Doklady Mathematics, Vol. 71, No. 1, 2005, pp. 46–48. Translated from Doklady Akademii Nauk, Vol. 400, No. 2, 2005, pp. 169–172.

MATHEMATICS

On a Problem of Gowers

I. D. Shkredov

Presented by Academician V.V. Kozlov May 26, 2004

Received May 27, 2004

1. INTRODUCTION

In 1927, Van der Waerden proved his celebrated theorem about arithmetic progressions [1].

Theorem 1. *Let h and k be positive integers. There exists a number* $N(h, k)$ *such that, for any positive integer* $N \geq N(h, k)$ *and any partitioning of the set* $1, 2, \ldots, N$ *into h subsets, one of the subsets contains an arithmetic progression of length k.*

Let N be a positive integer. We set

$$a_k(N) = \frac{1}{N}\max\{|A|: A \subseteq [1, N] \text{ and}$$
$$A \text{ contains no arithmetic progressions of length } k\},$$

where $|A|$ denotes the cardinality of A. In [2], Erdös and Turan conjectured that any set of positive density contains an arithmetic progression of a given length. In other words, they assumed that, for any $k \geq 3$,

$$a_K(N) \to 0 \quad \text{as} \quad N \to \infty. \tag{1}$$

Clearly, this conjecture implies the Van der Waerden theorem.

For the simplest case of $k = 3$, conjecture (1) was proved in [3] by Roth, who applied the Hardy–Littlewood method to show that $a_3(N) \ll \frac{1}{\ln\ln N}$. At present, the best upper bound for $a_3(N)$ is due to Bourgain [4], who proved that

$$a_3(N) \ll \sqrt{\frac{\ln\ln N}{\ln N}}. \tag{2}$$

For an arbitrary k, conjecture (1) was proved by Szemeredi in 1975 [5]. However, Szemeredi's proof uses difficult combinatorial arguments. An alternative proof was suggested by Furstenberg in [12]. His approach uses the methods of ergodic theory. Furstenberg showed that Szemeredi's theorem is equivalent to the multiple recurrence of almost all the points in any dynamical system.

Behrend [7] obtained the following lower bound for $a_3(N)$:

$$a_3(N) \gg \exp\left(-C(\ln N)^{\frac{1}{2}}\right),$$

where C a computable positive constant. A lower bound on $a_k(N)$ for an arbitrary k is given in [8].

Unfortunately, Szemeredi's methods give very approximate upper estimates for $a_k(N)$, while the ergodic approach gives no estimates at all. Only in 2001 did Gowers [6] obtain a quantitative result concerning the rate at which $a_k(N)$ approaches zero for $k \geq 4$. He proved the following theorem.

Theorem 2. *Suppose that* $\delta > 0$, $k \geq 4$, *and* $N \geq \exp\exp(C\delta^{-K})$, *where* C, $K > 0$ *are effective constants. Let A be any subset of* $\{1, 2, \ldots, N\}$ *with* $|A| \geq \delta N$.

Then, A contains an arithmetic progression of length k.

In other words, Gowers proved that, for any $k \geq 4$, $a_k(N) \ll \frac{1}{(\ln\ln N)^{c_k}}$, where the constant c_k depends only on k.

In this paper, we solve the following problem: Consider the two-dimensional lattice $[1, N]^2$ with the basis $\{(1, 0), (0, 1)\}$. Let

$$L(N) = \frac{1}{N^2}\max\{|A|: A \subseteq [1, N^2] \text{ and}$$
$$A \text{ contains no triples of the form}$$
$$(k, m), (k + d, m), (k, m + d), \text{ where } d > 0\}. \tag{3}$$

We refer to triples of the form specified in (3) as corners. In [9, 12], it was proved that $L(N)$ tends to 0 as N tends to infinity. Gowers (see [6]) then asked what the rate of convergence of $L(N)$ to 0 was.

In [11], Vu suggested the following solution this problem. We set $\ln_{[1]} = \ln N$ and $\ln_{[l]}N = \ln(\ln_{[l-1]}N)$ for $l \geq 2$. Let k be the maximum positive integer such that $\ln_{[k]}N \geq 2$. We then set $\ln_* N = k$. Vu proved that

$$L(N) \leq \frac{100}{\ln_*^{1/4} N}.$$

The main result of this paper is the following theorem.

Moscow State University, Leninskie gory, Moscow, 119992 Russia
e-mail: ishkredov@rambler.ru

Theorem 3. *Suppose that* $\delta > 0$; $N \geq \exp\exp\exp(\delta^{-c})$, *where* $c > 0$ *is an absolute constant; and* $A \subseteq \{1, 2, \ldots, N\}^2$ *is an arbitrary subset with a cardinality of, at most,* δN^2.

Then, A contains a triple of the form (k, m), $(k + d, m)$, $(k, m + d)$, *where* $d > 0$.

Thus, we prove the estimate $L(N) \ll \frac{1}{(\ln\ln\ln N)^{C_1}}$, where $C_1 > 0$ is an absolute constant.

Moreover, we obtain the simplest lower bound for $L(N)$.

Proposition 1. *For any* $\varepsilon > 0$, *there exists an* $N_\varepsilon \in \mathbf{N}$ *such that*

$$L(N) \geq N^{-\frac{\ln 2 + \varepsilon}{\ln\ln N}}$$

for all the positive integers $N \geq N_\varepsilon$.

The constructions that we have used develop the approach of [6, 10].

2. SCHEME OF THE PROOF

First, we give several definitions (see [6]).

Let A be an arbitrary set from $\mathbf{Z}_N$ with $|A| = \delta N$. Let us denote the characteristic function of A by $\chi_A(s)$. The function $f(s) = \chi_A(s) - \delta$ is called the balance function of the set A. Let D denote the closed disk of radius 1 centered at 0 in the complex plane.

Definition 1. A function f from $\mathbf{Z}_N$ to D is said to be α uniform if

$$\sum_k \left| \sum_s f(s)\overline{f(s-k)} \right|^2 \leq \alpha N^3. \qquad (4)$$

We say that a set A is α uniform if its balance function is α uniform.

Let $E_1 \times E_2$ be a subset of $\mathbf{Z}_N^2$, and let $f: \mathbf{Z}_N^2 \to D$ be a function. We write $f: E_1 \times E_2 \to D$ if f is zero outside $E_1 \times E_2$.

Definition 2. Let $\alpha \in [0, 1]$ be a real number, and let $E_1 \times E_2$ be a subset of $\mathbf{Z}_N^2$. A function $f: E_1 \times E_2 \to D$ is said to be α uniform with respect to a basis $(\mathbf{e}_1, \mathbf{e}_2)$ if

$$\sum_{\mathbf{s}, \mathbf{u}} \sum_r f(\mathbf{s})\overline{f(\mathbf{s} + u\mathbf{e}_2) f(\mathbf{s} + r\mathbf{e}_1)} f(\mathbf{s} + u\mathbf{e}_2 + r\mathbf{e}_1) \leq \alpha |E_1|^2 |E_2|^2. \qquad (5)$$

Suppose that a set A is contained in some $E_1 \times E_2$ and $\chi(\mathbf{s}) = \chi_A(\mathbf{s})$ is its characteristic function. Let us define the densities $\delta_m = \delta_m^{\mathbf{e}_1}$ and $\gamma_k = \gamma_k^{\mathbf{e}_2}$ as

$$\delta_m = \frac{1}{|E_1|} \sum_{p=1}^{N} \chi(m\mathbf{e}_2 + p\mathbf{e}_1),$$

$$\gamma_k = \frac{1}{|E_2|} \sum_{p=1}^{N} \chi(k\mathbf{e}_1 + p\mathbf{e}_2).$$

The function $f(\mathbf{s}) = (\chi(\mathbf{s}) - \delta_m)\chi_{E_1 \times E_2}(\mathbf{s})$ is called the balance function of the set A.

We say that a set $A \subseteq E_1 \times E_2$ is α uniform with respect to a basis $(\mathbf{e}_1, \mathbf{e}_2)$ if its balance function is α uniform with respect to this basis.

Our first proposition is concerned with the situation where $A \subseteq E_1 \times E_2$ is α uniform with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$ and the sets E_1 and E_2 themselves are α uniform in the sense of Definition 4.

Theorem 4. *Suppose that a set A is contained in* $E_1 \times E_2$ *and has the cardinality* $|A| = \delta|E_1| \cdot |E_2|$, $|E_1| = \beta_1 N$, $|E_2| = \beta_2 N$, *and the sets* E_1 *and* E_2 *are* $10^{-330}\beta_1^{24}\beta_2^{24}\delta^{132}$ *uniform. In addition, suppose that A is* α *uniform with respect to the basis* $(\mathbf{e}_1, \mathbf{e}_2)$, $\alpha = 10^{-108}\delta^{44}$, $N \geq 10^{10}(\delta^4\beta_1\beta_2)^{-1}$, *and* $\sum_m |\delta_m - \delta|^2 \leq \alpha\beta_2 N$.

Then, A contains a corner.

If $A \subseteq E_1 \times E_2$ is not α uniform and E_1 and E_2 are arbitrary, then the following assertion is valid.

Proposition 2. *If* $A \subseteq E_1 \times E_2$ *has the cardinality* $\delta|A| = |E_1| \cdot |E_2|$, $\alpha > 0$ *is a real number, and A is not* α *uniform with respect to the basis* $(\mathbf{e}_1, \mathbf{e}_2)$, *then there exist the sets* $G_1 \subseteq E_1$ *and* $G_2 \subseteq E_2$ *such that*

$$|A \cap (G_1 \times G_2)| > (\delta + 2^{-500}\alpha^{70})|G_1| \cdot |G_2|, \qquad (6)$$

$$|G_1|, |G_2| > 2^{-500}\alpha^{70}\min\{|E_1|, |E_2|\}. \qquad (7)$$

We say that a two-dimensional arithmetic progression P is a regular square if $P = P_1 \times P_2$, where P_1 and P_2 are one-dimensional progressions of the same cardinality with equal differences.

Proposition 3. *Suppose that* $W_1, W_2 \subseteq \mathbf{Z}_N$ *are sets;* $|W_1| = \beta_1 N$; $|W_2| = \beta_2 N$; $\zeta \in (0, 1)$ *is a number;* $\alpha(s) = Ks^\rho$, $K \in (0, 1]$, $\rho \geq 4$; *and* $a = \alpha(\zeta\beta_1\beta_2)$. *In addition, suppose that a set A is contained in* $W_1 \times W_2$, $|A| = \delta|W_1| \cdot |W_2|$, *and* $N \geq (Ca^{c_1})^{-(1/c_2)^{1/a}}$, *where* $C = 2^{1000\rho}$, $c_1 = 100\rho$, *and* $c_2 = 2^{-128}$.

Then, there exist a regular square $P = P_1 \times P_2$, *where* $|P| \geq N^{c_2^{1/a}}$, *and sets* R_1 *and* R_2 *such that* $R_1 \subseteq (W_1 \cap P_1)$; $R_2 \subseteq (W_2 \cap P_2)$; $|R_1 \times R_2| \geq \zeta\beta_1\beta_2|P|$; R_1 *and* R_2 *are*

$\alpha(\delta_{P_1}(R_1))^{1/2}$ *and* $\alpha(\delta_{P_2}(R_2))^{1/2}$ *uniform in* P_1 *and* P_2, *respectively*; *and* $\delta_{R_1 \times R_2}(A) \geq \delta - 4\zeta$.

Scheme of the proof of Theorem 3. Suppose that $N_1 \in \mathbf{N}$ and $J_1, J_2 \subseteq \mathbf{Z}_{N_1}$ are sets such that $|J_1| = \omega_1 N_1$ and $|J_2| = \omega_2 N_1$. In addition, suppose that the set A is contained in $J_1 \times J_2$, contains no corners, and has the cardinality $|A| = \delta|J_1| \cdot |J_2|$. Finally, suppose that J_1 and J_2 are $10^{-330}\omega_1^{24}\omega_2^{24}\delta^{132}$ uniform and $N_1 \geq 10^{10}(\delta^4\omega_1\omega_2)^{-1}$. Let us prove that there exist subsets J_1 and J_2 of the sets I_1 and I_2 and a subset A' of the set A such that

(i) $A' \subseteq I_1 \times I_2$,

(ii) $|A'| \geq (\delta + 10^{-10000}\delta^{3500})|I_1| \cdot |I_2|$,

(iii) $|I_1|, |I_2| \geq 10^{-10000}\delta^{3500}\min(\omega_1 N_1, \omega_2 N_1)$.

Let $\alpha_1 = 10^{-108}\delta^{44}$. If $\sum_m |\delta_m - \delta|^2 > \alpha\omega_2 N$, then the existence of the sets I_1, I_2, and A' with the required properties is obvious. If the set A is $10^{-108}\delta^{44}$ uniform with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$, then A contains a corner by Theorem 4. If the set A is not $10^{-108}\delta^{44}$ uniform with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$, then, by Proposition 2, the sets J_1 and J_2 have the subsets I_1 and I_2 such that these subsets and the subset $A' = A \cap (I_1 \times I_2)$ of A satisfy conditions (i)–(iii). Moreover, condition (ii) can be replaced by the stronger condition $|A'| \geq (\delta + 10^{-9000}\delta^{3500})|I_1| \cdot |I_2|$.

We then proceed to prove the theorem. Suppose that the set $A \subseteq \{1, 2, \ldots, N\}^2$ has the cardinality δN^2 and contains no corners. We set $E_1 = \{1, 2, \ldots, N\}$ and $E_2 = \{1, 2, \ldots, N\}$. Then E_1 and E_2 are 0 uniform. By assumption, $N \geq 10^{10}\delta^{-4}$. Arguing in the same way as at the beginning of the proof of Theorem 3, we find the subset A' of A, $G_1 \subseteq E_1$, and $G_2 \subseteq E_2$ such that conditions (i)–(iii) hold, $|G_1| = \beta_1 N$, and $|G_2| = \beta_2 N$. The set A', as well as A, contains no corners, and its density δ_1 in $G_1 \times G_2$ is at least $\delta + 10^{-9000}\delta^{3500}$.

Let $\zeta = 10^{-10000}\delta^{3500}$. Consider the function $\alpha(s) = 10^{-660}(\zeta\beta_1\beta_2)^{48}\delta^{264}s^{48}$, and let $a = \alpha(\zeta\beta_1\beta_2)$. According to the conditions of the theorem, $N \geq (Ca^{c_1})^{-(1/c_2)^{1/a}}$; therefore, we can apply Proposition 3 to the sets G_1, G_2, and A'. According to this proposition, there exist a regular square $P = P_1 \times P_2$, with $|P| \geq N^{c_2^{1/a}}$, and the sets R_1 and R_2 such that $R_1 \subseteq (G_1 \cap P_1)$; $R_2 \subseteq (G_2 \cap P_2)$; $|R_1| = \gamma_1|P_1|$; $|R_2| = \gamma_2|P_2|$; $|R_1 \times R_2| \geq \zeta\beta_1\beta_2|P|$; R_1 and R_2 are $10^{-330}\gamma_1^{24}\gamma_2^{24}\delta^{132}$ uniform in P_1 and P_2, respectively; and $\delta_{R_1 \times R_2}(A') \geq \delta_1 - 4\zeta$. The density of A in $R_1 \times R_2$ is at least $\delta_1 - 4 \cdot 10^{-10000}\delta^{3500} \geq \delta + 10^{-10000}\delta^{3500}$.

We apply the same argument to the regular square P; the $10^{-330}\gamma_1^{24}\gamma_2^{24}\delta^{132}$ uniform sets R_1 and R_2, where $R_1 \times R_2 \subseteq P$; and the set $A'' = A' \cap (R_1 \times R_2)$. We can repeat this procedure several times.

Since the density of A in the sets $R_1^{(i)} \times R_2^{(i)}$ increases by $10^{-10000}\delta^{3500}$ at each step, the density of A in these sets tends to 1. In other words, after a certain number of steps, we obtain a regular square $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$ and $10^{-330}\pi_1^{24}\pi_2^{24}\delta^{132}$ uniform sets $\mathbf{R}_1$ and $\mathbf{R}_2$ such that $|\mathbf{R}_1| = \pi_1|\mathbf{P}_1|$, $|\mathbf{R}_2| = \pi_2|\mathbf{P}_2|$, and $A \cap (\mathbf{R}_1 \times \mathbf{R}_2)$ is $10^{-108}\delta^{44}$ uniform with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$ in $\mathbf{R}_1 \times \mathbf{R}_2$. If

$$|\mathbf{P}_1| = |\mathbf{P}_2| \geq 10^{10}(\delta^4\pi_1\pi_2)^{-1}, \tag{8}$$

then A contains a corner by Theorem 4. It is easy to verify that the total number of steps in this iterative procedure does not exceed $O(\delta^{-\bar{c}})$, where $\bar{c} > 0$ is an absolute constant. Moreover, the cardinalities of the progressions $P^{(i)}$ and $P^{(i+1)}$ at the ith and $(i + 1)$th steps are related by

$$|P^{(i+1)}| \geq |P^{(i)}|^{\kappa_0^{(1/\delta)^{\delta^{-K}}}},$$

where $0 < \kappa_0 < 1$ and $K > 0$. We have $N \geq \exp\exp\exp(\delta^{-c})$. A simple calculation shows that this condition ensures the fulfillment of inequality (8) at the last step of the procedure.

ACKNOWLEDGMENTS

The author is very grateful to Doctor N.G. Moshchevitin for his invariable interest in this work and his patience.

This work was supported by the program "Leading Scientific Schools" (project no. 136.2003.1) and INTAS (grant no. 03-51-5070).

REFERENCES

1. B. L. Van der Waerden, Nieuw Arch. Wisk, No. 15, 212–216 (1927).
2. P. Erdös and P. Turan, J. London Math. Soc., No. 11, 261–264 (1936).
3. K. F. Roth, J. London Math. Soc., No. 28, 245–252 (1953).
4. J. Bourgain, Geom. Funct. Anal., No. 9, 968–984 (1999).
5. E. Szemeredi, Acta Arith., No. 27, 299–345 (1975).
6. W. T. Gowers, Geom. Funct. Anal. **11**, 465–588 (2001).
7. F. A. Behrend, Proc. Nat. Acad. Sci. USA, No. 23, 331–332 (1946).
8. R. A. Rankin, Proc. Roy. Soc. Edinburgh Sec. A **65** (4), 332–344 (1961).
9. M. Ajtai and E. Szemeredi, Stud. Sci. Math. Hung., No. 9, 9–11 (1974).
10. F. R. K. Chung, R. L. Graham, and R. M. Wilson, Combinatorica, No. 9 (4), 345–362 (1989).
11. V. H. Vu, Ann. Combin., No. 6, 229–233 (2002).
12. H. Furstenberg, *Reccurence in Ergodic Theory and Combinatorial Number Theory* (Princeton Univ. Press, Princeton, N. J., 1981).