ON A TWO–DIMENSIONAL ANALOG OF SZEMERÉDI'S THEOREM IN ABELIAN GROUPS

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Abstract

Let G be a finite Abelian group and $A \subseteq G \times G$ be a set of cardinality at least $|G|^2/(\log\log|G|)^c$, where $c > 0$ is an absolute constant. We prove that A contains a triple $\{(k,m),(k+d,m),(k,m+d)\}$, where $d \neq 0$. This theorem is a two-dimensional generalization of Szemerédi's theorem on arithmetic progressions.

1. *Introduction.*

Szemerédi's theorem [**29**] on arithmetic progressions states that an arbitrary set $A \subseteq \mathbf{Z}$ of positive density contains arithmetic progression of any length. This remarkable theorem has played a significant role in the development of two fields in mathematics : additive combinatorics (see e.g. [**31**]) and combinatorial ergodic theory (see e.g. [**10**]) A more precise statement of the theorem is as follows.

Let N be a natural number. We set

$$a_k(N) = \frac{1}{N} \max\{|A| \ : \ A \subseteq \{1,2,\dots,N\},$$

$$A \text{ contains no arithmetic progressions of length } k\},$$

where $|A|$ denotes the cardinality of A.

Theorem 1.1 (Szemerédi, 1975). *For any $k \geq 3$ the following holds*

$$a_k(N) \to 0 \text{ as } N \to \infty\,. \tag{1.1}$$

Clearly, this result implies van der Waerden's theorem [**33**].

In the simplest case $k = 3$ of Theorem 1.1 was proven by K.F. Roth [**22**] in 1953, who applied the Hardy – Littlewood method to show that

$$a_3(N) \ll \frac{1}{\log\log N}\,.$$

At present, the best upper bound for $a_3(N)$ is due to J. Bourgain [**4**]. He proved that

$$a_3(N) \ll \frac{(\log\log N)^2}{(\log N)^{2/3}}\,. \tag{1.2}$$

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Szemerédi's proof uses difficult combinatorial arguments. An alternative proof was suggested by Furstenberg in [**10**] (see also [**10**]). His approach uses the methods of ergodic theory. Furstenberg showed that Szemerédi's theorem is equivalent to the multiple recurrence of almost all points in any dynamical system.

A. Behrend [**2**] obtained the following lower bound for $a_3(N)$

$$a_3(N) \gg \exp(-C(\log N)^{\frac{1}{2}})\,,$$

where C is an absolute constant. A lower bound on $a_k(N)$ for an arbitrary k was given in [**21**].

Unfortunately, Szemerédi's methods give very weak upper estimates for $a_k(N)$. The ergodic approach gives no estimates at all. Only in 2001 W.T. Gowers [**11**] obtained a quantitative result concerning the rate at which $a_k(N)$ approaches zero for $k \ge 4$. He proved the following theorem.

Theorem 1.2. *For any $k \ge 4$, we have*

$$a_k(N) \ll 1/(\log\log N)^{c_k}\,,$$

where the constant c_k depends on k only.

In paper [**1**] and book [**10**] the following problem was considered. Let $\{1,2,\dots,N\}^2$ be the two–dimensional lattice with basis $\{(1,0),\ (0,1)\}$. Let also

$$L(N) = \frac{1}{N^2} \max\{\ |A|\ :\ A \subseteq \{1,2,\dots,N\}^2 \ \text{ and}$$

$$A \text{ contains no triples of the form } \{(k,m),\ (k+d,m),\ (k,m+d)\}$$

$$\text{with positive } d\}. \tag{1.3}$$

A triple from (1.3) is called a *"corner"*. In [**1**, **10**] it was proven that $L(N)$ tends to 0 as N tends to infinity. W.T. Gowers (see [**11**]) asked the question of what is the rate of convergence of $L(N)$ to 0.

The following theorem was proven in [**26**, **27**] (see also [**28**, **32**, **24**, **25**]).

Theorem 1.3. *Let $\delta > 0$, and $N \gg \exp\exp(\delta^{-73})$. Let also A be a subset of $\{1,\dots,N\}^2$ of cardinality at least δN^2. Then A contains a corner.*

Thus, we have the estimate $L(N) \ll 1/(\log\log N)^{1/73}$.

The question on upper estimates for $L(N)$ in the group $\mathbf{F}_3^n$ was considered in [**15**] and [**18**].

A natural generalization of Theorem 1.3 above is replace $\{1,\dots,N\}$ or $\mathbf{Z}/N\mathbf{Z}$ to an arbitrary Abelian group. Such generalizations of Roth's theorem and Theorems 1.1, 1.2 were obtained in papers [**5**, **8**, **20**, **19**, **17**].

The main result of this paper is the following theorem.

Let G be a finite Abelian group with additive group operation $+$. In the case any triple of the form $\{(k,m),(k+d,m),(k,m+d)\}$, where $d \ne 0$ is called a corner.

Theorem 1.4. *Let G be a finite Abelian group and $A \subseteq G \times G$ be a set of cardinality at least $|G|^2/(\log\log|G|)^c$, where $c > 0$ is an absolute constant. Then A contains a corner.*

NOTE. The constant c in Theorem 1.4 might be taken as $1/22$.

The proof of Theorem 1.4 is contained in §3,4,5,6 and proceeds by an iteration scheme as in all known effective proofs of Szemerédi–type theorems.

Let G be an Abelian group, $A \subseteq G \times G$, $|A| \gg |G|^2/(\log\log|G|)^c$ and we want to find a corner in A. At each step of our procedure we prove the following : either A is "sufficiently regular" or its "density" can be increased. A suitable definition of "sufficiently regular" sets (so–called uniform sets) is one of the main aims of our proof.

If A is a random set and A has cardinality $\delta|G|^2$ then it is easy to see that A contains approximately $\delta^3 N^3$ corners. We shall say A is regular (or in other words *α–uniform*) if A contains the same approximate number of corners.

Let E_1, E_2 be subsets of Λ, where $\Lambda \subseteq G$ to be chosen later. Let A be a subset of $E_1 \times E_2$ of cardinality $\delta|E_1||E_2|$. We shall say that A is *rectilinearly α–uniform* if, roughly speaking, the number of quadruples $\{(x,y),(x+d,y),(x,y+s),(x+d,y+s)\}$ in A^4 is at most $(\delta^4+\alpha)|E_1|^2|E_2|^2$, $\alpha > 0$ (in fact we need a slightly different definition of α–uniformity, which depends on the set Λ). In §3 we prove that if E_1, E_2 has small Fourier coefficients and A is rectilinearly α–uniform then A has about the expected number of corners. Simple observation shows (see e.g. [**27**]) that the notion of rectilinearly α–uniformity cannot be expressed in terms of Fourier transform, more precisely, there is a set, say A_0, with really small Fourier coefficients but large number of quadruples $\{(x,y),(x+d,y),(x,y+s),(x+d,y+s)\} \in A_0^4$. On the other hand, we can define a rectilinearly α–uniform set using so–called rectilinear norm (see §3).

Suppose that A fails to be rectilinearly α–uniform. Roughly speaking, it means that A has no random properties. The last observation can be expressed precisely by showing that A has increased density $\delta + c(\delta)$, $c(\delta) > 0$ on some product set $F_1 \times F_2$, $F_1 \subseteq E_1$, $F_2 \subseteq E_2$ (see §4). Clearly, this density increment can only occur finitely many times, because the density of any set does not exceed one. Thus, our iteration scheme must stop after finite number of steps. It means that we find a rectilinearly α–uniform subset of the set A and consequently a corner in A.

Unfortunately, the structure of $F_1 \times F_2$ need not be regular and we cannot make the next step of our procedure directly. To make $F_1 \times F_2$ regular, we pass to a subset of Λ, say, Λ' and a vector $\vec{t} = (t_1, t_2) \in G \times G$ such that $(F_1 - t_1) \cap \Lambda'$, $(F_2 - t_2) \cap \Lambda'$ has small Fourier coefficients.

We are now in the situation we started with, but A has a larger density and we iterate the procedure. This also can only occur finitely many times. In §6 we combine the arguments from the earlier sections and show that they give the bound that we stated in Theorem 1.4.

In our prove we chose Λ to be a *Bohr set* (see [**3**, **4**, **14**] and others). Note that the best upper bound for $a_3(N)$ was proven by J. Bourgain in [**4**] using exactly these very sets. The properties of Bohr sets will be considered in §2.

The constructions which we use develop the approach of [**3**, **11**, **24**, **27**]. We improve our constant c by more accurate calculations than in [**27**].

In our forthcoming papers we are going to obtain a multidimensional analog of Theorem 1.4.

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2. *On Bohr sets.*

Let $G = (G, +)$ be a finite Abelian group with additive group operation $+$. Suppose that A is a subset of G. It is very convenient to write $A(x)$ for such a function. Thus $A(x) = 1$ if $x \in A$ and $A(x) = 0$ otherwise. By $\widehat{G}$ denote the Pontryagin dual of G, in other words the space of homomorphisms ξ from G to $\mathbf{T}$, $\xi : x \to \xi \cdot x$. It is well known that $\widehat{G}$ is an additive group which is isomorphic to G. Also denote by N the cardinality of G.

One of the crucial moments in [**3**] was the notion of Bohr set.

Let S be a subset of $\widehat{G}$, $|S| = d$, $\varepsilon > 0$ be a real number.

Definition 2.1. Define the Bohr set $\Lambda = \Lambda(S, \varepsilon)$ by

$$\Lambda(S, \varepsilon) = \{n \in G \mid \|\xi \cdot n\| < \varepsilon \text{ for all } \xi \in S\}\,.$$

We shall say that the set $S \subseteq \widehat{G}$ is *generative set* of Bohr set Λ. The number d is called *dimension* of Bohr set Λ and is denoted by $\dim \Lambda$. If $M = \Lambda + n$, $n \in G$ is a translation of Λ, then, by definition, put $\dim M = \dim \Lambda$.

Another construction of Bohr set (so–called *smoothed* Bohr set) was given in [**30**] and [**14**].

Definition 2.2. Let $0 < \kappa < 1$ be a real number. A Bohr set $\Lambda = \Lambda(S, \varepsilon)$ is called *regular*, if for an arbitrary ε' such that

$$|\varepsilon - \varepsilon'| < \frac{\kappa}{100d}\varepsilon$$

we have

$$1 - \kappa < \frac{|\Lambda(S, \varepsilon')|}{|\Lambda(S, \varepsilon)|} < 1 + \kappa\,.$$

We need several results concerning Bohr sets (see [**3**] and [**14**]).

Lemma 2.1. *Let $\Lambda(S, \varepsilon)$ be a Bohr set, $|S| = d$. Then*

$$|\Lambda(S, \varepsilon)| \ge \varepsilon^d N\,.$$

Lemma 2.2. *Let $0 < \kappa < 1$ be a real number, and $\Lambda(S, \varepsilon)$ be a Bohr set. Then there exists ε_1 such that $\frac{\varepsilon}{2} < \varepsilon_1 < \varepsilon$ and such that $\Lambda(S, \varepsilon_1)$ is a regular Bohr set.*

All Bohr sets will be regular in the article.

Definition 2.3. Let f, g be functions from G to $\mathbf{C}$. By $f * g$ define the function

$$(f * g)(n) = \sum_{s \in G} f(s) g(n - s)\,.$$

Definition 2.4. Let $\varepsilon \in (0, 1]$ be a real number, and $\Lambda(S, \varepsilon_0)$ be a Bohr set, $S \subseteq \widehat{G}$, $|S| = d$. A regular Bohr set $\Lambda' = \Lambda(S', \varepsilon')$ is called an ε*–attendant* of Λ if $S \subseteq S'$ and $\varepsilon\varepsilon_0/2 \le \varepsilon' \le \varepsilon\varepsilon_0$.

Lemma 2.2 implies that for an arbitrary Bohr set there exists its ε–attendant.

We shall consider that $S' = S$ unless stated otherwise.

Let n be an arbitrary element of the group G, and Λ be a Bohr set. We shall say that a Bohr set Λ' is an ε–attendant of $\Lambda + n$, if Λ' is an ε–attendant of Λ.

The following lemma is also due to J. Bourgain [**3**]. We give his proof for the sake of completeness.

Lemma 2.3. *Let $\kappa > 0$ be a real number, $S \subseteq \widehat{G}$, $\Lambda = \Lambda(S, \varepsilon)$ be a regular Bohr set, and $\Lambda' = \Lambda(S, \varepsilon')$ its $\kappa/(100d)$–attendant. Then the number of n's such that $(\Lambda * \Lambda')(n) > 0$ does not exceed $|\Lambda|(1+\kappa)$, the number of n's such that $(\Lambda * \Lambda')(n) = |\Lambda'|$ is greater than $|\Lambda|(1-\kappa)$ and*

$$\Big\| \frac{1}{|\Lambda'|}(\Lambda * \Lambda')(n) - \Lambda(n) \Big\|_1 < 2\kappa |\Lambda| \,. \tag{2.1}$$

Proof. If $(\Lambda * \Lambda^{'})(n) > 0$ then there exists m such that for any $\xi \in S$, we have

$$\|\xi \cdot m\| < \frac{\kappa}{100d}\varepsilon, \quad \|\xi \cdot (n-m)\| < \varepsilon \,. \tag{2.2}$$

Using (2.2), we get for all $\xi \in S$

$$\|\xi \cdot n\| < \Big(1 + \frac{\kappa}{100d}\Big)\varepsilon \,, \tag{2.3}$$

for all $\xi \in S$. It follows that

$$n \in \Lambda^{+} := \Lambda \left(S, \Big(1 + \frac{\kappa}{100d}\Big)\varepsilon\right) \,. \tag{2.4}$$

By Lemma 2.2 we have $|\Lambda^{+}| \le (1+\kappa)|\Lambda|$.

On the other hand, if

$$n \in \Lambda^{-} := \Lambda \left(S, \Big(1 - \frac{\kappa}{100d}\Big)\varepsilon\right) \tag{2.5}$$

then $(\Lambda * \Lambda^{'})(n) = |\Lambda'|$. Using Lemma 2.2, we obtain $|\Lambda^{-}| \ge (1-\kappa)|\Lambda|$.

Let us prove (2.1). We have

$$\begin{aligned}\Big\| \frac{1}{|\Lambda'|}(\Lambda * \Lambda')(n) - \Lambda(n) \Big\|_1 &= \Big\| \frac{1}{|\Lambda'|}(\Lambda * \Lambda')(n) - \Lambda(n) \Big\|_{l^1(\Lambda^{+} \setminus \Lambda^{-})} \\ &\le |\Lambda^{+}| - |\Lambda^{-}| < 2\kappa|\Lambda|\end{aligned}$$

as required.

Corollary. *Lemma 2.3 implies that $|\Lambda| \le |\Lambda + \Lambda'| \le (1+2\kappa)|\Lambda|$.*

Note. Let $\Lambda^{x}(n) = \Lambda(n-x)$. Since $(\Lambda^{x} * \Lambda')(n) = (\Lambda * \Lambda')(n-x)$, it follows that (2.1) takes place for translations $\Lambda + x$.

Definition 2.5. By Λ^{+} and Λ^{-} denote the Bohr sets defined in (2.4) and (2.5), respectively, $\Lambda^{-} \subseteq \Lambda \subseteq \Lambda^{+}$.

By Lemma 2.3 we have $|\Lambda^{+}| \le |\Lambda|(1+\kappa)$ and $|\Lambda^{-}| \ge |\Lambda|(1-\kappa)$. Note that for any $s \in \Lambda^{'}$, we get $\Lambda^{-} \subseteq \Lambda + s$.

Suppose $\Lambda \subseteq G$ is a Bohr set, and $\vec{x} = (x_1, x_2)$ belongs to $G \times G$. By $\Lambda + \vec{x}$

denote the set $(\Lambda + x_1) \times (\Lambda + x_2) \subseteq G \times G$. Let $\vec{n} \in G \times G$. Let $\Lambda(\vec{n})$ denote the characteristic function of $\Lambda \times \Lambda$. We shall write $\vec{s} \in \Lambda$, $\vec{s} = (s_1, s_2)$, if $s_1 \in \Lambda$ and $s_2 \in \Lambda$.

Lemma 2.4. *Suppose Λ is a Bohr set, $\Lambda^{'}$ is its ε–attendant, $\varepsilon = \kappa/(100d)$, $\vec{x}$ is a vector, and $E \subseteq G \times G$. Then*

$$\Big| \delta_{\Lambda + \vec{x}}(E) - \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda + \vec{x}} \delta_{\Lambda' + \vec{n}}(E) \Big| \le 4\kappa \,. \tag{2.6}$$

Proof. We have

$$\sigma = \frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda + \vec{x}} \delta_{\Lambda' + \vec{n}}(E) = \frac{1}{|\Lambda|^2 |\Lambda'|^2} \sum_{\vec{s}} E(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n} - \vec{x}) \Lambda^{'}(\vec{s} - \vec{n})$$

$$= \frac{1}{|\Lambda|^2 |\Lambda'|^2} \sum_{\vec{s}} E(\vec{s}) \sum_{\vec{n}} \Lambda(\vec{n}) \Lambda^{'}(\vec{s} - \vec{x} - \vec{n})$$

Using Lemma 2.3, we get

$$\sigma = \frac{1}{|\Lambda|^2} \sum_{\vec{s}} E(\vec{s}) \Lambda(\vec{s} - \vec{x}) + 4\vartheta\kappa = \delta_{\Lambda + \vec{x}}(E) + 4\vartheta\kappa \,,$$

where $|\vartheta| \le 1$. This completes the proof.

Note. Clearly, the one–dimension analog of Lemma 2.4 takes place.

Let $\Lambda_1 = \Lambda(S_1, \varepsilon_1)$, $\Lambda_2 = \Lambda(S_2, \varepsilon_2)$ be two Bohr sets, $S_1, S_2 \subseteq \widehat{G}$. We shall write $\Lambda_1 \le \Lambda_2$, if $S_1 \subseteq S_2$ and $\varepsilon_1 \le \varepsilon_2$.

3. *On α–uniformity.*

Let f be a function from G to $\mathbf{C}$, $N = |G|$. By $\widehat{f}(\xi)$ denote the Fourier transformation of f

$$\widehat{f}(\xi) = \sum_{x \in G} f(x) e(-\xi \cdot x) \,,$$

where $e(x) = e^{2\pi i x}$. We shall use the following basic facts

$$\sum_{x \in G} |f(x)|^2 = \frac{1}{N} \sum_{\xi \in \widehat{G}} |\widehat{f}(\xi)|^2 \,. \tag{3.1}$$

$$\sum_{x \in G} f(x) \overline{g(x)} = \frac{1}{N} \sum_{\xi \in \widehat{G}} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \,. \tag{3.2}$$

$$\sum_{y \in G} | \sum_{x \in G} f(x) g(y - x)|^2 = \frac{1}{N} \sum_{\xi \in \widehat{G}} |\widehat{f}(\xi)|^2 |\widehat{g}(\xi)|^2 \,. \tag{3.3}$$

Let Λ be a Bohr set, and A be an arbitrary subset of Λ. Let $|A| = \delta|\Lambda|$. Define the *balanced* function of A to be $f(s) = (A(s) - \delta)\Lambda(s) = A(s) - \delta\Lambda(s)$.
Let $\mathbf{D}$ denote the closed disk of radius 1 centered at 0 in the complex plane. Let R

be an arbitrary set. We write $f : R \to \mathbf{D}$ if f is zero outside R.
The following definition is due to Gowers [**11**].

Definition 3.1. A function $f : \Lambda \to \mathbf{D}$ is called *α–uniform* if

$$\|\widehat{f}\|_\infty \le \alpha|\Lambda|\,. \tag{3.4}$$

We say that A is α–uniform if its balanced function is.
We shall write $\sum_s$ instead of $\sum_{s\in G}$ and $\sum_\xi$ instead of $\sum_{\xi\in\widehat{G}}$.
Let us prove an analog of Lemma 2.2 from [**11**].

Lemma 3.1. *Let Λ be a Bohr set, and let $f : \Lambda \to \mathbf{D}$ be an α–uniform function. Then we have*

$$\sum_k |\sum_s f(s)g(k-s)|^2 \le \alpha^2|\Lambda|^2\|g\|_2^2\,,$$

for an arbitrary function g, $g : G \to \mathbf{D}$.

Proof. By (3.3) we get

$$\sum_k |\sum_s f(s)g(k-s)|^2 = \sum_\xi |\widehat{f}(\xi)|^2|\widehat{g}(\xi)|^2\,. \tag{3.5}$$

Since the function f is α–uniform, it follows that $\|\widehat{f}\|_\infty \le \alpha|\Lambda|$. Using this inequality and (3.2), we have

$$\sum_k |\sum_s f(s)g(k-s)|^2 \le \alpha^2|\Lambda|^2\frac{1}{N}\sum_\xi |\widehat{g}(\xi)|^2 = \alpha^2|\Lambda|^2\|g\|_2^2\,. \tag{3.6}$$

This completes the proof.

Corollary 3.1. *Let $S \subseteq G$ be a set, and $\Lambda^{'}$ be a Bohr set. Suppose $E \subseteq \Lambda^{'}$ is α–uniform, and E have the cardinality $\delta|\Lambda^{'}|$. Let g be a function from S to $[-1,1]$. Then for all but $\alpha^{2/3}|S|$ choices of k we have*

$$\Big|(E*g)(k) - \delta(\Lambda^{'}*g)(k)\Big| \le \alpha^{2/3}|\Lambda^{'}|\,.$$

Proof. Let f be the balanced function of $E\cap\Lambda^{'}$. Using Lemma 3.1, we get

$$\sum_k |(E*g)(k) - \delta(\Lambda^{'}*g)(k)|^2 = \sum_k |\sum_s f(s)g(k-s)|^2 \le \tag{3.7}$$

$$\le \alpha^2|\Lambda^{'}|^2\|g\|_2^2 \le \alpha^2|\Lambda^{'}|^2|S|\,. \tag{3.8}$$

This concludes the proof.

Let Λ_1 and Λ_2 be Bohr sets, and $E_1\times E_2$ be a subset of $\Lambda_1\times\Lambda_2$. Suppose $f : \Lambda_1\times\Lambda_2 \to \mathbf{D}$ is a function.

Definition 3.2. Let α be a real number, $\alpha\in[0,1]$. A function $f : E_1\times E_2 \to \mathbf{D}$

is called *rectilinearly α–uniform* if

$$\sum_{x,x',y,y'} f(x,y)\overline{f(x',y)f(x,y')}f(x',y') \le \alpha|E_1|^2|E_2|^2. \qquad (3.9)$$

Note that the function f is α–uniform iff

$$\sum_{m,p} |\sum_{k} f(k,m)\overline{f(k,p)}|^2 \le \alpha|E_1|^2|E_2|^2. \qquad (3.10)$$

Let A be a subset of $E_1 \times E_2$, $|A| = \delta|E_1||E_2|$. Define the *balanced* function of A to be $f(x,y) = (A(x,y) - \delta) \cdot (E_1 \times E_2)(x,y)$. We say that $A \subseteq E_1 \times E_2$ is rectilinearly α–uniform if its balanced function is.

Let f be an arbitrary function, $f : G \times G \to \mathbf{C}$. Define $\|f\|$ by the formula

$$\|f\| = \Big| \sum_{x,x',y,y'} f(x,y)\overline{f(x',y)f(x,y')}f(x',y') \Big|^{\frac{1}{4}} \qquad (3.11)$$

Lemma 3.2. *$\|\cdot\|$ is a norm.*

Proof. See [**24**].

Definition 3.3. Let Λ be a Bohr set, $Q \subseteq \Lambda$, $|Q| = \delta|\Lambda|$, α, ε are positive numbers. A set Q is called *(α,ε)–uniform* if there exists Λ' such that Λ' is an ε–attendant set of Λ and the set

$$B := \{m \in \Lambda \mid \|(Q \cap (\Lambda' + m) - \delta(\Lambda' + m))\widehat{}\,\|_\infty \ge \alpha|\Lambda'|\}$$

has the cardinality at most $\alpha|\Lambda|$

$$|B| \le \alpha|\Lambda|\,, \qquad (3.12)$$

further

$$\frac{1}{|\Lambda|} \sum_{m \in \Lambda} |\delta_{\Lambda'+m}(Q) - \delta|^2 \le \alpha^2\,. \qquad (3.13)$$

and

$$\|(Q \cap \Lambda - \delta\Lambda)\widehat{}\,\|_\infty \le \alpha|\Lambda|\,. \qquad (3.14)$$

Certainly, this definition depends on Λ. and Λ'. We do not assume that Λ' has the same generative set as Λ. If Q is (α,ε)–uniform and Λ' is an ε–attendant set of Λ then we shall mean sometimes that Λ' is an ε–attendant set of Λ such that (3.12) — (3.14) hold.

Note. Let

$$B^* = \{m \in \Lambda \mid |\delta_{\Lambda'+m}(Q) - \delta| \ge \alpha^{2/3}\}\,.$$

Condition (3.13) implies that $|B^*| \le \alpha^{2/3}|\Lambda|$.

Note. Condition (3.14) is not so important as (3.12) and (3.13). The inequality

$$\|(Q \cap \Lambda - \delta\Lambda)\widehat{}\,\|_\infty \le 4\alpha|\Lambda|$$

follows from (3.12), (3.13) (see Proposition 3.1).

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, $\varepsilon > 0$ be a real number. Let also E_1, E_2 be subsets of Λ_1, Λ_2, respectively, and $|E_1| = \beta_1 |\Lambda_1|$, $E_2 = \beta_2 |\Lambda_2|$.

DEFINITION 3.4. A function $f : E_1 \times E_2 \to \mathbf{D}$ is called *rectilinearly* (α,ε)–*uniform* if there exists Λ' such that Λ' is an ε–attendant of Λ_1 and

$$\|f\|^4_{\Lambda_1\times\Lambda_2,\varepsilon} = \sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m,u} \Lambda'(m-k-i)\Lambda'(u-k-i) \times$$

$$|\sum_r \Lambda'(k+r-j) f(r,m) f(r,u)|^2 \le \alpha\beta_1^2\beta_2^2 |\Lambda'|^4 |\Lambda_1|^2 |\Lambda_2| \,. \tag{3.15}$$

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$. Let also E_1, E_2 be subsets of Λ_1, Λ_2, respectively, and $|E_1| = \beta_1 |\Lambda_1|$, $E_2 = \beta_2 |\Lambda_2|$.

DEFINITION 3.5. Let $A \subseteq E_1 \times E_2$, $|A| = \delta\beta_1\beta_2|\Lambda_1||\Lambda_2|$, and $f(\vec{s}) = A(\vec{s}) - \delta(E_1\times E_2)(\vec{s})$. A is called *rectilinearly* $(\alpha,\alpha_1,\varepsilon)$–*uniform* if there exist Λ', Λ'_ε such that Λ' is an ε–attendant of Λ_1, Λ'_ε is an ε–attendant of Λ' and the set

$$B = \{ l \in \Lambda_1 ~|~ \|f_l\|^4_{\Lambda'\times\Lambda_2,\varepsilon} > \alpha\beta_1^2\beta_2^2 |\Lambda'_\varepsilon|^4 |\Lambda'|^2 |\Lambda_2| \} \,,$$

where $f_l(\vec{s}) := f(s_1+l, s_2)\Lambda'(s_1)$, $l \in \Lambda_1$ has the cardinality at most $\alpha_1|\Lambda_1|$,

Note that

$$\|f_l\|^4_{\Lambda'\times\Lambda_2,\varepsilon} = \sum_{i\in\Lambda'}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m,u} \Lambda''(m-k-i)\Lambda''(u-k-i)|\sum_r \Lambda''(k+r-j) f_l(r,m) f_l(r,u)|^2$$

$$= \sum_{i\in\Lambda'+l}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m,u} \Lambda''(m-k-i)\Lambda''(u-k-i) \times |\sum_r \Lambda''(k+r-j) \tilde{f}_l(r,m) \tilde{f}_l(r,u)|^2 \,,$$

where $\Lambda'' = \Lambda'_\varepsilon$ and $\tilde{f}$ is a restriction of f to $(\Lambda'+l)\times\Lambda_2$.

NOTE. We need parameter α_1 to decrease the constant c in Theorem 1.4. To obtain Theorem 1.4 with c equals, say, 1000, one can put $\alpha_1 = \alpha$.

LEMMA 3.3. *Let Λ be a Bohr set. Suppose Λ' is an ε–attendant of Λ, Λ'' is an ε–attendant Λ' and an ε^2–attendant of Λ, $\varepsilon = \alpha^2/4(100d)$, $Q \subseteq \Lambda$, $|Q| = \delta\Lambda$, and $\alpha > 0$. Let*

$$\Omega_1 = \{ s \in \Lambda ~|~ |\delta_{\Lambda'+s}(Q) - \delta| \ge 4\alpha^{1/2} ~~ or ~~ \frac{1}{|\Lambda'|}\sum_{n\in\Lambda'+s} |\delta_{\Lambda''+n}(Q)-\delta|^2 \ge 4\alpha^{1/2} \} \,.$$

$$\Omega_2 = \{ s\in\Lambda ~|~ \|(Q\cap(\Lambda'+s) - \delta(\Lambda'+s))\widehat{~}\|_\infty \ge 4\alpha^{1/4}|\Lambda'| \} \,.$$

1) If

$$\frac{1}{|\Lambda|}\sum_{n\in\Lambda} |\delta_{\Lambda''+n}(Q)-\delta|^2 \le \alpha^2 \,, \tag{3.16}$$

then $|\Omega_1| \le 4\alpha^{1/2}|\Lambda|$.
2) If

$$\Omega^* = \{ s\in\Lambda ~|~ \|(Q\cap(\Lambda''+s) - \delta(\Lambda''+s))\widehat{~}\|_\infty \ge \alpha|\Lambda''| \} \tag{3.17}$$

has the cardinality at most $\alpha|\Lambda|$, then $|\Omega_2| \le 4\alpha^{1/2}|\Lambda|$.

3) Suppose Q is (α, ε^2)–uniform subset of Λ and Λ'' is an ε^2–attendant of Λ such that (3.12) — (3.14) hold. Let

$$\tilde{\Omega} = \{ s \in \Lambda \mid \ \text{Set } (Q-s) \cap \Lambda' \text{ is not } (8\alpha^{1/4}, \varepsilon)\text{–uniform } \}\,.$$

Then $|\tilde{\Omega}| \le 8\alpha^{1/2} |\Lambda|$.

Proof. Let us prove 1). Let $\delta'_n = \delta_{\Lambda'+n}(Q)$, $\delta''_n = \delta_{\Lambda''+n}(Q)$, $\kappa = \alpha^2/4$, and $\epsilon = \alpha^{1/2}$. Consider the sets

$$B_s = \{ n \in \Lambda' + s \mid |\delta''_n - \delta| \ge \epsilon \}\,, \quad G_s = \{ n \in \Lambda' + s \mid |\delta''_n - \delta| < \epsilon \}\,, s \in \Lambda$$

and the sets

$$B = \{ s \in \Lambda \mid |B_s| \ge \epsilon |\Lambda'| \}\,, \quad G = \{ s \in \Lambda \mid |B_s| < \epsilon |\Lambda'| \}\,.$$

If $s \in G$ then $|B_s| < \epsilon|\Lambda'|$. Using Lemma 2.4, we have

$$|\delta'_s - \delta| \le \Big| \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'+s} \delta''_x - \delta \Big| + 4\kappa \le \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'+s} |\delta''_x - \delta| + 4\kappa \le$$

$$\le \frac{1}{|\Lambda'|} \sum_{x \in B_s} |\delta''_x - \delta| + \frac{1}{|\Lambda'|} \sum_{x \in G_s} |\delta''_x - \delta| + 4\kappa < \epsilon + \frac{\epsilon |G_s|}{|\Lambda'|} + 4\kappa \le 4\epsilon\,. \qquad (3.18)$$

Besides that for $s \in G$, we get

$$\frac{1}{|\Lambda'|} \sum_{x \in \Lambda'+s} |\delta''_x - \delta|^2 \le \frac{1}{|\Lambda'|} \sum_{x \in B_s} |\delta''_x - \delta|^2 + \frac{1}{|\Lambda'|} \sum_{x \in G_s} |\delta''_x - \delta|^2 \le \epsilon + \epsilon^2 \le 2\epsilon\,. \quad (3.19)$$

Let us estimate the cardinality of B. We have

$$\alpha^2 \ge \frac{1}{|\Lambda|} \sum_{s \in B} |\delta''_s - \delta|^2 \ge \frac{1}{|\Lambda'||\Lambda|} \sum_{s \in B} \sum_{n \in \Lambda'+s} |\delta''_n - \delta|^2 - 4\kappa \ge$$

$$\ge \frac{1}{|\Lambda'||\Lambda|} \sum_{s \in B} \sum_{n \in B_s} |\delta''_n - \delta|^2 - 4\kappa \ge \frac{|B| \epsilon^3 |\Lambda'|}{|\Lambda'||\Lambda|} - 4\kappa\,.$$

It follows that, $|B| \le 4\alpha^{1/2}|\Lambda|$. Using (3.18), (3.19) we get $\Omega_1 \subseteq B$ and 1) is proven. To prove 2) it suffices to note that

$$\frac{1}{|\Lambda||\Lambda'|} \sum_{s \in \Lambda} \| (Q \cap (\Lambda''+s) - \delta(\Lambda''+s) \widehat{\ } \|_\infty = \frac{1}{|\Lambda||\Lambda'|} \sum_{s \in \Omega^*} \| (Q \cap (\Lambda''+s) - \delta(\Lambda''+s) \widehat{\ } \|_\infty +$$

$$+ \frac{1}{|\Lambda||\Lambda'|} \sum_{s \in (\Lambda \setminus \Omega^*)} \| (Q \cap (\Lambda''+s) - \delta(\Lambda''+s) \widehat{\ } \|_\infty \le \alpha + \frac{\alpha|\Lambda'|}{|\Lambda||\Lambda'|} |\Lambda \setminus \Omega^*| \le 2\alpha\,.$$

and define the sets B'_s, G'_s, B', G' :

$$B'_s = \{ n \in \Lambda' + s \mid \| (Q \cap (\Lambda''+n) - \delta(\Lambda''+n) \widehat{\ } \|_\infty \ge \epsilon_1 |\Lambda''| \}\,,$$

$$G'_s = \{ n \in \Lambda' + s \mid \| (Q \cap (\Lambda''+n) - \delta(\Lambda''+n) \widehat{\ } \|_\infty < \epsilon_1 |\Lambda''| \}, \;\; s \in \Lambda\,.$$

$$B' = \{ s \in \Lambda \mid |B_s| \ge \epsilon_1 |\Lambda'| \} \quad \text{and} \quad G' = \{ s \in \Lambda \mid |B_s| < \epsilon_1 |\Lambda'| \}\,,$$

where $\epsilon_1 = \alpha^{1/4}$. After that we can apply the same arguments as above, using Lemma 2.3 instead of Lemma 2.4.

Let us prove 3). Since Q is (α, ε^2)–uniform subset of Λ, it follows that Q satisfies

(3.16). Also we have $|\Omega^*| \le \alpha|\Lambda|$, and $|B|, |B'| \le 4\alpha^{1/2}|\Lambda|$ (see above). It is easily shown that for all $s \notin B \cup B'$ the set $(Q-s) \cap \Lambda'$ is $(8\alpha^{1/4}, \varepsilon)$–uniform. This completes the proof.

In the same way we can prove

PROPOSITION 3.1. *Let Λ be a Bohr set, and $E \subseteq \Lambda$, $|Q| = \delta|\Lambda|$ be (α,ε)–uniform, $\varepsilon = \alpha/4(100d)$. Then*

$$\|(Q \cap \Lambda - \delta\Lambda)\widehat{\ }\|_\infty < 4\alpha|\Lambda| \,. \tag{3.20}$$

We will not, however, use this fact.

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, and $E_1 \subseteq \Lambda_1$, $E_2 \subseteq \Lambda_2$, $|E_1| = \beta_1|\Lambda_1|$, $|E_2| = \beta_2|\Lambda_2|$. By $\mathcal{P}$ denote the $E_1 \times E_2$. Let $A \subseteq \mathcal{P}$, $|A| = \delta|E_1||E_2|$. Denote by H and W two copies of the set A.

THEOREM 3.1. *Let $f : \mathcal{P} \to \mathbf{D}$ be a rectilinearly (α,ε)–uniform function. Suppose that sets E_1, E_2 are (α_0,ε)–uniform, $\alpha_0 = 2^{-50}\alpha^2\beta_1^{12}\beta_2^{12}$, $\varepsilon = 2^{-10}\varepsilon_0^2$, $\varepsilon_0 = (2^{-10}\alpha_0^2)/(100d)$. Let Λ_1 be an ε_0–attendant of Λ_2. Then either*

$$|\sum_{s_1,s_2,r} H(s_1,s_2)W(s_1+r,s_2+r)f(s_1,s_2+r)| \le 16\alpha^{1/4}\delta^{3/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2| \tag{3.21}$$

or there exists a Bohr set Λ', two sets F_1, F_2 and a vector $\vec{y} = (y_1,y_2) \in G \times G$, $F_1 \subseteq E_1 \cap (\Lambda' + y_1)$, $F_2 \subseteq E_2 \cap (\Lambda' + y_2)$ such that Λ' is an ε_0–attendant of Λ_1 and

$$|F_1| \ge 2^{-2}\beta_1|\Lambda'|, \quad |F_2| \ge 2^{-2}\beta_2|\Lambda'| \quad \textit{and} \tag{3.22}$$

$$\delta_{F_1\times F_2}(A) \ge 2\delta \,. \tag{3.23}$$

Proof. Let Λ' be an ε_0–attendant of Λ_1 to be chosen later.

Let

$$\Omega_1^{(1)} = \{ s \in \Lambda_1 \mid \|(E_1 \cap (\Lambda'+s) - \delta(\Lambda'+s))\widehat{\ }\|_\infty \ge \alpha_0 \} \,,$$

$$\Omega_2^{(1)} = \{ s \in \Lambda_1 \mid |\delta_{\Lambda'+s}(E_1) - \beta_1| \ge \alpha_0^{2/3} \} \,,$$

and

$$\Omega_1^{(2)} = \{ s \in \Lambda_2 \mid \|(E_2 \cap (\Lambda'+s) - \delta(\Lambda'+s))\widehat{\ }\|_\infty \ge \alpha_0 \} \,,$$

$$\Omega_2^{(2)} = \{ s \in \Lambda_2 \mid |\delta_{\Lambda'+s}(E_2) - \beta_2| \ge \alpha_0^{2/3} \} \,,$$

Let also $\Omega_1 = \Omega_1^{(1)} \cup \Omega_2^{(1)}$, and $\Omega_2 = \Omega_1^{(2)} \cup \Omega_2^{(2)}$. By assumption the sets E_1, E_2 are (α_0,ε)–uniform. Let Λ' be ε_0–attendant of Λ_1 such that (3.12) — (3.14) hold. Using definitions and Lemma 3.3, we get $|\Omega_l^{(1)}| \le \alpha_0^{2/3}|\Lambda_1|$, $|\Omega_l^{(2)}| \le \alpha_0^{2/3}|\Lambda_2|$, $l = 1,2$. Hence $|\Omega_1| \le 2\alpha_0^{2/3}|\Lambda_1|$ and $|\Omega_2| \le 2\alpha_0^{2/3}|\Lambda_2|$.

Let $g_i(\vec{s}) = g_i(k,m) = W(k,m)\Lambda'(k-i)$, $i \in \Lambda_1$, and $h_j(\vec{s}) = h_j(k,m) = H(k,m)\Lambda'(m-j)$, $j \in \Lambda_2$. We have $k \in \Lambda_1$, $m \in \Lambda_2$ and $k+r \in \Lambda_1$ in (3.21). It follows that the sum (3.21) does not exceed $|\Lambda_1|^2|\Lambda_2|$. Let also $\lambda_i = \Lambda' + i$, and

$\mu_j = \Lambda' + j$. Using Lemma 2.3, we get

$$\sigma_0 = \sum_{s_1,s_2,r} H(s_1,s_2) W(s_1+r, s_2+r) f(s_1, s_2+r) =$$

$$= \sum_{k,m} \sum_{r} H(k,m) W(k+r, m+r) f(k, m+r) \Lambda_1(k+r) \Lambda_2(m) =$$

$$\frac{1}{|\Lambda'|^2} \sum_{k,m} \sum_{r} H(k,m) W(k+r,m+r) f(k,m+r) (\Lambda_1 * \Lambda')(k+r) (\Lambda_2 * \Lambda')(m) + 16\vartheta_0 \kappa |\Lambda_1|^2 |\Lambda_2| =$$

$$\frac{1}{|\Lambda'|^2} \sum_{i\in\Lambda_1} \sum_{j\in\Lambda_2} \sum_{k,m} \sum_{r} h_j(k,m) g_i(k+r, m+r) f(k, m+r) + 16\vartheta_0 \kappa |\Lambda_1|^2 |\Lambda_2| \,, \quad (3.24)$$

where $|\vartheta_0| \le 1$ and $\kappa \le 2^{-10}\alpha_0^2$. Split the sum σ_0 as

$$\sigma_0 = \widetilde{\sigma}_0 + \sigma_0^{'} + \sigma_0^{''} + \sigma_0^{'''} + R \,, \quad (3.25)$$

The sum $\widetilde{\sigma}_0$ is taken over $i \notin \Omega_1, j \notin \Omega_2$, the sum $\sigma_0^{'}$ is taken over $i \in \Omega_1, j \notin \Omega_2$, the sum $\sigma_0^{''}$ is taken over $i \notin \Omega_1, j \in \Omega_2$, the sum $\sigma_0^{'''}$ is taken over $i \in \Omega_1, j \in \Omega_2$ and $|R| \le 16\varepsilon |\Lambda_1|^2 |\Lambda_2|$. Let us estimate $\sigma_0^{'}$, $\sigma_0^{''}$ and $\sigma_0^{'''}$. Rewrite σ_0 as

$$\sigma_0 = \frac{1}{|\Lambda'|^2} \sum_{i\in\Lambda_1} \sum_{j\in\Lambda_2} \sum_{k,m} \sum_{r} h_j(k-r, m) g_i(k, m+r) f(k-r, m+r) + R \,. \quad (3.26)$$

Let i and j in the sum (3.26) be fixed. We have $k \in \lambda_i$ and $m \in \mu_j$. Further if $f(k-r, m+r)$ is not zero, then $k - r \in \Lambda_1$. It follows that $r \in \lambda_i - \Lambda_1 = \Lambda' - \Lambda_1 + i$. The set Λ' is ε_0–attendant of Λ_1. Using Lemma 2.3, we obtain that r belongs to a set of cardinality at most $2|\Lambda_1|$. Hence

$$|\sigma_0^{'}| \le \frac{1}{|\Lambda'|^2} 2|\Omega_1| \cdot |\Lambda_2| \cdot |\Lambda'|^2 |\Lambda_1| \le 2\alpha_0^{2/3} |\Lambda_1|^2 |\Lambda_2| \,. \quad (3.27)$$

In the same way $|\sigma_0^{''}| \le 2\alpha_0^{2/3} |\Lambda_1|^2 |\Lambda_2|$ and $|\sigma_0^{'''}| \le 2\alpha_0^{2/3} |\Lambda_1|^2 |\Lambda_2|$.

Take i and j such that $i \notin \Omega_1$, $j \notin \Omega_2$. Let $g(\vec{s}) = g_i(\vec{s})$, $h(\vec{s}) = h_j(\vec{s})$, and $\Lambda_1 \times \mu_j = \Lambda_1^{(1)} \times \Lambda_2^{(1)}$, $\lambda_i \times \Lambda_2 = \Lambda_1^{(2)} \times \Lambda_2^{(2)}$. Let $E_2^{(1)} = E_2 \cap \Lambda_2^{(1)}$, $E_1^{(2)} = E_1 \cap \Lambda_1^{(2)}$, $\beta_2^{(1)} = |E_2^{(1)}| / |\Lambda_2^{(1)}|$, and $\beta_1^{(2)} = |E_1^{(2)}| / |\Lambda_1^{(2)}|$. We have

$$\sigma = \sigma_{i,j} = \sum_{s_1,s_2,r} h(s_1,s_2) g(s_1+r, s_2+r) f(s_1, s_2+r) = \quad (3.28)$$

$$= \sum_{k,m} h(k,m) E_2^{(1)}(m) \sum_{r} g(k+r, m+r) f(k, m+r) \quad (3.29)$$

Note that k in (3.29) belongs to $\Lambda_1^{(2)}$. Using the Cauchy–Schwartz inequality, we obtain

$$|\sigma|^2 \le \|h\|_2^2 \sum_{k,m} E_2^{(1)}(m) |\sum_{r} g(k+r, m+r) f(k, m+r)|^2 = \quad (3.30)$$

$$= \|h\|_2^2 \sum_{k,m} E_2^{(1)}(m) \sum_{r,p} g(k+r, m+r) f(k, m+r) g(k+p, m+p) f(k, m+p) =$$

$$= \|h\|_2^2 \sum_{k,m,u} g(k,m) g(k+u, m+u) \sum_{r} E_2^{(1)}(m-r) f(k-r, m) f(k-r, m+u) =$$

$$= \|h\|_2^2 \sum_{k,m,u} g(k,m)g(k+u,m+u)E_1^{(2)}(k)E_1^{(2)}(k+u)$$

$$\cdot \sum_r E_2^{(1)}(m-r)f(k-r,m)f(k-r,m+u)\,.$$

We have $k \in \Lambda_1^{(2)}$ and $k-r \in \Lambda_1$. It follows that $r \in k - \Lambda_1 \in \Lambda_1^{(2)} - \Lambda_1$. Since $m - r \in \Lambda_2^{(1)}$ it follows that $m \in \Lambda_2^{(1)} + r \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1$. On the other hand $k+u \in \Lambda_1^{(2)}$. Hence $u \in \Lambda_1^{(2)} - \Lambda_1^{(2)}$ and $m+u \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1 + \Lambda_1^{(2)} - \Lambda_1^{(2)}$. Let $\tilde{\Lambda}_i = \Lambda' + \Lambda' + \Lambda' + \Lambda' + \Lambda_1 + i$. Then $m, m+u \in \tilde{\Lambda}_i + j = Q_{ij} = Q$. Using Lemma 2.3 for the Bohr set Λ_1 and its ε_0–attendant Λ', we obtain that the cardinality of $\tilde{\Lambda}_i$ does not exceed $5|\Lambda_1|$. Using the Cauchy–Schwartz inequality, we get

$$|\sigma|^4 \le \|h\|_2^4 \left(\sum_k \sum_{m,u} g(k,m)g(k+u,m+u) \right) \tag{3.31}$$

$$\cdot \Big(\sum_{k,m,u} E_2^{(1)}(k)E_1^{(2)}(k+u) \sum_{r,r'} E_1^{(2)}(m-r)E_1^{(2)}(m-r') \times$$

$$f(k-r,m)f(k-r,m+u)f(k-r',m)f(k-r',m+u)\Big)\,.$$

Let $\sigma^* = \sigma^*_{ij} = \sum_k \sum_{m,u} g(k,m)g(k+u,m+u)$. Let

$$\Omega' = \{ s \in \Lambda_2 \mid |\delta_{\Lambda_1+s}(E_2) - \beta_2| \ge 4\alpha_0^{1/2} \text{ or}$$

$$\frac{1}{|\Lambda_1|} \sum_{n \in \Lambda_1 + s} |\delta_{\Lambda'+n}(E_2) - \beta_2|^2 \ge 4\alpha_0^{1/2} \},\ \text{and } G' = \Lambda_2 \setminus \Omega'\,.$$

By assumption Λ_1 is an ε_0–attendant of Λ_2 and E_2 is an (α_0, ε)–uniform subset of Λ_2. Using Lemma 3.3, we get $|\Omega'| \le 8\alpha_0^{1/2}|\Lambda_2|$. Let $\tilde{\Lambda} = \Lambda' + \Lambda' + \Lambda' + \Lambda' + \Lambda_1$. Since Λ' is an ε_0–attendant of Λ_1, it follows that for any $s \in G'$ we have $|\delta_{\tilde{\Lambda}+s}(E_2) - \beta_2| < 8\alpha_0^{1/2}$ and $\sum_{n \in \tilde{\Lambda}+s} |\delta_{\Lambda'+n}(E_2) - \beta_2|^2 < 8\alpha_0^{1/2}|\tilde{\Lambda}|$. For an arbitrary $i \in \Lambda_1$ consider the set

$$\Omega^* = \Omega^*_i = \{\ j \in \Lambda_2 \mid |\delta_{\tilde{\Lambda}_i + j}(E_2) - \beta_2| \ge 8\alpha_0^{1/2} \text{ or}$$

$$\frac{1}{|\tilde{\Lambda}_i|} \sum_{n \in \tilde{\Lambda}_i + j} |\delta_{\Lambda'+n}(E_2) - \beta_2|^2 \ge 8\alpha_0^{1/2}\ \}\,. \tag{3.32}$$

Since $(\Lambda_2 \setminus \Omega^*_i) \supseteq (\Lambda_2 \cap (G' - i))$ it follows that $\Omega^*_i \subseteq (\Lambda_2 \setminus (G' - i))$. Since Λ_1 is an ε_0–attendant of Λ_2, it follows that $|\Lambda_2 \setminus (G' - i)| = |(\Lambda_2 + i) \setminus G'| \ge |\Lambda_2^- \cap G'| \ge (1 - 8\alpha_0^{1/2} - 8\kappa_0)|\Lambda_2|$, $\kappa_0 \le \alpha_0^2$. Hence $|\Omega^*_i| \le 8\alpha_0^{1/2}|\Lambda_2| + 8\kappa_0|\Lambda_2| \le 16\alpha_0^{1/2}|\Lambda_2|$. This yields

$$\frac{1}{|\Lambda'|^2} \sum_{i \notin \Omega_1, j \in \Omega^*_i} |\sigma_{ij}| \le \frac{1}{|\Lambda'|^2} \sum_{i \notin \Omega_1} (16\alpha_0^{1/2}|\Lambda_2|2|\Lambda'|^2|\Lambda_1|) \le 32\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2|\,. \tag{3.33}$$

We have $j \notin \Omega_2$. Suppose in addition that $j \notin \Omega^*_i$. Let $\Omega'_2 = \Omega'_2(i) = \Omega_2 \cup \Omega^*_i$.

Lemma 3.4. *For any $i \notin \Omega_1$ and any $j \notin \Omega^*_i$ the following holds we have either*

$$|\sigma^*_{ij}| \le 16\delta\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_1|^2|\Lambda_2|\,. \tag{3.34}$$

or there exist two sets F_1, F_2 *and a vector* $\vec{y} = (y_1, y_2) \in G \times G$, $F_1 \subseteq E_1 \cap (\tilde{\Lambda} + y_1)$, $F_2 \subseteq E_2 \cap (\tilde{\Lambda} + y_2)$ *such that (3.22) and (3.23) hold.*

NOTE. Let T be a subset of G, $|T| = \delta |G|$, $E_1 = E_2 = G$, $\beta_1 = \beta_2 = 1$ and let g be the characteristic function of the set $\mathcal{A} = \bigsqcup_{x \in G} (\{x\} \times \{T + x\})$. Then it is easy to see that inequality (3.34) is best possible in the case (up to constants). On the other hand (3.22), (3.23) does not hold with A equals $\mathcal{A}$.

Proof. Let $\tilde{E}_2^{(2)} = E_2 \cap Q$ and $\overline{E}_2^{(2)}(x) = \tilde{E}_2^{(2)}(-x)$. We have

$$\sigma^*_{ij} = \sum_{k,m,u} g(k,m) g(k+u, m+u) \le \sum_{k,m,u} g(k,m) E_1^{(2)}(k+u) \tilde{E}_2^{(2)}(m+u)$$

$$= \sum_{k,m} g(k,m) (E_1^{(2)} * \overline{E}_2^{(2)})(k-m) = \sum_{k',m} g(k'+m, m) (E_1^{(2)} * \overline{E}_2^{(2)})(k') \,. \quad (3.35)$$

If k' is fixed then the variable m in (3.35) belongs to the set of the cardinality $|\Lambda'|$. Recall that $|Q_{ij}| \le 5|\Lambda_1|$. Lemma 2.3 implies that k' in the sum (3.35) belongs to a set of cardinality at most $8|\Lambda_1|$. Since $i \notin \Omega_1$, it follows that the set $E_1^{(2)}$ is α_0–uniform. Using Corollary 3.1, we get

$$\sigma^*_{ij} \le \beta_1^{(2)} \sum_{k',m} g(k'+m, m) (\lambda_i * \overline{E}_2^{(2)})(k') + 16 \alpha_0^{2/3} |\Lambda'|^2 |\Lambda_1| \,.$$

We have $j \notin \Omega^*_i$. Hence

$$\sigma^*_{ij} \le \beta_1^{(2)} \sum_{k',m} g(k'+m, m) (\lambda_i * E_2)(k') + 16 \alpha_0^{2/3} |\Lambda'|^2 |\Lambda_1|$$

$$\le \beta_1^{(2)} \beta_2 |\Lambda'| \sum_{k,m} g(k,m) + 32 \alpha_0^{1/6} |\Lambda'|^2 |\Lambda_1| \,. \quad (3.36)$$

Suppose that $\sigma^*_{ij} > 16 \delta \beta_1^2 \beta_2^2 |\Lambda'|^2 |\Lambda_1|$. Since $i \notin \Omega_1$, it follows that $\beta_1/2 \le \beta_1^{(2)} \le 2\beta_1$. Using this and (3.36), we get

$$\sum_{k,m} g(k,m) \ge 8 \beta_1^2 \beta_2^2 |\Lambda'| |\Lambda_1| \,. \quad (3.37)$$

Recall that m belongs to the set $\tilde{\Lambda}_i + j$ in (3.37). By Lemma 2.3, we find

$$\sum_{k,m} A(k,m) \Lambda'(k-i) \Lambda_1(m-i-j) \ge 4 \beta_1^2 \beta_2^2 |\Lambda'| |\Lambda_1| \,. \quad (3.38)$$

We have $i \notin \Omega_1$ and $j \notin \Omega^*_i$. Using this fact, inequality (3.38) and simple average arguments it is easy to see that there is a vector $\vec{y} = (y_1, y_2) \in G \times G$ and two sets $F_1 \subseteq E_1 \cap (\Lambda' + y_1)$, $F_2 \subseteq E_2 \cap (\Lambda' + y_2)$ such that (3.22), (3.23) hold. This completes the proof of the lemma. $\square$

We have

$$|\sigma|^4 \le \|h\|_2^4 \cdot \sigma^* \cdot \sum_{m,u} \sum_{r,r'} f(r,m) f(r,u) f(r',m) f(r',u) \;\cdot \quad (3.39)$$

$$\sum_k E_1^{(2)}(k) E_1^{(2)}(k-m+u) E_2^{(1)}(m-k+r) E_2^{(1)}(m-k+r') = \quad (3.40)$$

$$= \|h\|_2^4 \cdot \sigma^* \cdot \sum_{m,u} \sum_{r,r'} f(r,m)f(r,u)f(r',m)f(r',u) \ \cdot \tag{3.41}$$

$$\sum_k E_1^{(2)}(m-k)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r') = \|h\|_2^4 \cdot \sigma^* \cdot \sigma' \,. \tag{3.42}$$

Rewrite σ' as

$$\sigma' = \sum_k \sum_{r,r'} E_2^{(1)}(k+r)E_2^{(1)}(k+r')|\sum_m E_1^{(2)}(m-k)f(r,m)f(r',m)|^2 \tag{3.43}$$

We have $r \in \Lambda_1$ and $k+r \in \Lambda_2^{(1)}$. It follows that $k \in \Lambda_2^{(1)} - \Lambda_1$. On the other hand $m-k \in \Lambda_1^{(2)}$. Hence $m \in \Lambda_1^{(2)} + k \in \Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1$. By symmetry u belongs to $\Lambda_2^{(1)} + \Lambda_1^{(2)} - \Lambda_1$. Using Lemma 2.3 for Λ_1 and its ε_0–attendant Λ', we obtain that k and m,u belongs to some translations of *Bohr* sets $W_1 = \Lambda_1^+$ and $W_2 = W_1^+$, respectively, and the cardinalities of these sets do not exceed $3|\Lambda_1|$.

If k is fixed, then m,u,r,r' in (3.42) run some sets of the cardinalities at most $|\Lambda'|$.

Let $\Phi^1_{r,r'}(m) = f(r,-m)f(r',-m)W_2(m-i-j)$, $\Phi^2_{r,r'}(u) = f(r,-u)f(r',-u)W_2(u-i-j)$, $\Phi^3_{m,u}(r) = f(-r,m)f(-r,u)$, and $\Phi^4_{m,u}(r') = f(r',m)f(r',u)$. Consider the sets

$$B_1 = \{k \mid |(\Phi^1_{r,r'} * E_1^{(2)})(-k) - \beta_1^{(2)}(\Phi^1_{r,r'} * \Lambda_1^{(2)})(-k)| \ge \alpha_0^{2/3}|\Lambda'|\}$$

$$B_2 = \{k \mid |(\Phi^2_{r,r'} * E_1^{(2)})(-k) - \beta_1^{(2)}(\Phi^2_{r,r'} * \Lambda_1^{(2)})(-k)| \ge \alpha_0^{2/3}|\Lambda'|\}$$

$$B_3 = \{k \in \Lambda_1 \mid |(\Phi^3_{m,u} * E_2^{(1)})(k) - \beta_2^{(1)}(\Phi^3_{m,u} * \Lambda_2^{(1)})(k)| \ge \alpha_0^{2/3}|\Lambda'|\}$$

$$B_4 = \{k \in \Lambda_1 \mid |(\Phi^4_{m,u} * E_2^{(1)})(k) - \beta_2^{(1)}(\Phi^4_{m,u} * \Lambda_2^{(1)})(k)| \ge \alpha_0^{2/3}|\Lambda'|\} \,.$$

We have $i \notin \Omega_1$, $j \notin \Omega_2$. Using Corollary 3.1, we get $|B_1|,|B_2| \le 3\alpha_0^{2/3}|\Lambda_1|$ and $|B_3|,|B_4| \le \alpha_0^{2/3}|\Lambda_1|$. Let $B = B_1 \bigcup B_2 \bigcup B_3 \bigcup B_4$. Then $|B| \le 8\alpha_0^{2/3}|\Lambda_1|$. Split σ' as

$$\sigma' = \sum_{k\in B} \sum_{r,r'} E_2^{(1)}(k+r)E_2^{(1)}(k+r')|\sum_m E_1^{(2)}(m-k)f(r,m)f(r',m)|^2 +$$

$$+ \sum_{k\notin B} \sum_{r,r'} E_2^{(1)}(k+r)E_2^{(1)}(k+r')|\sum_m E_1^{(2)}(m-k)f(r,m)f(r',m)|^2 = \sigma_1 + \sigma_2$$

Let us estimate σ_1. Since $|B| \le 8\alpha_0^{2/3}|\Lambda_1|$, it follows that

$$|\sigma_1| \le 8\alpha_0^{2/3}|\Lambda'|^4|\Lambda_1| \,. \tag{3.44}$$

If $k \notin B$, then $k \notin B_1$. This implies that

$$\sigma_2 = \sum_{k\notin B} \sum_u \sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r') \ \cdot$$

$$\sum_m f(r,m)f(r',m)E_1^{(2)}(m-k) =$$

$$= \sum_{k\notin B} \sum_u \sum_{r,r'} f(r,u)f(r',u)E_1^{(2)}(u-k)E_2^{(1)}(k+r)E_2^{(1)}(k+r')(\Phi^1_{r,r'} * E_1^{(2)})(-k)$$

$$= \beta_1^{(2)} \sum_{k \notin B} \sum_u \sum_{r,r'} f(r,u) f(r',u) E_1^{(2)}(u-k) E_2^{(1)}(k+r) E_2^{(1)}(k+r') \cdot$$

$$\sum_m f(r,m) f(r',m) \Lambda_1^{(2)}(m-k) +$$

$$+ \vartheta \alpha_0^{2/3} |\Lambda'| \sum_{k \notin B} \sum_u \sum_{r,r'} f(r,u) f(r',u) E_1^{(2)}(u-k) E_2^{(1)}(k+r) E_2^{(1)}(k+r')$$

$$= \beta_1^{(2)} \sum_{k \notin B} \sum_u \sum_{r,r'} f(r,u) f(r',u) E_1^{(2)}(u-k) E_2^{(1)}(k+r) E_2^{(1)}(k+r') \cdot$$

$$\sum_m f(r,m) f(r',m) \Lambda_1^{(2)}(m-k) + 4\vartheta \alpha_0^{2/3} |\Lambda'|^4 |\Lambda_1| \,, \tag{3.45}$$

where $|\vartheta| \le 1$. Using these arguments for B_2, B_3 and B_4, we get

$$|\sigma_2| \le (\beta_1^{(2)})^2 (\beta_2^{(1)})^2 \sum_{m,u} \sum_{r,r'} f(r,m) f(r,u) f(r',m) f(r',u) \cdot$$

$$\sum_k \Lambda_1^{(2)}(m-k) \Lambda_1^{(2)}(u-k) \Lambda_2^{(1)}(k+r) \Lambda_2^{(1)}(k+r') + 16 \alpha_0^{2/3} |\Lambda'|^4 |\Lambda_1| \,, \tag{3.46}$$

It follows that

$$|\sigma'| \le |\sigma_1| + |\sigma_2| \le (\beta_1^{(2)})^2 (\beta_2^{(1)})^2 \sum_{m,u} \sum_{r,r'} f(r,m) f(r,u) f(r',m) f(r',u) \cdot$$

$$\sum_k \Lambda_1^{(2)}(m-k) \Lambda_1^{(2)}(u-k) \Lambda_2^{(1)}(k+r) \Lambda_2^{(1)}(k+r') + 32 \alpha_0^{2/3} |\Lambda'|^4 |\Lambda_1| \,. \tag{3.47}$$

Using (3.42), we obtain

$$|\sigma|^4 \le \|h\|_2^4 \cdot \sigma^* \cdot (\beta_1^{(2)})^2 (\beta_2^{(1)})^2 \sum_k \sum_{r,r'} \Lambda_2^{(1)}(k+r) \Lambda_2^{(1)}(k+r') \cdot$$

$$\Big| \sum_m \Lambda_1^{(2)}(m-k) f(r,m) f(r',m) \Big|^2 + 32 \|h\|_2^4 \cdot \sigma^* \cdot \alpha_0^{2/3} |\Lambda'|^4 |\Lambda_1| \tag{3.48}$$

Since $i \notin \Omega_1$, $j \notin \Omega_2$, it follows that $\beta_1^{(2)} \le 2\beta_1$ and $\beta_2^{(1)} \le 2\beta_2$. Whence

$$|\sigma_{ij}|^4 \le 2^4 \beta_1^2 \beta_2^2 \cdot \|h\|_2^4 \cdot \sigma^*_{ij} \cdot \sum_k \sum_{r,r'} \Lambda_2^{(1)}(k+r) \Lambda_2^{(1)}(k+r') \cdot$$

$$\Big| \sum_m \Lambda_1^{(2)}(m-k) f(r,m) f(r',m) \Big|^2 + 2^5 \alpha_0^{2/3} \cdot \|h\|_2^4 \cdot \sigma^*_{ij} \cdot |\Lambda'|^4 |\Lambda_1| \,. \tag{3.49}$$

Let $\alpha_{ij} = \sum_k \sum_{r,r'} \Lambda_2^{(1)}(k+r) \Lambda_2^{(1)}(k+r') | \sum_m \Lambda_1^{(2)}(m-k) f(r,m) f(r',m)|^2$.

Suppose that there are $i \notin \Omega_1, j \notin \Omega'_2(i)$ such that $\|h_j\|_2^2 \ge 8\delta\beta_1\beta_2 |\Lambda'||\Lambda_1|$. It follows that

$$\sum_{k,m} A(k,m) \Lambda'(m-j) \ge 8\delta\beta_1\beta_2 |\Lambda'||\Lambda_1| \,. \tag{3.50}$$

Let $F'_1 = E_1$, $F'_2 = E_2^{(1)}$. We have $j \notin \Omega'_2(i)$. Using this and (3.50), we get

$$|A \cap F'_1 \times F'_2| \ge 4\delta\beta_1\beta_2 |F'_1||F'_2|$$

and

$$|F_1'| = \beta_1|\Lambda_1|\,, \quad |F_2'| \ge 2^{-1}\beta_2|\Lambda'|\,.$$

Using simple average arguments it is easy to see that there are a vector $\vec{y} = (y_1, y_2) \in G \times G$ and two sets $F_1 \subseteq E_1 \cap (\Lambda' + y_1)$, $F_2 \subseteq E_2 \cap (\Lambda' + y_2)$ such that (3.22), (3.23) hold.

Using Lemma 3.4, we obtain

$$\sum_{i \notin \Omega_1, j \notin \Omega_2'(i)} |\sigma_{ij}| \le 8\delta^{3/4}\beta_1^{3/2}\beta_2^{3/2}|\Lambda'||\Lambda_1|^{3/4} \cdot \left(\sum_{i\in\Lambda_1, j\in\Lambda_2} \alpha_{ij} \right)^{1/4} (|\Lambda_1||\Lambda_2|)^{3/4} +$$

$$+4\alpha_0^{1/6}|\Lambda'|^2|\Lambda_1|^2|\Lambda_2|\,.$$

By assumption the function f is rectilineary (α,ε)–uniform. Clearly,

$$\sum_{i\in\Lambda_1, j\in\Lambda_2} \alpha_{ij} = \sum_{i\in\Lambda_1, j\in\Lambda_2} \sum_k \sum_{r,r'} \mu_j(k{+}r)\mu_j(k{+}r') \cdot \Big| \sum_m \lambda_i(m{-}k) f(r,m) f(r',m) \Big|^2 .$$

It follows that

$$\sum_{i \notin \Omega_1, j \notin \Omega_2'(i)} |\sigma_{ij}| \le 8\alpha^{1/4}\delta^{3/4}\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_1|^2|\Lambda_2| +$$

$$+ 4\alpha_0^{1/6}|\Lambda'|^2|\Lambda_1|^2|\Lambda_2|\,. \tag{3.51}$$

Using (3.25), (3.27), (3.33) and (3.51), we have

$$|\sigma_0| \le 16\kappa|\Lambda_1|^2|\Lambda_2| + 8\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2| + 32\alpha_0^{1/2}|\Lambda_1|^2|\Lambda_2| + 4\alpha_0^{1/6}|\Lambda'|^2|\Lambda_1|^2|\Lambda_2|$$

$$+8\alpha^{1/4}\delta^{3/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2| \le 16\alpha^{1/4}\delta^{3/4}\beta_1^2\beta_2^2|\Lambda_1|^2|\Lambda_2|$$

as required.

The next result is the main in this section.

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, $\Lambda_1 = \Lambda(S, \varepsilon_1)$, $S \subseteq \widehat{G}$ and let $E_1 \subseteq \Lambda_1$, $E_2 \subseteq \Lambda_2$, $|E_1| = \beta_1|\Lambda_1|$, $|E_2| = \beta_2|\Lambda_2|$. By $\mathcal{P}$ denote the product set $E_1 \times E_2$.

THEOREM 3.2. *Let A be an arbitrary subset of $E_1 \times E_2$ of cardinality $\delta|E_1||E_2|$. Suppose that the sets E_1, E_2 are $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform, $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$, $\varepsilon = (2^{-100}\alpha_0^2)/(100d)$. Let A be rectilineary $(\alpha, \alpha_1, \varepsilon)$–uniform, $\alpha = 2^{-100}\delta^9$, $\alpha_1 = 2^{-7}$, and*

$$\log N \ge 2^{10} d \log \frac{1}{\varepsilon_1 \varepsilon}\,. \tag{3.52}$$

Then either A contains a triple $\{(k,m), (k+d,m), (k,m+d)\}$, where $d \ne 0$ or there exists a Bohr set $\tilde{\Lambda}$, two sets F_1, F_2 and a vector $\vec{y} = (y_1, y_2) \in G \times G$, $F_1 \subseteq E_1 \cap (\tilde{\Lambda} + y_1)$, $F_2 \subseteq E_2 \cap (\tilde{\Lambda} + y_2)$ such that $\tilde{\Lambda}$ is an $2^{-4}\varepsilon^2$–attendant of Λ_1 and

$$|F_1| \ge 2^{-20}\beta_1|\tilde{\Lambda}|, \quad |F_2| \ge 2^{-20}\beta_2|\tilde{\Lambda}| \quad \textit{and} \tag{3.53}$$

$$\delta_{F_1 \times F_2}(A) \ge \frac{3}{2}\delta\,. \tag{3.54}$$

Proof. Let Λ' be an ε–attendant set of Λ_1 to be chosen later, and $\lambda_i = \Lambda' + i$, $i \in \Lambda_1$. Let $G_i = (\lambda_i \times \Lambda_2) \cap A$, $f_i(\vec{s}) = f(s_1 + i, s_2)\Lambda'(s_1, s_2)$, $i \in \Lambda_1$. By G_i denote the characteristic functions of the sets G_i. Let

$$B_1 = \{ i \in \Lambda_1 \mid E_1 \cap \lambda_i \text{ is not } (8\alpha_0^{1/4}, \varepsilon)\text{–uniform} \},$$

$$B_2 = \{ i \in \Lambda_1 \mid |\delta_{\lambda_i}(E_1) - \beta_1| \ge 4\alpha_0^{1/2} \},$$

$$B_3 = \{ i \in \Lambda_1 \mid \|f_i\|^4_{\Lambda' \times \Lambda_2, \varepsilon} > \alpha\beta_1^2\beta_2^2 |\Lambda'_\varepsilon|^4 |\Lambda'|^2 |\Lambda_2| \}, \text{ and } B = B_1 \cup B_2 \cup B_3 .$$

By assumption E_1 is (α_0, ε)–uniform. By Lemma 3.3, we get $|B_1| \le 8\alpha_0^{1/4}|\Lambda_1|$. and $|B_2| \le 8\alpha_0^{1/4}|\Lambda_1|$. Since A is rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform, it follows that $|B_3| \le \alpha_1|\Lambda_1|$. Hence $|B| \le 16\alpha_0^{1/4}|\Lambda_1| + \alpha_1|\Lambda_1| \le 2\alpha_1|\Lambda_1|$.

Using Lemma 2.3, we obtain

$$A(\vec{s}) = \frac{1}{|\Lambda'|} \cdot \sum_{i \in \Lambda_1} G_i(\vec{s}) + \epsilon(\vec{s}) , \qquad (3.55)$$

where $\|\epsilon\|_1 \le 2\kappa |\Lambda_1||\Lambda_2|$, $\kappa = \alpha_0^2$. Consider the sum

$$\sigma = \frac{1}{|\Lambda'|} \sum_{i \in \Lambda_1} \sum_{x,y} G_i(x + y, y) . \qquad (3.56)$$

We have $|A| = \delta\beta_1\beta_2|\Lambda_1||\Lambda_2|$. Using (3.55), we get

$$\sigma \ge \frac{7\delta\beta_1\beta_2}{8} |\Lambda_1||\Lambda_2| . \qquad (3.57)$$

Split σ as

$$\sigma = \frac{1}{|\Lambda'|} \sum_{i \in B} \sum_{x,y} G_i(x + y, y) + \frac{1}{|\Lambda'|} \sum_{i \notin B} \sum_{x,y} G_i(x + y, y) = \sigma_1 + \sigma_2 . \qquad (3.58)$$

Let us estimate σ_1. We have

$$\sigma_1 = \frac{1}{|\Lambda'|} \sum_{i \in B_3 \setminus (B_1 \cup B_2)} \sum_{x,y} G_i(x + y, y) + \frac{1}{|\Lambda'|} \sum_{i \in B_1 \cup B_2} \sum_{x,y} G_i(x + y, y) \le \qquad (3.59)$$

$$\le \frac{1}{|\Lambda'|} \sum_{i \in B_3 \setminus (B_1 \cup B_2)} \sum_{x,y} G_i(x + y, y) + 16\alpha_0^{1/4} |\Lambda_1||\Lambda_2| . \qquad (3.60)$$

Suppose that there exists $i \notin B_1 \cup B_2$ such that

$$\sum_{x,y} G_i(x + y, y) \ge 4\delta\beta_1\beta_2|\Lambda'||\Lambda_2| .$$

In other words

$$\sum_{x,y} G_i(x, y) \ge 4\delta\beta_1\beta_2|\Lambda'||\Lambda_2| .$$

Put $y_1 = i$, $y_2 = 0$ and $F_1 = (\Lambda' + i) \cap E_1$. Since $i \notin B_2$, it follows that $|F_1| \ge \beta_1|\Lambda'|/2$. Using simple average arguments we see that there exists an element a such that $F_2 = (\Lambda' + a) \cap E_2$ has the cardinality at least $\beta_2|\tilde{\Lambda}_1|/2$ and for $\vec{y} = (i, a)$ we have

$$|A \cap (F_1 \times F_2)| > 2\delta |F_1||F_2| .$$

Thus we get (3.53), (3.54) and the theorem is proven in the case.

We have $\alpha_1 = 2^{-7}$. Using $|B_3| \le \alpha_1|\Lambda_1|$ and $\alpha_0^{1/4} \le 2^{-4}\alpha_1\beta_1\beta_2$, we obtain

$$\sigma_1 \le 4\delta\beta_1\beta_2|\Lambda'||B_3||\Lambda_2| + 16\alpha_0^{1/4}|\Lambda_1||\Lambda_2| \le 2^{-3}\delta\beta_1\beta_2|\Lambda'||\Lambda_1||\Lambda_2|\,. \tag{3.61}$$

Using this and (3.57), (3.58), we obtain

$$\frac{1}{|\Lambda'|}\sum_{i\notin B}\sum_{x,y} G_i(x+y,y) \ge \frac{3\delta\beta_1\beta_2}{4}|\Lambda_1||\Lambda_2|\,. \tag{3.62}$$

The formula (3.62) implies that there exists $i_0 \notin B$ such that

$$\sum_{x,y} G_{i_0}(x+y,y) \ge \frac{3}{4}\delta\beta_1\beta_2|\Lambda'||\Lambda_2|\,. \tag{3.63}$$

Let $G'(\vec{s}) = G_{i_0}(\vec{s})$. We have

$$\sum_{k}\sum_{m} G'(k+m,m) \ge 2^{-3}\delta\beta_1\beta_2|\Lambda'||\Lambda_2|\,. \tag{3.64}$$

We have $m \in \Lambda_2$ and $k+m \in \lambda_i$. It follows that $k \in \lambda_i - \Lambda_2$. Using Lemma 2.3 we obtain that k belongs to a set of cardinality at most $2|\Lambda_2|$. By the Cauchy–Schwartz inequality, we get

$$2^{-6}\delta^2\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|^2 \le \sum_k \Big(\sum_m G'(k+m,m)\Big)^2 \cdot 2|\Lambda_2|\,. \tag{3.65}$$

It follows that

$$\sum_k \Big(\sum_m G'(k+m,m)\Big)^2 = \sum_k\sum_{m,p} G'(k+m,m)G'(k+p,p) \ge$$
$$\ge 2^{-7}\delta^2\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|\,. \tag{3.66}$$

Consider the sum

$$\sigma_0 = \sum_{s_1,s_2,r} G'(s_1,s_2)G'(s_1+r,s_2+r)A(s_1,s_2+r)\,. \tag{3.67}$$

We have

$$G'(s_1,s_2)G'(s_1+r,s_2+r)f(s_1,s_2+r) =$$
$$= G'(s_1,s_2)G'(s_1+r,s_2+r)f_{i_0}(s_1,s_2+r)\,, \tag{3.68}$$

where f_{i_0} is the restriction of the function f to G'. It follows that

$$\sigma_0 = \delta\sum_{s_1,s_2,r} G'(s_1,s_2)G'(s_1+r,s_2+r)\mathcal{P}(s_1,s_2+r)+$$
$$+\sum_{s_1,s_2,r} G'(s_1,s_2)G'(s_1+r,s_2+r)f(s_1,s_2+r) =$$
$$= \delta\sum_{s_1,s_2,r} G'(s_1,s_2)G'(s_1+r,s_2+r)+\sum_{s_1,s_2,r} G'(s_1,s_2)G'(s_1+r,s_2+r)f_{i_0}(s_1,s_2+r)\,. \tag{3.69}$$

The inequality (3.66) implies that the first term in (3.69) is greater than $2^{-7}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$. Since $i_0 \notin B$, it follows that $\|f_{i_0}\|^4 \le \alpha\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$ and $\delta_{\lambda_{i_0}}(E_1) \le 2\beta_1$. By assumption $\alpha = 2^{-100}\delta^9$. Using Theorem 3.1 and (3.63), we obtain that either the second term in (3.69) does not exceed

$$2^{10}\alpha^{1/4}\delta^{3/4}\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2| \le 2^{-8}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$$

or there is a vector $\vec{y} = (y_1, y_2) \in G \times G$ and two sets $F_1 \subseteq E_1 \cap (\tilde{\tilde{\Lambda}} + y_1)$, $F_2 \subseteq E_2 \cap (\tilde{\tilde{\Lambda}} + y_2)$ such that (3.53), (3.54) hold. If we have the second situation then we are done and $\tilde{\tilde{\Lambda}}$ is an ε–attendant of Λ'. In the other case $\sigma_0 \ge 2^{-7}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2|$.

The sum (3.67) is the number of triples $\{(k,m),(k+d,m),(k,m+d)\}$, where $k \in \Lambda_{i_0}$, $m \in \Lambda_2$, $d \in G$. The number of triples with $d = 0$ does not exceed $|\Lambda'||\Lambda_2|$. By assumption $\log N \ge 2^{10} d \log \frac{1}{\varepsilon_1 \varepsilon}$. Using Lemma 2.1, we get $|\Lambda'| > 2^8(\delta^3\beta_1^2\beta_2^2)^{-1}$. Hence, $2^{-8}\delta^3\beta_1^2\beta_2^2|\Lambda'|^2|\Lambda_2| > |\Lambda'||\Lambda_2|$. It follows that A contains a triple $\{(k,m),(k+d,m),(k,m+d)\}$ with $d \ne 0$. This completes the proof.

4. *Non–uniform case.*

LEMMA 4.1. *Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, and Λ' be an ε–attendant set of Λ_1, $\varepsilon = \kappa/(100d)$. Let set A be a subset of $C \subseteq \Lambda_1 \times \Lambda_2$ of cardinality $\delta|C|$. By B define the set of $s \in \Lambda_1$ such that $|A \cap ((\Lambda'+s)\times\Lambda_2)| < (\delta-\eta)|C \cap ((\Lambda'+s)\times\Lambda_2)|$, where $\eta > 0$. Then*

$$\sum_{s \in (\Lambda_1 \setminus B)} |A \cap ((\Lambda'+s)\times\Lambda_2)| \ge \delta \sum_{s \in (\Lambda_1 \setminus B)} |C \cap ((\Lambda'+s)\times\Lambda_2)| +$$

$$+\eta \sum_{s \in B} |C \cap ((\Lambda'+s)\times\Lambda_2)| - 4\kappa|\Lambda'||\Lambda_1||\Lambda_2|\,.$$

Proof. Using Lemma 2.3, we get

$$\delta|C| = \sum_{\vec{s}} A(\vec{s})\Lambda_1(k)\Lambda_2(m) = \frac{1}{|\Lambda'|}\sum_{n\in\Lambda_1}\sum_{\vec{s}} A(\vec{s})((\Lambda'+n)\times\Lambda_2)(\vec{s}) + 2\vartheta\kappa|\Lambda_1||\Lambda_2|\,, \tag{4.1}$$

where $|\vartheta| \le 1$. Split the sum (4.1) into a sum over $n \in B$ and a sum over $n \in \Lambda_1 \setminus B$. We have

$$\delta|C| < \frac{1}{|\Lambda'|}(\delta-\eta)\sum_{n\in B}|C \cap ((\Lambda'+n)\times\Lambda_2)| +$$

$$+\frac{1}{|\Lambda'|}\sum_{n\in(\Lambda_1\setminus B)}|A \cap ((\Lambda'+n)\times\Lambda_2)| + 2\kappa|\Lambda_1||\Lambda_2|\,. \tag{4.2}$$

In the same way

$$|C| = \frac{1}{|\Lambda'|}\sum_{n\in B}|C\cap((\Lambda'+n)\times\Lambda_2)| + \frac{1}{|\Lambda'|}\sum_{n\in(\Lambda_1\setminus B)}|C\cap((\Lambda'+n)\times\Lambda_2)| + 2\vartheta_1\kappa|\Lambda_1||\Lambda_2|\,, \tag{4.3}$$

where $|\vartheta_1| \le 1$. Combining (4.2) and (4.3), we obtain the required result.

Let X be a finite set, μ be a measure on X and let $Z : X \to \mathbf{R}$ be a function. By $\mathbf{E}Z$ denote the sum $\frac{1}{|X|}\sum_{x\in X} Z(x)$. The following lemma is well–known (see e.g. [**18**]).

LEMMA 4.2. *Let p be a real number. Suppose that $Z : X \to [-1,1]$ is a function* *such that* $\mathbf{E}Z = 0$ *and* $\mathbf{E}|Z|^p = \sigma^p$. *Then*

$$\mu\left\{x \in X \ : \ Z > \frac{\sigma^p}{5}\right\} \ge \frac{\sigma^p}{5}\,. \tag{4.4}$$

Proof. Suppose that (4.4) does not hold. Since $\mathbf{E}Z = 0$ it follows that

$$-\mathbf{E}Z\mathbf{1}_{\{Z<0\}} = \mathbf{E}Z\mathbf{1}_{\{Z>0\}} \le \mu\{x \ : \ Z > 5^{-1}\sigma^p\} + \mathbf{E}Z\mathbf{1}_{\{0<Z\le 5^{-1}\sigma^p\}} \le \frac{2}{5}\sigma^p\,,$$

where $\mathbf{1}_{\{Z<0\}}$, $\mathbf{1}_{\{Z>0\}}$ are the characteristics functions of the sets $\{x \ : \ Z(x) < 0\}$, $\{x \ : \ Z(x) > 0\}$ respectively. We have $|Z(x)| \le 1$ for all $x \in X$. Hence

$$\sigma^p = \mathbf{E}|Z|^p = \mathbf{E}|Z|^p\mathbf{1}_{\{Z<0\}} + \mathbf{E}|Z|^p\mathbf{1}_{\{Z>0\}} \le 2\mathbf{E}Z\mathbf{1}_{\{Z>0\}} \le \frac{4}{5}\sigma^p \tag{4.5}$$

with contradiction. □

We need in the proposition concerning the properties of not rectilinearly α–uniform sets. The similar proposition was proven in [**24**, **26**, **15**, **18**].

PROPOSITION 4.1. *Let A be a subset of $E_1 \times E_2$ of cardinality $|A| = \delta|E_1||E_2|$. Suppose that $\alpha > 0$ is a real number, $\alpha \le \delta^4/8$, and A is not rectilinearly α–uniform. Then there are two sets $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$ such that*

$$|A\bigcap(F_1 \times F_2)| > (\delta + 2^{-15}\cdot\min\{\alpha^2\delta^{-5}, \alpha\delta^{-2}\})|F_1||F_2| \quad \textit{and} \tag{4.6}$$

$$|F_1| \ge 2^{-15}\min\{\alpha^2\delta^{-5}, \alpha\delta^{-2}\}\cdot|E_1|\,, \quad |F_2| \ge 2^{-15}\min\{\alpha^2\delta^{-5}, \alpha\delta^{-2}\}\cdot|E_2|\,. \tag{4.7}$$

Proof. Denote by f the balanced function of A. Suppose that

$$\sum_x |\sum_y f(x,y)|^2 \le \alpha\delta^{-2}|E_1||E_2|^2/16 \tag{4.8}$$

and

$$\sum_y |\sum_x f(x,y)|^2 \le \alpha\delta^{-2}|E_1|^2|E_2|/16\,. \tag{4.9}$$

If (4.8) or (4.9) is not true then we can use Lemma 4.2 and find two sets F_1, F_2 such that (4.6), (4.7) hold. Let us prove that

$$\|A\|^4 \ge (\delta^4 + \alpha/2)|E_1|^2|E_2|^2\,. \tag{4.10}$$

By assumption $\|f\|^4 \ge \alpha|E_1|^2|E_2|^2$. Using the obvious formulas $A = f + \delta(E_1 \times E_2)$ and $\sum_{x,y} f(x,y) = 0$, we get

$$\|A\|^4 \ge (\delta^4 + \alpha)|E_1|^2|E_2|^2 + \tag{4.11}$$

$$+\,\delta\sum_{x,x',y,y'} f(x,y)f(x',y)f(x,y') + \delta\sum_{x,x',y,y'} f(x,y)f(x',y)f(x',y') + \tag{4.12}$$

$$+\,\delta\sum_{x,x',y,y'} f(x,y)f(x,y')f(x',y') + \delta\sum_{x,x',y,y'} f(x',y)f(x,y')f(x',y') + \tag{4.13}$$

$$+\,\delta^2 \sum_{x,x',y,y'} f(x,y)f(x',y') + \delta^2 \sum_{x,x',y,y'} f(x',y)f(x,y') + \tag{4.14}$$

$$+\,\delta^2 \sum_{x,x',y,y'} f(x,y)f(x',y)E_2(y') + \delta^2 \sum_{x,x',y,y'} f(x,y)f(x,y')E_1(x') + \tag{4.15}$$

$$+\,\delta^2 \sum_{x,x',y,y'} f(x',y)f(x',y')E_1(x) + \delta^2 \sum_{x,x',y,y'} f(x,y')f(x',y')E_2(y)\,. \tag{4.16}$$

It is easy to see that two summands in (4.14) equal zero. Using (4.8) and (4.9), we see that the sum of four terms in (4.15) — (4.16) does not exceed $\alpha|E_1|^2|E_2|^2/4$. Let us prove that any term in (4.12) — (4.13) at most $\alpha/(16\delta)$. Without loss of generality it can be assumed that the first summand in (4.12) is greater than $\alpha/(16\delta)$. We have

$$\frac{\alpha}{16\delta} \le \left(\sum_{x,y} |f(x,y)|^3 \right)^{1/3} \cdot \left(\sum_{x,y} |\sum_{x'} f(x',y)|^{3/2} \cdot |\sum_{y'} f(x,y')|^{3/2} \right)^{2/3}$$

$$\le 2\delta^{1/3} \cdot \left(\sum_{y} |\sum_{x'} f(x',y)|^{3/2} \cdot \sum_{x} |\sum_{y'} f(x,y')|^{3/2} \right)^{2/3}$$

Thus, we have for example

$$\sum_{y} |\sum_{x'} f(x',y)|^{3/2} \ge \frac{\alpha^{3/4}}{16\delta} \ge \frac{\alpha^2}{16\delta^5}\,.$$

Using Lemma 4.2 and find two sets F_1, F_2 such that (4.6), (4.7) hold. So any term in (4.12) — (4.13) does not exceed $\alpha/(16\delta)$ and we have proved (4.10).

Let $e(x,y) = \{(\overline{x},\overline{y}) \in A \mid (\overline{x},y) \in A \text{ and } (x,\overline{y}) \in A\}$ and $N_x = \{y \mid (x,y) \in A\}$, $N_y = \{x \mid (x,y) \in A\}$. Clearly,

$$\|A\|^4 = \sum_{(x,y)\in A} e(x,y)\,. \tag{4.17}$$

Let $\tilde{X} = \{x \in E_1 \,:\, |\sum_y f(x,y)| \le \alpha|E_2|/(32\delta^3)\}$ and $\tilde{Y} = \{y \in E_2 \,:\, |\sum_x f(x,y)| \le \alpha|E_1|/(32\delta^3)\}$. Let also $X^c = E_1 \setminus \tilde{X}$ and $Y^c = E_2 \setminus \tilde{Y}$. Note that $|X^c| \le \zeta|E_1|$, $|Y^c| \le \zeta|E_2|$, where $\zeta = \alpha/(128\delta^2)$. Indeed, if $|X^c| > \zeta|E_1|$ then $\sum_x |\sum_y f(x,y)| > \alpha^2|E_1||E_2|/(2^{12}\delta^5)$. Using Lemma 4.2 and find two sets F_1, F_2 such that (4.6), (4.7) hold.

Let us prove that

$$\sum_{x\in\tilde{X},y\in\tilde{Y}} A(x,y)e(x,y) \ge (\delta^4 + \alpha/4)|E_1|^2|E_2|^2\,. \tag{4.18}$$

We have

$$\|A\|^4 = \sum_{x\in\tilde{X},y\in\tilde{Y},x',y'} A(x,y)A(x',y)A(x,y')A(x',y') +$$

$$+ \sum_{x\in\tilde{X},y\in Y^c,x',y'} A(x,y)A(x',y)A(x,y')A(x',y') +$$

$$+ \sum_{x\in X^c, y\in \tilde{Y}, x', y'} A(x,y)A(x',y)A(x,y')A(x',y') +$$

$$+ \sum_{x\in X^c, y\in Y^c, x', y'} A(x,y)A(x',y)A(x,y')A(x',y') = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 \,.$$

Clearly, $\sigma_3 \le |X^c||Y^c| \cdot \delta |E_1||E_2| \le \alpha |E_1|^2 |E_2|^2/16$. Further,

$$\sigma_1 \le |Y^c| \sum_{x,x',y'} A(x,y')A(x',y') = |Y^c| \sum_{y'\in\tilde{Y}} |\sum_x A(x,y')|^2 + |Y^c| \sum_{y'\in Y^c} |\sum_x A(x,y')|^2$$

$$\le 4\delta^2 |Y^c||E_1|^2|E_2| + |Y^c|^2 |E_1|^2 \le \alpha |E_1|^2 |E_2|^2/16 \,.$$

In the same way $\sigma_2 \le \alpha |E_1|^2|E_2|^2/16$. Using (4.10) and (4.17), we get (4.18).

By (4.18), we find $(x_0, y_0) \in A \cap (\tilde{X} \times \tilde{Y})$ such that

$$e(x_0,y_0) \ge (\delta^3 + \frac{\alpha}{4\delta})|E_1||E_2| \,. \tag{4.19}$$

Put $F_1 = N_{y_0}$, $F_2 = N_{x_0}$. By definition of $\tilde{X}, \tilde{Y}$, we get $\big||N_x| - \delta|E_2|\big| \le \alpha|E_2|/(32\delta^3)$ and $\big||N_y| - \delta|E_1|\big| \le \alpha|E_1|/(32\delta^3)$. In particular $|F_1|, |F_2| \ge \delta/2$ and (4.6) holds. Obviously, $e(x,y) = |(N_y \times N_x) \cap A|$. Using (4.19) and $\alpha \le \delta^4/8$, we obtain

$$|A \bigcap (F_1 \times F_2)| \ge (\delta + \frac{\alpha}{4\delta^3}) \left(1 + \frac{\alpha}{32\delta^4}\right)^{-2} |F_1||F_2| \ge$$

$$\ge (\delta + \frac{\alpha}{4\delta^3})(1 - \frac{\alpha}{16\delta^4})|F_1||F_2| \ge (\delta + \frac{\alpha}{8\delta^3})|F_1||F_2| \,.$$

and we get (4.7). This concludes the proof.

Let Λ_1, Λ_2 be Bohr sets, $\Lambda_1 \le \Lambda_2$, $\Lambda_1 = \Lambda(S, \varepsilon_0)$, $|S| = d$, and $E_1 \subseteq \Lambda_1$, $E_2 \subseteq \Lambda_2$, $|E_1| = \beta_1 |\Lambda_1|$, $|E_2| = \beta_2 |E_2|$. Let $\mathcal{P}$ be a product set $E_1 \times E_2$.

THEOREM 4.1. *Let A be a subset of $\mathcal{P}$ of cardinality $|A| = \delta |E_1||E_2|$. Suppose that A has no triples $\{(k,m), (k+d,m), (k,m+d)\}$ with $d \ne 0$, E_1, E_2 are $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform, $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$, $\varepsilon = (2^{-100}\alpha_0^2)/(100d)$, $\varepsilon' = 2^{-10}\varepsilon^2$, and*

$$\log N \ge 2^{10} d \log \frac{1}{\varepsilon_0 \varepsilon} \,.$$

Then there exists a Bohr set $\tilde{\Lambda}$, two sets F_1, F_2 and a vector $\vec{y} = (y_1, y_2) \in G \times G$, $F_1 \subseteq E_1 \cap (\tilde{\Lambda} + y_1)$, $F_2 \subseteq E_2 \cap (\tilde{\Lambda} + y_2)$ such that

$$|F_1| \ge 2^{-500}\delta^{22}\beta_1|\tilde{\Lambda}|, \quad |F_2| \ge 2^{-500}\delta^{22}\beta_2|\tilde{\Lambda}| \quad \textit{and} \tag{4.20}$$

$$\delta_{F_1 \times F_2}(A) \ge \delta + 2^{-500}\delta^{22} \,. \tag{4.21}$$

Besides that for $\tilde{\Lambda} = \Lambda(\tilde{S}, \tilde{\varepsilon})$ we have $\tilde{S} = S$ and $\tilde{\varepsilon} \ge 2^{-5}\varepsilon'\varepsilon_0$.

Proof. Let Λ' be an ε–attendant of Λ_1, and Λ'' be an ε–attendant of Λ' to be chosen later. Suppose that A is rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform, $\alpha = 2^{-100}\delta^9$, $\alpha_1 = 2^{-7}$. Using Theorem 3.2, we obtain that either A contains a triple $\{(k,m), (k+d,m), (k,m+d)\}$ with $d \ne 0$ or (4.20), (4.21) hold. At the first case we get a contradiction, at the second case we obtain the required result. Hence the set A is not rectilinearly $(\alpha, \alpha_1, \varepsilon)$–uniform.

Let

$$B_1 = \{ s \in \Lambda_1 ~|~ |\delta_{\Lambda'+s}(E_1) - \beta_1| \ge 4\alpha_0^{1/2} \} \,,$$

$$B_2 = \{ s \in \Lambda_1 ~|~ \Lambda' \cap (E_1 - s) \mbox{ is not } (8\alpha_0^{1/4}, \varepsilon)\mbox{–uniform} \} \,,$$

and

$$B = \{ i \in \Lambda_1 ~|~ \| f_i \|^4_{\Lambda' \times \Lambda_2, \varepsilon} > \alpha \beta_1^2 \beta_2^2 |\Lambda'_\varepsilon|^4 |\Lambda'|^2 |\Lambda_2| \} \,.$$

Since A is not rectilinearly $(\alpha, \alpha_1, \varepsilon')$–uniform, it follows that $|B| > \alpha_1 |\Lambda_1|$. By assumption E_1, E_2 are (α_0, ε')–uniform. Using Lemma 3.3, we obtain $|B_1| \le 4\alpha_0^{1/2} |\Lambda_1|$, $|B_2| \le 8\alpha_0^{1/2} |\Lambda_1|$. Let $B_3 = B_1 \cup B_2$. Then $|B_3| \le 12\alpha_0^{1/2} |\Lambda_1|$. Let $B' = B \setminus B_3$. Since $32\alpha_0^{1/2} < \alpha_1$, it follows that $|B'| \ge \alpha_1 |\Lambda_1|/2$. Note that for all $l \in B'$ we have

$$|\delta_{\Lambda'+s}(E_1) - \beta_1| < 4\alpha_0^{1/2} \,. \tag{4.22}$$

Let $\eta = 2^{-100} \alpha^{3/2}$. Let $\lambda_l = \Lambda' + l$, $l \in \Lambda_1$. Suppose that for any $l \in B'$ we have

$$|A \cap (\lambda_l \times \Lambda_2)| \le (\delta - \eta) |\lambda_l \cap E_1| |\Lambda_2 \cap E_2| \,. \tag{4.23}$$

Let $B'^c = \Lambda_1 \setminus B'$. Using Lemma 4.1 and (4.22), we get

$$\sum_{l \in B'^c} |A \cap (\lambda_l \times \Lambda_2)| \ge \delta |\Lambda_2 \cap E_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + \eta |\Lambda_2 \cap E_2| \sum_{l \in B'} |\lambda_l \cap E_1| - \alpha_0^2 |\Lambda'| |\Lambda_1| |\Lambda_2|$$

$$\ge \delta \beta_2 |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + \eta \frac{\alpha_1 |\Lambda_1|}{2} \frac{\beta_1 |\Lambda'|}{4} \beta_2 |\Lambda_2| \ge$$

$$\ge \delta \beta_2 |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + 2^{-3} \alpha_1 \eta \beta_1 \beta_2 |\Lambda'| |\Lambda_1| |\Lambda_2| \,. \tag{4.24}$$

We have

$$\sum_{l \in B_1} |A \cap (\lambda_l \times \Lambda_2)| \le 4\alpha_0^{1/2} |\Lambda_1| |\Lambda'| |\Lambda_2| \le 2^{-4} \alpha_1 \eta \beta_1 \beta_2 |\Lambda'| |\Lambda_1| |\Lambda_2| \,. \tag{4.25}$$

Combining (4.24) and (4.25), we obtain

$$\sum_{l \in (B'^c \setminus B_1)} |A \cap (\lambda_l \times \Lambda_2)| \ge \delta \beta_1 |\Lambda_2| \sum_{l \in B'^c} |\lambda_l \cap E_1| + 2^{-4} \alpha_1 \eta \beta_1 \beta_2 |\Lambda'| |\Lambda_1| |\Lambda_2| \,. \tag{4.26}$$

This implies that, there exists a number $l \in B'^c \setminus B_1$ such that

$$|A \cap (\lambda_l \times \Lambda_2)| > (\delta + 2^{-5} \alpha_1 \eta) |\lambda_l \cap E_1| |\Lambda_2 \cap E_2| \,. \tag{4.27}$$

Put $\tilde{\Lambda} = \Lambda'$, $y_1 = l_0$ and $F_1 = (\tilde{\Lambda} + l_0) \cap E_1$. Since $l_0 \notin B_1$, it follows that $|F_1| \ge \beta_1 |\tilde{\Lambda}|/2$. The set E_2 is $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform. This yields that there exists a number a such that $F_2 = (\tilde{\Lambda} + a) \cap E_2$ has the cardinality at least $\beta_2 |\tilde{\Lambda}_1|/2$ and for $\vec{y} = (l_0, a)$ we have

$$|A \cap (\tilde{\Lambda} + \vec{y})| > (\delta + 2^{-6} \alpha_1 \eta) |F_1| |F_2| \,.$$

and the theorem is proven.

Let $f(\vec{x})$ be the balanced function of A. There exists $l_0 \in B'$ such that

$$|A \cap (\lambda_{l_0} \times \Lambda_2)| > (\delta - \eta) |\lambda_{l_0} \cap E_1| |\Lambda_2 \cap E_2| \,.$$

If

$$|A \cap (\lambda_{l_0} \times \Lambda_2)| \ge (\delta + \eta) |\lambda_{l_0} \cap E_1| |\Lambda_2 \cap E_2| \,, \tag{4.28}$$

then the theorem is proven.

Hence there exists $l_0 \in B'$ such that

$$|\sum_{r,m} f(r,m)\lambda_{l_0}(r)\Lambda_2(m)| < \eta|\lambda_{l_0} \cap E_1||\Lambda_2 \cap E_2|\,. \qquad (4.29)$$

Let $\Lambda_0 = \Lambda' + l_0$. Put $\nu_i = \Lambda'' + i$, $i \in \Lambda_0$ and $\mu_j = \Lambda'' + j$, $j \in \Lambda_2$. Consider the sum

$$\sigma^* = \sum_{i\in\Lambda_0}\sum_{j\in\Lambda_2}\sum_{k}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r)\,. \qquad (4.30)$$

Suppose that i and j are fixed in the sum (4.30). Using Lemma 2.3, we obtain that k runs a set of cardinality at most $2|\Lambda_0|$. Besides that if i, j, k are fixed, then m, r run sets of size at most $|\Lambda''|$. Using Lemma 2.3 once again, we obtain

$$\sigma^* = |\Lambda''|^2 \sum_{k}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k)\Lambda_2(k+r) + \vartheta\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|\,, \quad (4.31)$$

where $|\vartheta| \le 1$. Let $\Lambda_3 = \Lambda_2 - \Lambda' - l_0$. Using Lemma 2.3, we get $|\Lambda_2| \le |\Lambda_3| \le (1+\alpha_0^2)|\Lambda_2|$. Note that k belongs to the set Λ_3 in (4.31). If $k \in \Lambda_2^- - l_0$, then $\Lambda_2(k+r) = 1$, for all $r \in \Lambda_0$. If k is fixed in (4.31), then r and m run sets of cardinality at most $|\Lambda_0|$. It follows that

$$\frac{\sigma^*}{|\Lambda''|^2} = \sum_{k\in(\Lambda_2^- - l_0)}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k) +$$

$$+ \sum_{k\in(\Lambda_3\setminus(\Lambda_2^- - l_0))}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k)\Lambda_2(k+r) =$$

$$= \sum_{k\in(\Lambda_2^- - l_0)}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k) + \alpha_0^2\vartheta_1|\Lambda_0|^2|\Lambda_2| =$$

$$\sum_{k}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\Lambda_0(m-k) + 2\alpha_0^2\vartheta_2|\Lambda_0|^2|\Lambda_2| =$$

$$= |\Lambda_0|\sum_{m}\sum_{r\in\Lambda_0} f(r,m) + 2\alpha_0^2\vartheta_2|\Lambda_0|^2|\Lambda_2|\,,$$

where $|\vartheta_1|, |\vartheta_2| \le 1$. Using (4.29), we get

$$|\sigma^*| < \eta|\Lambda''|^2|\Lambda_0||\Lambda_0 \cap E_1||\Lambda_2 \cap E_2| + 4\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2| \qquad (4.32)$$

If j is fixed, then k runs a set $-\Lambda_0 + j + \Lambda''$ in (4.30). Clearly, the cardinality of this set does not exceed $(1+\alpha_0^2)|\Lambda'|$. Hence, replacing $4\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|$ in (4.32) by $8\alpha_0^2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|$, we can assume that k runs $-\Lambda_0 + j$ in (4.30).

Since $l \in B'$, it follows that $\beta_1|\Lambda_0|/2 \le |\Lambda_0 \cap E_1| \le 2\beta_1|\Lambda_0|$. Besides that $16\alpha_0^2 < \eta\beta_1\beta_2$. This implies that

$$|\sum_{i\in\Lambda_0}\sum_{j\in\Lambda_2}\sum_{k\in-\Lambda_0+j}\sum_{m}\sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r)| <$$

$$< 2\eta|\Lambda''|^2|\Lambda_0|\cdot|\Lambda_0 \cap E_1|\cdot|\Lambda_2 \cap E_2| \le 4\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|\,. \qquad (4.33)$$

Let

$$\Omega = \{ j \in \Lambda_2 ~|~ \frac{1}{|\Lambda'|} \sum_{k \in \Lambda' + j} |\delta_{\Lambda''+k}(E_2) - \beta_2|^2 \ge 4\alpha_0^{1/2} \},~ \text{and}~ G = \Lambda_2 \setminus \Omega\,.$$

Since E_2 is (α_0, ε')–uniform, it follows that $|\Omega| \le 8\alpha_0^{1/2}|\Lambda_2|$. Let $i \in \Lambda_0$ be fixed. Let

$$\Omega(i) = \{ j \in \Lambda_2 ~|~ \frac{1}{|\Lambda'|} \sum_{k \in -\Lambda_0 + j} |\delta_{\Lambda''+i+k}(E_2) - \beta_2|^2 \ge 4\alpha_0^{1/2} \},~ \text{and}~ G(i) = \Lambda_2 \setminus \Omega(i)\,.$$

Since

$$\sum_{k \in -\Lambda_0 + j} |\delta_{\Lambda''+i+k}(E_2) - \beta_2|^2 = \sum_{k \in \Lambda' + j + (i - l_0)} |\delta_{\Lambda''+k}(E_2) - \beta_2|^2\,,$$

it follows that $\Lambda_2 \cap (G + l_0 - i) \subseteq G(i)$. Hence, $|\Omega(i)| \le |\Lambda_2| - |\Lambda_2 \cap (G + l_0 - i)|$. Since i belongs to Λ_0, this implies that a number $a = l_0 - i$ belongs to Λ'. Using Lemma 2.3 for Λ_2 and its ε–attendant Λ', we get $(G \cap \Lambda_2^-) + a \subseteq \Lambda_2$ and

$$|\Lambda_2 \cap (G + a)| \ge |\Lambda_2 \cap ((G \cap \Lambda_2^-) + a)| \ge |(G \cap \Lambda_2^-) + a| = |G \cap \Lambda_2^-| \ge |G| - 8\alpha_0^2|\Lambda_2|\,.$$

Hence $|\Omega(i)| \le 8\alpha_0^{1/2}|\Lambda_2|$.

Since $l_0 \in B'$, it follows that

$$\frac{1}{|\Lambda'|} \sum_{k \in \Lambda'} |\delta_{\Lambda''+k}(E_1 - l_0 \cap \Lambda') - \beta_1|^2 \le 2^6 \alpha_0^{1/2} \tag{4.34}$$

It is clear that for any j the sum (4.34) equals

$$\frac{1}{|\Lambda'|} \sum_{k \in -\Lambda_0 + j} |\delta_{\Lambda''+j-k}(E_1 \cap \Lambda_0) - \beta_1|^2\,.$$

Indeed

$$\sum_{k \in -\Lambda_0 + j} |\delta_{\Lambda''+j-k}(E_1 \cap \Lambda_0) - \beta_1|^2 = \sum_{k \in \Lambda' + l_0} |\delta_{\Lambda''+k}(E_1 \cap \Lambda' + l_0) - \beta_1|^2 =$$

$$= \sum_{k \in \Lambda'} |\delta_{\Lambda''+k}(E_1 - l_0 \cap \Lambda') - \beta_1|^2$$

Let

$$\Omega_1(i,j) = \{ k \in -\Lambda_0 + j ~:~ |\delta_{\Lambda''+i+k}(E_2) - \beta_2| \ge 4\alpha_0^{1/8} \}\,,$$

$$\Omega_2(i,j) = \{ k \in -\Lambda_0 + j ~:~ |\delta_{\Lambda''+j-k}(E_1 \cap \Lambda_0) - \beta_1| \ge 4\alpha_0^{1/8} \},~ \text{and}$$

$$\Omega_3(i,j) = \Omega_1(i,j) \cup \Omega_2(i,j)\,.$$

For all $j \notin \Omega(i)$ we have $|\Omega_1(i,j)| \le 2\alpha_0^{1/4}|\Lambda'|$. The inequality (4.34) implies that $|\Omega_2(i,j)| \le 4\alpha_0^{1/4}|\Lambda'|$. Hence $|\Omega_3(i,j)| \le 8\alpha_0^{1/4}|\Lambda'|$ if $j \notin \Omega(i)$.

Since $l_0 \in B'$, it follows that

$$\sigma = \sum_{i \in \Lambda_0} \sum_{j \in \Lambda_2} \sum_{k} \sum_{m,u} \nu_i(m-k)\nu_i(u-k) \Big| \sum_r \mu_j(k+r) \tilde{f}_{l_0}(r,m) \tilde{f}_{l_0}(r,u) \Big|^2 \ge$$

$$\ge \alpha \beta_1^2 \beta_2^2 |\Lambda''|^4 |\Lambda_0|^2 |\Lambda_2|\,, \tag{4.35}$$

where $\tilde{f}_{l_0}$ is a restriction of f to $\lambda_{l_0} \times \Lambda_2$. If j is fixed, then k runs $-\Lambda_0 + j + \Lambda''$ in (4.35). Clearly, the cardinality of this set does not exceed $(1+\alpha_0^2)|\Lambda'|$. Hence, replacing α by $\alpha/2$ in (4.35), we can assume that k runs $-\Lambda_0 + j$ in (4.35). Using $|\Omega(i)| \le 8\alpha_0^{1/2}|\Lambda_2|$, we get

$$\sigma = \sum_{i\in\Lambda_0} \sum_{j\notin\Omega(i)} \sum_k \sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big|\sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge$$

$$\ge \frac{\alpha}{4}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.36}$$

Now we can prove the theorem.

Let

$$J = \{(i,j,k) \mid i\in\Lambda_0,\ j\notin\Omega(i),\ k\notin\Omega_3(i,j) \text{ such that}$$

$$\sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big|\sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge \frac{\alpha}{64}\beta_1^2\beta_2^2|\Lambda''|^4\}\,.$$

Using (4.36), we get

$$\sum_{i\in\Lambda_0} \sum_{j\notin\Omega(i)} \sum_{k\notin\Omega_3(i,j)} \sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big|\sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge$$

$$\ge \frac{\alpha}{8}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.37}$$

It follows that

$$\sum_{(i,j,k)\in J} \sum_{m,u} \nu_i(m-k)\nu_i(u-k)\Big|\sum_r \mu_j(k+r)\tilde{f}_{l_0}(r,m)\tilde{f}_{l_0}(r,u)\Big|^2 \ge$$

$$\ge \frac{\alpha}{16}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.38}$$

Let us estimate the cardinality of J. For any triple (i,j,k) belongs to J we have $|E_2\cap(\nu_i+k)| - \beta_2|\Lambda''|| \le 4\alpha_0^{1/8}|\Lambda''|$ and $|(E_1\cap\Lambda_0)\cap(\mu_j-k)| - \beta_1|\Lambda''|| \le 4\alpha_0^{1/8}|\Lambda''|$. Using (4.38), we get

$$32|J|\cdot|\Lambda''|^4\beta_1^2\beta_2^2 \ge \frac{\alpha}{16}\beta_1^2\beta_2^2|\Lambda''|^4|\Lambda_0|^2|\Lambda_2|\,. \tag{4.39}$$

This yields that $|J| \ge 2^{-12}\alpha|\Lambda_0|^2|\Lambda_2|$.

Let us assume that for all $(i,j,k)\in J$ we have

$$\sum_m \sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) < -2^{15}\frac{\eta}{\alpha}\beta_1\beta_2|\Lambda''|^2\,. \tag{4.40}$$

Using (4.33), we get

$$\sum_{(i,j,k)\in\overline{J}} \sum_m \sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge 4\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|\,, \tag{4.41}$$

where $\overline{J} = \{(i,j,k) \ :\ (i,j,k)\in(\Lambda_0\times\Lambda_2\times(-\Lambda_0+j))\setminus J\}$. Since $|\Omega(i)| \le 8\alpha_0^{1/2}|\Lambda_2|$, $i\in\Lambda_0$, it follows that

$$\sum_{(i,j,k)\in\overline{J},\, j\notin\Omega(i)} \sum_m \sum_{r\in\Lambda_0} f(r,m)\nu_i(m-k)\mu_j(k+r) \ge 2\eta\beta_1\beta_2|\Lambda''|^2|\Lambda_0|^2|\Lambda_2|\,. \tag{4.42}$$

Hence, there exist i and j, $j \notin \Omega(i)$ such that

$$\sum_{k \in Q(i,j)} \sum_m \sum_{r \in \Lambda_0} f(r,m) \nu_i(m-k) \mu_j(k+r) \geq \frac{\eta}{2} \beta_1 \beta_2 |\Lambda'|^2 |\Lambda_0| \,, \tag{4.43}$$

where $Q(i,j)$ is a subset of $-\Lambda_0 + j$. Since $j \notin \Omega(i)$, it follows that $|\Omega_3(i,j)| \leq 8\alpha_0^{1/4} |\Lambda'|$. Hence

$$\sum_{k \in Q(i,j) \setminus \Omega_3(i,j)} \sum_m \sum_{r \in \Lambda_0} f(r,m) \nu_i(m-k) \mu_j(k+r) \geq \frac{\eta}{4} \beta_1 \beta_2 |\Lambda''|^2 |\Lambda_0| \,. \tag{4.44}$$

This implies that there exists $k \notin \Omega_3(i,j)$ such that

$$\sum_m \sum_{r \in \Lambda_0} f(r,m) \nu_i(m-k) \mu_j(k+r) \geq \frac{\eta}{8} \beta_1 \beta_2 |\Lambda''|^2 \,. \tag{4.45}$$

Put $\tilde{\Lambda} = \Lambda''$, $\vec{y} = (j-k, k+i)$ and $F_1 = (\tilde{\Lambda} + y_1) \cap (E_1 \cap \Lambda_0)$, $F_2 = (\tilde{\Lambda} + y_2) \cap E_2$. Since $k \notin \Omega_3(i,j)$, it follows that $\beta_1 |\Lambda''|/2 \leq |F_1| \leq 2\beta_1 |\Lambda''|$, $\beta_2 |\Lambda''|/2 \leq |F_2| \leq 2\beta_2 |\Lambda''|$. Using this and (4.45), we get

$$|A \cap (F_1 \times F_2)| = |A \cap (((\mu_j - k) \cap \Lambda_0) \times ((\nu_i + k) \cap \Lambda_2))| \geq$$
$$\geq \delta |(\mu_j - k) \cap E_1 \cap \Lambda_0| |(\nu_i + k) \cap E_2| + \frac{\eta}{8} \beta_1 \beta_2 |\Lambda''|^2 \geq$$
$$\geq (\delta + \frac{\eta}{32}) |F_1| |F_2| \,.$$

Hence, if for all $(i,j,k) \in J$ we have (4.40), then the theorem is proven.

Now assume that there exists a triple $(i,j,k) \in J$ such that

$$\sum_m \sum_{r \in \Lambda_0} f(r,m) \nu_i(m-k) \mu_j(k+r) \geq -2^{15} \frac{\eta}{\alpha} \beta_1 \beta_2 |\Lambda''|^2 \,. \tag{4.46}$$

We can assume that for all $(i,j,k) \in J$ we have

$$|\sum_m \sum_{r \in \Lambda_0} f(r,m) \nu_i(m-k) \mu_j(k+r)| \leq 2^{15} \frac{\eta}{\alpha} \beta_1 \beta_2 |\Lambda''|^2 \,. \tag{4.47}$$

Indeed, if

$$\sum_m \sum_{r \in \Lambda_0} f(r,m) \nu_i(m-k) \mu_j(k+r) > 2^{15} \frac{\eta}{\alpha} \beta_1 \beta_2 |\Lambda''|^2 \,,$$

then we might apply the same reasoning as above. For sets $\tilde{\Lambda}_1 = \Lambda''$, $\tilde{\Lambda}_2 = \Lambda''$, a vector $\vec{y} = (j-k, k+i)$ and $F_1 = (\tilde{\Lambda}_1 + y_1) \cap (E_1 \cap \Lambda_0)$, $F_2 = (\tilde{\Lambda}_2 + y_2) \cap E_2$ we have $|F_1| \geq \beta_1 |\tilde{\Lambda}_1|/2$, $|F_2| \geq \beta_2 |\tilde{\Lambda}_2|/2$ and

$$|A \cap (F_1 \times F_2)| \geq (\delta + 2^6 \frac{\eta}{\alpha}) |F_1| |F_2|.$$

Since $(i,j,k) \in J$, it follows that

$$\sum_{m,u \in \nu_i + k} \Big| \sum_{r \in \mu_j - k} \tilde{f}_{l_0}(r,m) \tilde{f}_{l_0}(r,u) \Big|^2 \geq 2^{-6} \alpha \beta_1^2 \beta_2^2 |\Lambda''|^4 \,. \tag{4.48}$$

Note that m, u belong to $\nu_i + k \cap \Lambda_2$ in (4.48) and r belongs to a set $\mu_j - k \cap \Lambda_0$. Put $\mathcal{L}_1 = \mu_j - k \cap \Lambda_0$, $\mathcal{L}_2 = \nu_i + k \cap \Lambda_2$, $E_1' = E_1 \cap \mathcal{L}_1$ and $E_2' = E_2 \cap \mathcal{L}_2$. We can assume that $\tilde{f}_{l_0}$ is zero outside $\mathcal{L}_1 \times \mathcal{L}_2$ in (4.48). Let $A_1 = A \cap (\mathcal{L}_1 \times \mathcal{L}_2)$, $\delta_1 = \delta_{E_1' \times E_2'}(A)$, and f_1 be a balanced function of A_1. Using (4.47), we get $|\delta_1 - \delta| \le 2^{20}\frac{\eta}{\alpha}$. We have $k \notin \Omega_3(i,j)$. Using this, we obtain

$$\|\tilde{f}_{l_0} - f_1\|^4 = |E_1'|^2|E_2'|^2(\delta_1-\delta)^2 \le 2^{44}\beta_1^2\beta_2^2\frac{\eta^2}{\alpha^2}|\Lambda''|^4\,. \qquad (4.49)$$

Using Lemma 3.2, we get

$$\sum_{m,u\in\nu_i+k}\Big|\sum_{r\in\mu_j-k} f_1(r,m)f_1(r,u)\Big|^2 \ge 2^{-7}\alpha\beta_1^2\beta_2^2|\Lambda''|^4\,. \qquad (4.50)$$

Since $k\notin\Omega_3(i,j)$, it follows that $2^{-1}\beta_1|\Lambda''| \le |E_1'| \le 2\beta_1|\Lambda''|$, $2^{-1}\beta_2|\Lambda''| \le |E_2'| \le 2\beta_2|\Lambda''|$. Hence

$$\sum_{m,u\in\nu_i+k}\Big|\sum_{r\in\mu_j-k} f_1(r,m)f_1(r,u)\Big|^2 \ge 2^{-11}\alpha|E_1'|^2|E_2'|^2\,. \qquad (4.51)$$

Using Proposition 4.1, we obtain sets $F_1 \subseteq E_1' \subseteq \mu_j - k$, $F_2 \subseteq E_2' \subseteq \nu_i + k$ such that

$$|A\cap(F_1\times F_2)| \ge |A_1\cap(F_1\times F_2)| \ge (\delta_1 + 2^{-37}\frac{\alpha^2}{\delta_1^5})|F_1||F_2| \ge$$

$$\ge (\delta + 2^{-40}\frac{\alpha^2}{\delta^5})|F_1||F_2| \ge (\delta + 2^{-240}\delta^{13})|F_1||F_2|\,. \qquad (4.52)$$

and

$$|F_i| \ge 2^{-40}\frac{\alpha^2}{\delta^5}|E_i'| \ge 2^{-300}\delta^{13}\beta_i|\Lambda''|, \quad i=1,2\,.$$

Put $\tilde\Lambda = \Lambda''$, $\vec{y} = (j-k, k+i)$ and $F_1 = (\tilde\Lambda_1 + y_1)\cap(E_1\cap\Lambda_0)$, $F_2 = (\tilde\Lambda_2+y_2)\cap E_2$. The sets $\tilde\Lambda$ and F_1, F_2 satisfy (4.20), (4.21). This concludes the proof.

5. *On dense subsets of Bohr sets.*

The following lemmas were proven in [**27**].

LEMMA 5.1. *Let Λ be a Bohr set, Λ' be an ε–attendant of Λ, $\varepsilon = \kappa/(100d)$, and Q be a subset of Λ. Let $g : 2^G \times (G\times G) \to \mathbf{D}$ be the function such that $g(\Lambda,\vec{x}) = \delta^2_{\Lambda+\vec{x}}(Q)$. Then*

$$\frac{1}{|\Lambda|^2}\sum_{\vec{x}\in\Lambda} g(\Lambda',\vec{x}) \ge g(\Lambda,0) - 8\kappa\,. \qquad (5.1)$$

LEMMA 5.2. *Let Λ be a Bohr set, Λ' be an ε–attendant of Λ, $\varepsilon = \kappa/(100d)$, $\alpha > 0$ be a real number, and Q be a subset of Λ, $|Q| = \delta|\Lambda|$. Suppose that*

$$\frac{1}{|\Lambda|^2}\sum_{\vec{n}\in\Lambda} |\delta_{\Lambda'+\vec{n}}(Q) - \delta|^2 \ge \alpha\,. \qquad (5.2)$$

Then

$$\sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(Q) \ge \delta^2 + \alpha - 4\kappa\,. \qquad (5.3)$$

Note. Clearly, the one–dimension analogs of Lemma 5.1 and Lemma 5.2 take place.

Also, in [**27**] was proven a corollary.

Corollary 5.1. *Let Λ be a Bohr set, $\alpha > 0$ be a real number, and E_1, E_2 be sets, $|E_1 \cap \Lambda| = \beta_1 |\Lambda|$, $|E_2 \cap \Lambda| = \beta_2 |\Lambda|$. Suppose that either E_1 or E_2 does not satisfy (3.13). Let Λ' be an arbitrary $(2^{-10}\alpha^2\beta_1^2\beta_2^2)/(100d)$–attendant set of Λ. Then*

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n} \in \Lambda} \delta^2_{\Lambda' + \vec{n}} (E_1 \times E_2) \ge \beta_1^2 \beta_2^2 (1 + \frac{\alpha^2}{2})\,. \tag{5.4}$$

The following lemma was proven by J. Bourgain in [**3**]. We give his proof for the sake of completeness.

Lemma 5.3. *Let $\Lambda = \Lambda(S,\varepsilon)$ be a Bohr set, $|S| = d \in \mathbf{N}$, $\alpha > 0$ be a real number, and Q be a set, $|Q \cap \Lambda| = \delta |\Lambda|$. Suppose that*

$$\| (Q \cap \Lambda - \delta \Lambda)\widehat{\ } \|_\infty \ge \alpha |\Lambda|\,. \tag{5.5}$$

Then there exists a Bohr set $\Lambda' = \Lambda(S',\varepsilon')$, $|S'| = d+1$ such that Λ' is an ε_1–attendant of Λ, $\varepsilon_1 = \frac{\kappa}{100d}$, $\kappa \le \alpha/32$ and

$$\frac{1}{|\Lambda|} \sum_{n \in \Lambda} |\delta_{\Lambda' + n}(Q) - \delta|^2 \ge \frac{\alpha^2}{4}\,. \tag{5.6}$$

Proof. Let $Q_1 = Q \cap \Lambda$. Using (5.5), we obtain

$$|\widehat{Q}_1(\xi_0) - \delta \widehat{\Lambda}(\xi_0)| \ge \alpha |\Lambda|\,, \tag{5.7}$$

where $\xi_0 \in \widehat{G}$. We have $\Lambda = \Lambda_{S,\varepsilon}$, where $S \subseteq \widehat{G}$. Put $S' = S \cup \{\xi_0\} \subseteq \widehat{G}$ and

$$\Lambda' = \Lambda_{S',\varepsilon'}$$

be an ε_1–attendant of Λ. Using Lemma 2.3, we get

$$\widehat{Q}_1(\xi_0) = \sum_n Q(n)\Lambda(n) e^{-2\pi i(\xi_0 \cdot n)} = \frac{1}{|\Lambda'|} \sum_n (\Lambda * \Lambda')(n) Q(n) e^{-2\pi i (\xi_0 \cdot n)} + 2\kappa\vartheta |\Lambda|\,,$$

where $|\vartheta| \le 1$. We have

$$\widehat{Q}_1(\xi_0) = \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m) Q(n) e^{-2\pi i(\xi_0 \cdot n)} + 2\kappa\vartheta|\Lambda| =$$

$$= \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m) Q(n) e^{-2\pi i(\xi_0 \cdot m)} +$$

$$+ \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m) Q(n) [e^{-2\pi i(\xi_0 \cdot n)} - e^{-2\pi i(\xi_0 \cdot m)}] + 2\kappa\vartheta|\Lambda| =$$

$$= \sum_{m \in \Lambda} \delta_{\Lambda' + m}(Q) e^{-2\pi i(\xi_0 \cdot m)} + \vartheta_1 \frac{1}{|\Lambda'|} \sum_m \sum_n \Lambda'(n-m)\Lambda(m) Q(n) |e^{-2\pi i(\xi_0 \cdot (n-m))} - 1| +$$

$$+ 2\kappa\vartheta|\Lambda| = \sum_{m\in\Lambda} \delta_{\Lambda'+m}(Q)e^{-2\pi i(\xi_0\cdot m)} + (14\kappa\vartheta_1 + 2\kappa\vartheta)|\Lambda| \,, \tag{5.8}$$

where $|\vartheta_1| \le 1$. Using (5.5) and (5.8), we obtain

$$\Big| \sum_{m\in\Lambda} \delta_{\Lambda'+m}(Q)e^{-2\pi i(\xi_0\cdot m)} - \delta \sum_{m\in\Lambda} e^{-2\pi i(\xi_0\cdot m)} \Big| \ge \frac{\alpha}{2}|\Lambda| \,. \tag{5.9}$$

Hence

$$\sum_{m\in\Lambda} |\delta_{\Lambda'+m}(Q) - \delta| \ge \frac{\alpha}{2}|\Lambda| \,. \tag{5.10}$$

Using the Cauchy–Schwartz inequality, we get

$$\frac{1}{|\Lambda|} \sum_{\vec{n}\in\Lambda} |\delta_{\Lambda'+\vec{n}}(Q) - \delta|^2 \ge \frac{\alpha^2}{4} \,. \tag{5.11}$$

This completes the proof.

Corollary 5.2. *Let $\Lambda = \Lambda(S,\varepsilon)$ be a Bohr set, $\alpha > 0$ be a real number, and E_1, E_2 be sets, $|E_1 \cap \Lambda| = \beta_1|\Lambda|$, $|E_2 \cap \Lambda| = \beta_2|\Lambda|$ Suppose that either E_1 or E_2 satisfies (5.5). Then there exists $(2^{-10}\alpha^2\beta_1^2\beta_2^2)/(100d)$–attendant set $\Lambda' = \Lambda(S',\varepsilon')$ of the Bohr set Λ such that*

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1\times E_2) \ge \beta_1^2\beta_2^2(1 + \frac{\alpha^2}{8}) \tag{5.12}$$

and

$$|S'| = d+1 \,. \tag{5.13}$$

Proof. Let $\vec{n} = (x,y)$, and $\kappa = 2^{-10}\alpha^2\beta_1^2\beta_2^2$. We have

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1\times E_2) = \Big(\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta^2_{\Lambda'+x}(E_1) \Big) \Big(\frac{1}{|\Lambda|} \sum_{y\in\Lambda} \delta^2_{\Lambda'+y}(E_2) \Big) \tag{5.14}$$

We can assume without loss of generality that E_1 satisfies (5.5). Using Lemma 5.3 and Lemma 5.2, we obtain

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta^2_{\Lambda'+x}(E_1) \ge \beta_1^2 + \frac{\alpha^2}{4} - 4\kappa \,. \tag{5.15}$$

Let us estimate the second term in (5.14). Using Lemma 5.1, we get

$$\frac{1}{|\Lambda|} \sum_{y\in\Lambda} \delta^2_{\Lambda'+y}(E_2) \ge \beta_2^2 - 8\kappa \,. \tag{5.16}$$

Combining (5.15) and (5.16), we obtain

$$\frac{1}{|\Lambda|^2} \sum_{\vec{n}\in\Lambda} \delta^2_{\Lambda'+\vec{n}}(E_1\times E_2) \ge (\beta_1^2 + \frac{\alpha^2}{4} - 4\kappa)(\beta_2^2 - 8\kappa) \ge \beta_1^2\beta_2^2(1 + \frac{\alpha^2}{8}) \,.$$

This concludes the proof.

We shall say that the set S' from (5.13) is constructed by Corollary 5.2.

Clearly, all lemmas of this section apply to translations of Bohr sets.

Our further arguments and arguments from [**27**] are particulary the same.

Let $\mathbf{\Lambda}$ be a union of a family of Bohr sets $\Lambda^*_0, \Lambda^*_1(\vec{x}_0), \dots, \Lambda^*_n(\vec{x}_0, \dots, \vec{x}_{n-1})$ and a sequence of some translations of Bohr sets $\Lambda_0, \Lambda_1(\vec{x}_0), \dots, \Lambda_n(\vec{x}_0, \dots, \vec{x}_{n-1})$ such that

$$\Lambda_1(\vec{x}_0) \text{ and } \Lambda^*_1(\vec{x}_0) \text{ are defined iff } \vec{x}_0 \in \Lambda_0$$

$$\Lambda_2(\vec{x}_0, \vec{x}_1) \text{ and } \Lambda^*_2(\vec{x}_0, \vec{x}_1) \text{ are defined iff } \vec{x}_1 \in \Lambda_1(\vec{x}_0), \vec{x}_0 \in \Lambda_0$$

$$\dots$$

$$\Lambda_n(\vec{x}_0, \dots, \vec{x}_{n-1}) \text{ and } \Lambda^*_n(\vec{x}_0, \dots, \vec{x}_{n-1}) \text{ are defined iff}$$

$$\vec{x}_{n-1} \in \Lambda_{n-1}(\vec{x}_0, \dots, \vec{x}_{n-2}), \vec{x}_{n-2} \in \Lambda_{n-2}(\vec{x}_0, \dots, \vec{x}_{n-3}), \dots, \vec{x}_0 \in \Lambda_0 \,. \qquad (5.17)$$

Let $m \ge 0$ be an integer number and $\mathbf{\Lambda}$ be a family of Bohr sets satisfies (5.17). Let $g : 2^G \times (G \times G) \to \mathbf{D}$ be a function. Let us define the *index* of g, respect $\mathbf{\Lambda}$, for all $k = 0, \dots, m$ by

$$\mathrm{ind}_k(\mathbf{\Lambda})(g) = \frac{1}{|\Lambda_0|^2} \sum_{\vec{x}_0 \in \Lambda_0} \frac{1}{|\Lambda_1(\vec{x}_0)|^2} \sum_{\vec{x}_1 \in \Lambda_1(\vec{x}_0)} \dots$$

$$\frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) \,. \qquad (5.18)$$

Let $M_k = M_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ be the family of sets such that $M_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \subseteq \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ for all $(\vec{x}_0, \dots, \vec{x}_{k-1})$. For any $k = 0, \dots, m$ by $\mathrm{ind}_k(\mathbf{\Lambda}, M)(g)$ define the following expression

$$\mathrm{ind}_k(\mathbf{\Lambda}, M)(g) = \frac{1}{|\Lambda_0|^2} \sum_{\vec{x}_0 \in \Lambda_0} \frac{1}{|\Lambda_1(\vec{x}_0)|^2} \sum_{\vec{x}_1 \in \Lambda_1(\vec{x}_0)} \dots$$

$$\frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in M_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) \,. \qquad (5.19)$$

Clearly, we have $|\,\mathrm{ind}_k(\mathbf{\Lambda}, M)(g)| \le 1$, for any natural $k \ge 0$, a family M_k and a function $g : 2^G \times (G \times G) \to \mathbf{D}$.

The following simple lemma was proven in [**27**].

LEMMA 5.4. *Let Q be a subset of $\Lambda_0 \times \Lambda_0$, and $|Q| = \delta |\Lambda_0|^2$. Suppose that $\Lambda^*_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ is an arbitrary ε–attendant of $\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})$, $\varepsilon = \kappa/(100d)$. Let $g(M, \vec{x}) = \delta_{M+\vec{x}}(Q)$. Then for all $k = 0, \dots, n$ we have*

$$\Big|\, \mathrm{ind}_k(\mathbf{\Lambda})(g) - \delta \Big| \le 4\kappa(k+1) \,. \qquad (5.20)$$

The next result is the main in this section.

PROPOSITION 5.1. *Let $\Lambda = \Lambda(S, \varepsilon_0)$ be a Bohr set, $|S| = d$, and $\vec{s} = (s_1, s_2)$ be a vector. Let $\varepsilon, \sigma, \tau, \delta \in (0,1)$ be real numbers, E_1, E_2 be sets, $E_i = \beta_i |\Lambda|$, $i = 1, 2$.*

Suppose that $\mathbf{E} = E_1 \times E_2$ *is a subset of* $(\Lambda + s_1) \times (\Lambda + s_2)$, $A \subset \mathbf{E}$, $\delta_{\mathbf{E}}(A) = \delta + \tau$, *and* $\varepsilon \le \kappa/(100d)$, $\kappa = 2^{-100}(\tau\beta_1\beta_2)^5\sigma^3$. *Let*

$$N \ge (2^{-100}\varepsilon_0\varepsilon)^{-2^{100}((\tau\beta_1\beta_2)^{-5}\sigma^{-3}+d)^2}\,, \tag{5.21}$$

and $\sigma \le 2^{-100}\tau\beta_1\beta_2$. *Then there exists a Bohr set* $\Lambda' = \Lambda(S', \varepsilon')$, $|S'| = D$, $D \le 2^{30}(\tau\beta_1\beta_2)^{-5}\sigma^{-3} + d$, $\varepsilon' \ge (2^{-10}\varepsilon)^D\varepsilon_0$ *and a vector* $\vec{t} = (t_1, t_2)$ *such that if* $E_1' = (E_1 - t_1) \cap \Lambda'$, $E_2' = (E_2 - t_2) \cap \Lambda'$, $\mathbf{E}' = E_1' \times E_2'$, *then*
1) $|\mathbf{E}'| \ge \beta_1\beta_2\tau|\Lambda'|/16$*;*
2) E_1', E_2' *are* (σ, ε)*–uniform subsets of* Λ'*;*
3) $\delta_{\mathbf{E}'}(A - \vec{t}) \ge \delta + \tau/16$.

Proof. Let $\beta = \beta_1\beta_2$, and $\tilde{E}_1 = E_1 - s_1$, $\tilde{E}_2 = E_2 - s_2$, $\tilde{E} = \tilde{E}_1 \times \tilde{E}_2$. If the sets $\tilde{E}_1$, $\tilde{E}_2$ are (σ, ε)–uniform subsets of Λ, then Proposition 5.1 is proven.

Suppose that $\tilde{E}_1$, $\tilde{E}_2$ are not (σ, ε)–uniform subsets of Λ. We shall construct a family of Bohr sets $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}$ satisfies the conditions (5.17). The proof of Proposition 5.1 is a sort of an algorithm. At the first step of our algorithm we put $\Lambda_0 = \Lambda = \Lambda(S, \varepsilon_0)$. If either $\tilde{E}_1$ or $\tilde{E}_2$ does not satisfy (3.14) with $\alpha = \sigma/2$, then let Λ_0^* be an ε–attendant of Λ_0 such that Λ_0^* is constructed by Corollary 5.2. In the other cases let Λ_0^* be an ε–attendant of Λ_0 with the same set S to be chosen later. Define

$$R_0 = \{\vec{p} = (p_1, p_2) \in \Lambda_0 \mid \tilde{E}_1 - p_1, \tilde{E}_2 - p_2 \text{ are } (\sigma, \varepsilon)\text{–uniform in } \Lambda_0^*$$
$$\text{or } \delta_{\Lambda_0^* + \vec{p}}(\tilde{E}_1 \times \tilde{E}_2) < \beta\tau/16\}$$

and $\overline{R}_0 = (\Lambda_0 \times \Lambda_0) \setminus R_0$.

Let $\tilde{\Lambda}$ be an arbitrary Bohr set, and $\vec{n} \in G \times G$ be an arbitrary vector. Put $g(\tilde{\Lambda}, \vec{n}) = \delta^2_{\tilde{\Lambda}+\vec{n}}(\tilde{E})$, $g_1(\tilde{\Lambda}, \vec{x}) = \delta_{\tilde{\Lambda}+\vec{n}}(A)$, $g_2(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{E}\cap\tilde{\Lambda}+\vec{n}}(A)$ and $g_3(\tilde{\Lambda}, \vec{n}) = \delta_{\tilde{\Lambda}+\vec{n}}(\tilde{E})$. Clearly, $g(\tilde{\Lambda}, \vec{n}) = g_3^2(\tilde{\Lambda}, \vec{n})$ and $g_1(\tilde{\Lambda}, \vec{x}) \le g_3(\tilde{\Lambda}, \vec{n})$. Besides that, we have

$$g_1(\tilde{\Lambda}, \vec{n}) = g_2(\tilde{\Lambda}, \vec{n})g_3(\tilde{\Lambda}, \vec{n})\,.$$

Let $\mathbf{\Lambda}_0 = \{\Lambda_0\}$. If $\mathrm{ind}_0(\mathbf{\Lambda}_0, \overline{R}_0)(g_3) < \tau\beta/4$, then we stop the algorithm at step 0.

Using Lemma 2.4 and the Cauchy–Schwartz inequality, we get

$$\mathrm{ind}_0(\mathbf{\Lambda}_0)(g) \ge \Big(\frac{1}{|\Lambda_0|^2}\sum_{\vec{y}\in\Lambda_0}\delta_{\Lambda_0^*+\vec{y}}(\tilde{E})\Big)^2 \ge \beta/2\,. \tag{5.22}$$

Let after the kth step of the algorithm the family of Bohr sets $\mathbf{\Lambda}_k$ has been constructed, $k \ge 0$.

Let

$$\Lambda_{k+1}(\vec{x}_0, \dots, \vec{x}_k) = \Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}) + \vec{x}_k\,,\ \vec{x}_k \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})\,.$$

Let $\vec{x}_k = (a, b)$, and $\Lambda_k^* = \Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1})$. If either $(\tilde{E}_1 - a) \cap \Lambda_k^*$ or $(\tilde{E}_2 - b) \cap \Lambda_k^*$ does not satisfy (3.14) with $\alpha = \sigma/2$, then let $\Lambda_{k+1}^*(\vec{x}_0, \dots, \vec{x}_k)$ be an ε–attendant of $\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_k)$ such that $\Lambda_{k+1}^*(\vec{x}_0, \dots, \vec{x}_k)$ is constructed by Corollary 5.2. In the other cases let $\Lambda_{k+1}^*(\vec{x}_0, \dots, \vec{x}_k)$ be an ε–attendant of $\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_k)$ with the same generative vector.

By $R_{k+1}(\vec{x}_0, \dots, \vec{x}_k)$, $\overline{R}_{k+1}(\vec{x}_0, \dots, \vec{x}_k)$ denote the sets

$$R_{k+1}(\vec{x}_0, \dots, \vec{x}_k) = \{\vec{p} = (p_1, p_2) \in \Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}) + \vec{x}_k \mid \tilde{E}_1 - p_1, \tilde{E}_2 - p_2$$

$$\text{are } (\sigma,\varepsilon)\text{–uniform in } \Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$$

$$\text{or } \delta_{\Lambda^*_{k+1}(\vec{x}_0,\dots,\vec{x}_k)+\vec{p}}(\tilde{E}_1\times\tilde{E}_2) < \tau\beta/16\}$$

and $\overline{R}_{k+1}(\vec{x}_0,\dots,\vec{x}_k) = (\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})+\vec{x}_k)\setminus R_{k+1}(\vec{x}_0,\dots,\vec{x}_k)$.

By $E_k(\vec{x}_0,\dots,\vec{x}_{k-1})$ denote the sets

$$E_k(\vec{x}_0,\dots,\vec{x}_{k-1}) =$$

$$\{\vec{p}=(p_1,p_2)\in\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})+\vec{x}_{k-1} \mid \delta_{\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})+\vec{p}}(\tilde{E}_1\times\tilde{E}_2) < \tau\beta/16\}.$$

Obviously, $E_k(\vec{x}_0,\dots,\vec{x}_{k-1}) \subseteq R_k(\vec{x}_0,\dots,\vec{x}_{k-1})$, $k=0,1,\dots$

Let $\mathbf{\Lambda'}_{k+1} = \{\Lambda_{k+1}(\vec{x}_0,\dots,\vec{x}_k)\}$, $\vec{x}_k\in\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})$, and $\mathbf{\Lambda}_{k+1} = \{\mathbf{\Lambda}_k,\mathbf{\Lambda'}_{k+1}\}$. If $\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1},\overline{R}_{k+1})(g_3) < \tau\beta/4$, then we stop the algorithm at step $k+1$.

Let $\Lambda^*_{k-1} = \Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$, and $\beta'_k = \delta_{\Lambda^*_{k-1}}(\tilde{E}_1)$, $\beta''_k = \delta_{\Lambda^*_{k-1}}(\tilde{E}_2)$. Suppose $\vec{x}_{k-1}=(a',b')$ belongs to $\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$. Note that $\vec{x}_{k-1}$ does not belong to $E_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$. Let us consider three cases.
Case 1 : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.12).
Case 2 : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.13).
Case 3 : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.14).
Note that α equals σ in all these cases.
Let us consider the following situation : either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.14) with $\alpha=2^{-4}\sigma^{3/2}$. Let

$$S_0 = \frac{1}{|\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})|^2}\sum_{\vec{y}\in\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})} g(\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1}),\vec{y})\,, \tag{5.23}$$

where $\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})$ is an ε–attendant of $\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})$ such that $\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1})$ is constructed by Corollary 5.2. Using Corollary 5.2, we get

$$S_0 \ge g(\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1}),0)(1+2^{-11}\sigma^3) =$$

$$= g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-11}\sigma^3)\,. \tag{5.24}$$

Note that in this case, we have $\dim\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1}) = \dim\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})+1$.

Suppose that either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.13) with $\alpha=2^{-4}\sigma^{3/2}$. Using Corollary 5.1, we obtain

$$S_0 \ge g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-11}\sigma^3)\,. \tag{5.25}$$

In this case, we have $\dim\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_{k-1}) = \dim\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})$.

Finally, suppose that either $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ or $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ does not satisfy (3.12) with $\alpha=\sigma$. Note that $(\tilde{E}_1-a')\cap\Lambda^*_{k-1}$ *and* $(\tilde{E}_2-b')\cap\Lambda^*_{k-1}$ satisfy (3.13) with $\alpha=2^{-4}\sigma^{3/2}$. Let $\Lambda^*_k=\Lambda^*_k(\vec{x}_0,\dots,\vec{x}_k)$. Define

$$B_k(\vec{x}_0,\dots,\vec{x}_{k-1}) = \{\vec{p}=(p_1,p_2)\in\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1}) \ :$$

$$\|((\tilde{E}_1-p_1)-\beta'_k\Lambda^*_k)\widehat{\ }\|_\infty \ge \sigma|\Lambda^*_k| \text{ or } \|((\tilde{E}_2-p_2)-\beta''_k\Lambda^*_k)\widehat{\ }\|_\infty \ge \sigma|\Lambda^*_k|\}\,.$$

We have

$$|B_k(\vec{x}_0,\dots,\vec{x}_{k-1})| \ge \sigma|\Lambda_k(\vec{x}_0,\dots,\vec{x}_{k-1})|^2\,. \tag{5.26}$$

Let

$$\tilde{B}_k(\vec{x}_0,\dots,\vec{x}_{k-1}) = \{\vec{p}=(p_1,p_2)\in B_k(\vec{x}_0,\dots,\vec{x}_{k-1}) \ :$$

$$|\delta_{\Lambda_k^*}(\tilde{E}_1 - p_1) - \beta'_k| \le \sigma/8 \quad \text{and} \quad |\delta_{\Lambda_k^*}(\tilde{E}_2 - p_2) - \beta''_k| \le \sigma/8\}\,.$$

For all $\vec{p} \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$, we have either $(\tilde{E}_1 - p_1) \cap \Lambda_k^*$ or $(\tilde{E}_2 - p_2) \cap \Lambda_k^*$ does not $\sigma/2$–uniform. The sets $(\tilde{E}_1 - a') \cap \Lambda_{k-1}^*$ and $(\tilde{E}_2 - b') \cap \Lambda_{k-1}^*$ satisfy (3.13) with α equals $2^{-4}\sigma^{3/2}$. This implies that

$$|\tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})| \ge \frac{\sigma}{2} |\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2\,. \tag{5.27}$$

Suppose that

$$g_3(\Lambda_{k-1}^*, \vec{x}_{k-1}) = \beta'_k \beta''_k \ge \tau\beta/8\,. \tag{5.28}$$

It follows from (5.28) that

$$g_3(\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{p}) \ge \beta'_k \beta''_k - \sigma/2 \ge \tau\beta/16\,, \tag{5.29}$$

for all $\vec{p} \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$.

Let us consider the sum

$$S = S(\vec{x}_0, \dots, \vec{x}_{k-1}) = \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{x}_k \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})} \frac{1}{|\Lambda_{k+1}(\vec{x}_0, \dots, \vec{x}_k)|^2}$$

$$\cdot \sum_{\vec{y} \in \Lambda_{k+1}(\vec{x}_0, \dots, \vec{x}_k)} g(\Lambda_{k+1}^*(\vec{x}_0, \dots, \vec{x}_k), \vec{y})\,.$$

Write the sum S as $S' + S''$, where the summation in S' is taken over $\vec{x}_k \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$ and the summation in S'' is taken over $\vec{x}_k \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \setminus \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$. Note that if $\vec{x}_k \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})$, then the Bohr set $\Lambda_{k+1}^*(\vec{x}_0, \dots, \vec{x}_k)$ is constructed by Corollary 5.2. Using this corollary, we obtain

$$S' \ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y})(1 + \frac{\sigma^2}{32})\,. \tag{5.30}$$

Let us estimate the sum S''. Using Lemma 5.1, we get

$$S'' \ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1}) \setminus \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) - 8\kappa \tag{5.31}$$

Combining (5.29), (5.30), (5.31) and (5.27), we have

$$S \ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) +$$

$$+ \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \tilde{B}_k(\vec{x}_0, \dots, \vec{x}_{k-1})} 2^{-13}\tau^2\beta^2\sigma^2 - 2^4\kappa \ge$$

$$\ge \frac{1}{|\Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})|^2} \sum_{\vec{y} \in \Lambda_k(\vec{x}_0, \dots, \vec{x}_{k-1})} g(\Lambda_k^*(\vec{x}_0, \dots, \vec{x}_{k-1}), \vec{y}) + 2^{-14}\tau^2\beta^2\sigma^3 - 2^4\kappa$$

Using Lemma 5.1, we obtain

$$\begin{aligned} S &\ge g(\Lambda_{k-1}^*(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1}) + 2^{-14}\tau^2\beta^2\sigma^3 - 2^5\kappa \ge \\ &\ge g(\Lambda_{k-1}^*(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1}) + 2^{-15}\tau^2\beta^2\sigma^3 \ge \\ &\ge g(\Lambda_{k-1}^*(\vec{x}_0, \dots, \vec{x}_{k-2}), \vec{x}_{k-1})(1 + 2^{-15}\tau^2\beta^2\sigma^3)\,. \end{aligned} \tag{5.32}$$

On the other hand, S_0 is an estimate for S. Using Lemma 5.1, we get

$$S \ge S_0 - 8\kappa\,.$$

Thus if $\vec{x}_{k-1}$ belongs to $\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$ and $\vec{x}_{k-1}$ satisfies (5.28), then we have

$$S \ge g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-15}\tau^2\beta^2\sigma^3) - 8\kappa\,. \qquad (5.33)$$

Now suppose that $\vec{x}_{k-1}$ is an arbitrary vector, $\vec{x}_{k-1}\in\Lambda_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$. Using Lemma 5.1 twice, we have

$$S \ge g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1}) - 16\kappa\,. \qquad (5.34)$$

Let us consider $\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g)$. We have

$$\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g) =$$

$$\frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots\sum_{\vec{x}_{k-1}\in\Lambda_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})} S(\vec{x}_0,\dots,\vec{x}_{k-1})\,.$$

By assumption $\mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1},\overline{R}_{k-1})(g_3)\ge\tau\beta/4$. In other words

$$\frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\sum_{\vec{x}_{k-1}\in\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})} g_3(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1}) \ge \tau\beta/4\,. \qquad (5.35)$$

By $M_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$ denote the set of $\vec{x}_{k-1}\in\overline{R}_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})$ such that $\vec{x}_{k-1}$ satisfies (5.28). Using (5.35), we obtain

$$S_M := \frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\sum_{\vec{x}_{k-1}\in M_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})} g_3(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1}) \ge \tau\beta/8\,. \qquad (5.36)$$

Using (5.28), (5.33), (5.34) and (5.36), we get

$$\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g) \ge \frac{1}{|\Lambda_0|^2}\sum_{\vec{x}_0\in\Lambda_0}\frac{1}{|\Lambda_1(\vec{x}_0)|^2}\sum_{\vec{x}_1\in\Lambda_1(\vec{x}_0)}\dots$$

$$\Big\{\sum_{\vec{x}_{k-1}\in M_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})}(g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})(1+2^{-15}\tau^2\beta^2\sigma^3)-8\kappa)+$$

$$+\sum_{\vec{x}_{k-1}\in\Lambda_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})\setminus M_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2})}(g(\Lambda^*_{k-1}(\vec{x}_0,\dots,\vec{x}_{k-2}),\vec{x}_{k-1})-16\kappa)\Big\} \ge$$

$$\ge \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g) + 2^{-15}\tau^2\beta^2\sigma^3\Big(\frac{\tau\beta}{8}\Big)S_M - 24\kappa \ge$$

$$\ge \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g) + 2^{-24}\tau^4\beta^4\sigma^3 - 24\kappa \ge$$

$$\ge \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g) + 2^{-25}\tau^4\beta^4\sigma^3\,.$$

In other words, for all $k \ge 1$, we have

$$\mathrm{ind}_{k+1}(\mathbf{\Lambda}_{k+1})(g) \ge \mathrm{ind}_{k-1}(\mathbf{\Lambda}_{k-1})(g) + 2^{-25}\tau^4\beta^4\sigma^3 \,. \tag{5.37}$$

Since for any k we have $\mathrm{ind}_k(\mathbf{\Lambda}_k)(g) \le 1$, it follows that the total number of steps of the algorithm does not exceed $K_0 = 2^{30}\tau^{-4}\beta^{-4}\sigma^{-3}$.

Suppose that the algorithm stops at step K, $K \ge 1$, $K \le 2^{30}\tau^{-4}\beta^{-4}\sigma^{-3}$. We have

$$\mathrm{ind}_K(\mathbf{\Lambda}_K, \overline{R}_K)(g_3) < \frac{\tau\beta}{4} \,. \tag{5.38}$$

Using Lemma 5.4, we get

$$\mathrm{ind}_K(\mathbf{\Lambda}_K)(g_1) \ge (\delta+\tau)\beta - 8\kappa K \ge (\delta + \frac{7\tau}{8})\beta \,.$$

Using (5.38), we obtain

$$\mathrm{ind}_K(\mathbf{\Lambda}_K, R_K)(g_1) \ge (\delta + \frac{3\tau}{8})\beta \,. \tag{5.39}$$

The summation in (5.39) is taken over the sets $\Lambda^*_K(\vec{x}_0,\dots,\vec{x}_{K-1}) + \vec{y}$, where $\vec{y} \in R_K(\vec{x}_0,\dots,\vec{x}_{K-1})$.

Let E_K be the family of vectors $\vec{y}$ such that $\vec{y} \in E_K(\vec{x}_0,\dots,\vec{x}_{K-1})$, and R^*_K be the family of vectors $\vec{y}$ such that $\vec{y} \in R_K(\vec{x}_0,\dots,\vec{x}_{K-1})$, but $\vec{y}$ does not belong to $E_K(\vec{x}_0,\dots,\vec{x}_{K-1})$. We have

$$\mathrm{ind}_K(\mathbf{\Lambda}_K, E_K)(g_1) < \frac{\tau\beta}{16}\,\mathrm{ind}_K(\mathbf{\Lambda}_K)(1) \le \frac{\tau\beta}{16} \,. \tag{5.40}$$

Combining (5.39), (5.40), we get

$$\mathrm{ind}_K(\mathbf{\Lambda}_K, R^*_K)(g_1) > (\delta + \frac{\tau}{4})\beta \,. \tag{5.41}$$

Suppose that for all $\vec{y} \in R^*_K(\vec{x}_0,\dots,\vec{x}_{K-1})$, we have $g_2(\Lambda^*_K(\vec{x}_0,\dots,\vec{x}_{K-1}),\vec{y}) < (\delta+\tau/16)$. Then

$$(\delta + \frac{\tau}{4})\beta < \mathrm{ind}_K(\mathbf{\Lambda}_K, R^*_K)(g_1) \le (\delta + \frac{\tau}{16})\,\mathrm{ind}_K(\mathbf{\Lambda}_K, R^*_K)(g_3) \le$$

$$\le (\delta + \frac{\tau}{16})\,\mathrm{ind}_K(\mathbf{\Lambda}_K)(g_3) \,. \tag{5.42}$$

Using Lemma 5.4 once again, we obtain

$$(\delta + \frac{\tau}{4})\beta < (\delta + \frac{\tau}{16})\,\mathrm{ind}_K(\mathbf{\Lambda}_K)(g_3) \le (\delta + \frac{\tau}{16})(\beta + 8\kappa K) \le (\delta + \frac{\tau}{4})\beta$$

with contradiction. Whence there exist vectors $\vec{x}_0,\dots,\vec{x}_{K-1}$, $\vec{y}$ such that $g_2(\Lambda^*_K(\vec{x}_0,\dots,\vec{x}_{K-1}),\vec{y}) \ge (\delta+\tau/16)$ and $\vec{y} \in R_K(\vec{x}_0,\dots,\vec{x}_{K-1}) \setminus E_K(\vec{x}_0,\dots,\vec{x}_{K-1})$. Put $\vec{t} = \vec{y} + \vec{s}$ and $\Lambda' = \Lambda^*_K(\vec{x}_0,\dots,\vec{x}_{K-1})$. We obtain the vector $\vec{t}$, the sets $E'_1 = (\tilde{E}_1 - y_1) \cap \Lambda'$, $E'_2 = (\tilde{E}_2 - y_2) \cap \Lambda'$ and the Bohr set Λ' which satisfy the conditions 1)—3).

Let us estimate D and ε'. At the each step of the algorithm the dimension of Bohr sets increases at most 1. Since the total number of steps does not exceed K_0, it follows that $D \le d + 2^{30}\tau^{-5}\beta^{-5}\sigma^{-3}$ and $\varepsilon' \ge (2^{-10}\varepsilon)^D\varepsilon_0$. Using Lemma 2.1 and (5.21), we obtain that the set Λ' is not empty. This completes the proof.

6. *Proof of main result.*

Let us put Theorem 4.1 and Proposition 5.1 together in a single proposition.

Proposition 6.1. *Let $\Lambda = \Lambda(S, \varepsilon_0)$ be a Bohr set, $|S| = d$, and $\vec{s} = (s_1, s_2) \in G \times G$. Let E_1, E_2 be sets, $E_i = \beta_i |\Lambda|$, $i = 1, 2$, $\beta = \beta_1 \beta_2$. Suppose $\mathbf{E} = E_1 \times E_2$ is a subset of $(\Lambda + s_1) \times (\Lambda + s_2)$, E_1, E_2 are $(\alpha_0, 2^{-10}\varepsilon^2)$–uniform subsets of $\Lambda + s_1$, $\Lambda + s_2$, respectively, $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$, $\varepsilon = (2^{-100}\alpha_0^2)/(100d)$. Suppose that A is a subset of $\mathbf{E}$, $\delta_{\mathbf{E}}(A) = \delta$, and A has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$. Let*

$$\log N \geq 2^{1000000} (2^{250000} \delta^{-20000} \beta^{-200} + d)^3 \log \frac{1}{\delta \beta \varepsilon_0} \,. \tag{6.1}$$

Then there is a Bohr set $\tilde{\Lambda}$ and a vector $\vec{y} = (y_1, y_2) \in G \times G$ with the following properties : there exist sets $E_1' \subseteq (E_1 - y_1 \cap \tilde{\Lambda})$, $E_2' \subseteq (E_2 - y_2 \cap \tilde{\Lambda})$ such that

1) *Let $|E_1'| = \beta_1' |\tilde{\Lambda}|, |E_2'| = \beta_2' |\tilde{\Lambda}|$ and $\beta' = \beta_1' \beta_2'$. Then $\beta' \geq 2^{-1500}\delta^{100}\beta$.*
2) *E_1', E_2' are $(\alpha_0', 2^{-10}\varepsilon'^2)$–uniform, where $\alpha_0' = 2^{-2000}\delta^{96}\beta'^{48}$, $\varepsilon' = \frac{2^{-100}\alpha_0'^2}{100D'}$, $D \leq D' = 2^{250000}\delta^{-20000}\beta^{-200} + d$.*
3) *For $\tilde{\Lambda} = \Lambda(\tilde{S}, \tilde{\varepsilon})$ we have $|\tilde{S}| = D$, and $\tilde{\varepsilon} \geq (2^{-100}\varepsilon'^2)^D \varepsilon_0$.*
4) *$\delta_{E_1' \times E_2'}(A) \geq \delta + 2^{-600}\delta^{22}$.*

Proof of Theorem 1.4. Suppose that $A \subseteq G \times G$, $|A| = \delta N^2$ and A has no triples $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$.

The proof of Theorem 1.4 is a sort of an algorithm.

After the ith step of the algorithm a vector $\vec{s}_i = (s_i^{(1)}, s_i^{(2)})$ and sets : a regular Bohr set $\Lambda_i = \Lambda(S_i, \varepsilon_i)$, sets $E_i^{(1)} - s_i^{(1)} \subseteq \Lambda_i$, $E_i^{(2)} - s_i^{(2)} \subseteq \Lambda_i$, will be constructed. Let $|E_i^{(1)}| = \beta_i^{(1)} |\Lambda_i|$, $|E_i^{(2)}| = \beta_i^{(2)} |\Lambda_i|$, $\beta_i = \beta_i^{(1)} \beta_i^{(2)}$, $\mathbf{E_i} = E_i^{(1)} \times E_i^{(2)}$.

The sets Λ_i, $E_i^{(1)}$, $E_i^{(2)}$ satisfy the following conditions

1) $\beta_i \geq 2^{-1500}\delta^{100}\beta_{i-1}$.
2) $E_i^{(1)}, E_i^{(2)}$ are $(\alpha_0^{(i)}, 2^{-10}(\varepsilon_i')^2)$–uniform, $\alpha_0^{(i)} = 2^{-2000}\delta^{96}\beta_i^{48}$, $\varepsilon_i' = 2^{-100}(\alpha_0^{(i)})^2/(100 d_i)$.
3) $\Lambda_i = \Lambda(S_i, \varepsilon_i), |S_i| = d_i, d_i \leq 2^{250000}\delta^{-20000}\beta_{i-1}^{-200} + d_{i-1}, \varepsilon_i \geq (2^{-100}(\varepsilon_i')^2)^{d_i}\varepsilon_{i-1}$.
4) $\delta_{\mathbf{E_i}}(A') \geq \delta_{\mathbf{E_{i-1}}}(A') + 2^{-600}\delta^{22}$.

Proposition 6.1 allows us to carry the $(i + 1)$th step of the algorithm. By this Proposition there exists a new vector $\vec{s}_{i+1} = (s_{i+1}^{(1)}, s_{i+1}^{(2)}) \in G \times G$ and sets : a regular Bohr set $\Lambda_{i+1} = \Lambda(S_{i+1}, \varepsilon_{i+1})$, sets $E_{i+1}^{(1)} - s_i^{(1)} \subseteq \Lambda_{i+1}$, $E_{i+1}^{(2)} - s_{i+1}^{(2)} \subseteq \Lambda_{i+1}$, $\mathbf{E_{i+1}} = E_{i+1}^{(1)} \times E_{i+1}^{(2)}$, which satisfy 1) — 4).

Put $S_0 = \{0\}$, $\Lambda_0 = \Lambda(S_0, 1)$ and $E_1 = E_2 = G$, $\beta_0 = 1$. Clearly, E_1, E_2 are $(2^{-2000}\delta^{96}, 2^{-10000}\delta^{400})$–uniform. Hence we have constructed zeroth step of the algorithm.

Let us estimate the total number of steps of our procedure. For an arbitrary i we have $\delta_{\mathbf{E_i}}(A') \leq 1$. Using this and condition 4), we obtain that the total number of steps cannot be more then $2^{700}\delta^{-21} = K$.

Condition 3) implies $\beta_i \geq (2^{-1500}\delta^{100})^i$. Hence $d_i \leq (C_1\delta)^{-C_1' i}$, where $C_1, C_1' > 0$ are absolute constants.

To prove Theorem 1.4, we need to verify condition (6.1) at the last step of the algorithm. Condition (6.1) can be rewrite as

$$N \geq (C_2'\delta)^{-C_3'\delta^{-C_4' K}} = \exp(\delta^{-C'\delta^{-21}}) \,, \tag{6.2}$$

where $C_2', C_3', C_4', C' > 0$ are absolute constants. By assumption

$$\delta \gg \frac{1}{(\log\log N)^{1/22}}$$

and we get (6.2). Hence A' has a triple $\{(k,m),(k+d,m),(k,m+d)\}$, where $d \neq 0$. This contradiction concludes the proof.

NOTE. Certainly, the constant 14 in Theorem 1.4 can be slightly decreased. Nevertheless, it is the author's opinion that this constant cannot be lowered to anything like 1 without a new idea.

Using the following lemma of B. Green (see e.g. [**27**] or [**13**]) one can obtain a corollary of Theorem 1.4 concerning subsets of $\{-N,\dots,N\}^2$ without corners (see details in [**27**]).

LEMMA 6.1. *Let N be a natural number. Suppose A is a subset of $\{-N,\dots,N\}^2$, $|A| = \delta(2N+1)^2$, and A has no triples $\{(k,m),(k+d,m),(k,m+d)\}$ with $d > 0$. Then there exists a set $A_1 \subseteq A$ such that*
1) $|A_1| \geq \delta^2(2N+1)^2/4$ and
2) A_1 has no triples $\{(k,m),(k+d,m),(k,m+d)\}$ with $d \neq 0$.

COROLLARY 6.1. *Let $\delta > 0$, and $N \gg \exp\exp(\delta^{-43})$. Let A be a subset of $\{1,\dots,N\}^2$ of cardinality at least δN^2. Then A contains a triple $\{(k,m),(k+d,m),(k,m+d)\}$ with $d > 0$.*

References

1. M. AJTAI and E. SZEMERÉDI, 'Sets of lattice points that form no squares', *Stud. Sci. Math. Hungar.* **9** (1974) 9–11.
2. F.A. BEHREND, 'On sets of integers which contain no three terms in arithmetic progression', *Proc. Nat. Acad. Sci.* **23** (1946) 331–332.
3. J. BOURGAIN, 'On triples in arithmetic progression', *Geom. Funct. Anal.* **9** (1999) 968–984.
4. J. BOURGAIN, 'Roth's Theorem on Progressions Revisited', preprint.
5. T.C. BROWN and J.C. BUHLER, 'A density version of a geometric Ramsey theorem', *J. Combin. Theory Ser. A* **32** (1982) 20–34.
6. F.R.K. CHUNG, R.L. GRAHAM and R.M. WILSON, 'Quasi–random graphs', *Combinatorica* **9**, N 4 (1989) 345–362.
7. P. ERDÖS and P. TURÁN, 'On some sequences of integers', *J. London Math. Soc.* **11** (1936) 261–264.
8. P. FRANKL, G. GRAHAM and V. RÖDL, 'On sets of abelian groups with no 3–term arithmetic progressions', *J. Combin. Theory Ser. A* **45** (1987) 157–161.
9. G.R. FREIMAN, 'Foundations of a Structural Theory of Set Addition' (Kazan Gos. Ped. Inst., Kazan, 1966, in Russian).
10. H. FURSTENBERG, 'Recurrence in ergodic theory and combinatorial number theory' (Princeton N.J., 1981).
11. W.T. GOWERS, 'A new proof of Szemerédi's theorem' *Geom. Funct. Anal.* **11**, N 3 (2001) 465–588.
12. W.T. GOWERS, 'Rough structure and classification', *Geom. Funct. Anal.* Spec. Vol (2000).
13. W.T. GOWERS, 'Quasirandomness, counting and regularity for 3–uniform hypergraphs', *Combin. Probab. Comput.* **15**:1–2 (2006) 143–184.
14. B. GREEN, 'A Szemerédi–type regularity lemma in abelian groups', *Geom. Funct. Anal.* **15**, N 2 (2005) 340–376.
15. B. GREEN, 'Finite field models in additive number theory' *Surveys in Combinatorics 2005*, LMS lecture notes 327, 1–29.
16. B. GREEN and T. TAO, 'The primes contain arbitrarily long arithmetic progressions' to appear, Annals of Math., available at http:// www.arXiv:math.NT/0404188 (2004).

17. B. Green and T. Tao, 'An inverse theorem for the Gowers $U^3(G)$ norm, with applications', preprint, available at http:// www.arXiv : math.NT/0503014 v1 1 Mar 2005.
18. M.T. Lacey and W. McClain, 'On an argument of Shkredov on two–dimensional corners', preprint.
19. V.F. Lev, 'Progressions–free sets in finite abelian groups', preprint.
20. R. Meshulam, 'On subsets of finite abelian groups with no 3–term arithmetic progressions', *J. Combin. Theory Ser. A* **71** (1995) 168–172.
21. R.A. Rankin, 'Sets of Integers Containing not more than a Given Number of Terms in Arithmetic Progression', *Proc. Roy. Soc. Edinburgh* **65**, N 4, Sec. A (1961) 332–344.
22. K.F. Roth, 'On certain sets of integers', *J. London Math. Soc.* **28** (1953) 245–252.
23. G.N. Sárközy and S. Selkow, 'On a question of Gowers concerning isosceles right–angle triangles', preprint.
24. I.D. Shkredov, 'On one problem of Gowers', *Izvestiya of Russian Academy of Sciences* **70**:2 (2006), 176–217; English transl. Izvestiya : Mathematics **70**:2 (2006), 385–425.
25. I.D. Shkredov, 'On one problem of Gowers', *Doklady of Russian Academy of Sciences* **400**, N 2 (2005) 169–172.
26. I.D. Shkredov, 'On a Generalization of Szemerédi's Theorem', *Doklady of Russian Academy of Sciences* **405**:3 (2005), 315–319; English transl. Doklady Mathematics **72**:3 (2005), 899–902.
27. I.D. Shkredov, 'On a Generalization of Szemerédi's Theorem', *Proceedings London Math. Soc.* **93**:3 (2006), 723–760.
28. J. Solymosi, 'Note on a generalization of Roth's theorem', Discrete and computational geometry 825–827, Algorithms Combin., 25, Springer, Berlin, (2003).
29. E. Szemerédi, 'On sets of integers containing no k elements in arithmetic progression', *Acta Arith.* **27** (1975) 299–345.
30. T. Tao, 'Lecture notes 5 for Math 254A', UCLA 2003, available at http://math.ucla.edu∼tao/254a.1.03w/notes5.dvi
31. T. Tao, V.H. Vu, 'Additive Combinatorics' CUP 2006.
32. V.H. Vu, 'On a question of Gowers', *Ann. of Combinatorics* **6** (2002) 229–233.
33. B.L. van der Waerden, 'Beweis einer Baudetschen Vermutung', *Nieuw Arch. Wisk.* **15** (1927) 212–216.

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