{A}| < \beta^s m$. Whence

$$\sum_{\mathcal{A}\in B^{(s)}} T_2(\mathcal{A}) < (\beta^s m)^2 \sum_{\mathcal{A}\in\mathcal{P}^{(s)}} |\mathcal{A}| = \beta^{2s} m^3\,. \tag{48}$$

If $s = s_0$ then the last inequality contradicts (47). So for some $s < s_0$ inequality (46) holds. This completes the proof.

There is a difference between Theorem 2.11 and Theorem 4.9. In Theorem 4.9 we prove that there exists a *partition* of A into strongly β—connected components and some exceptional set Ω while Theorem 2.11 states that there is *one* connected subset of A. Besides, Theorem 4.9 implies that the remaining set Ω has small $T_2(\Omega)$. Indeed, by the property 2), we have $\sum_{i=1}^{t} T_2(A_i) \ge (1-\varepsilon)\cdot T_2(A)$, whence $T_2(\Omega) \le \varepsilon T_2(A)$.

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