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MATHEMATICS

On a Generalization of Szemeredi's Theorem

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1. Let N be a positive integer number. Define

$$a_k(N) = \frac{1}{N}\max\{|A|: A \subseteq [1, N] \text{ and } A \text{ contains no arithmetic progressions of length } k\},$$

where $|A|$ denotes the cardinality of A. In [1], Erdös and Turan conjectured that any set of positive density contains an arithmetic progression of a given length. In other words, they assumed that, for any $k \geq 3$,

$$a_k(N) \to 0 \text{ as } N \to \infty. \quad (1)$$

In the simplest case $k = 3$, conjecture (1) was proved in [2] by Roth, who used the Hardy–Littlewood method to show that $a_3(N) \ll \frac{1}{\ln\ln N}$. At present, the best upper bound for $a_3(N)$ is due to Bourgain [3], who proved that

$$a_3(N) \ll \sqrt{\frac{\ln\ln N}{\ln N}}. \quad (2)$$

For an arbitrary k, conjecture (1) was proved by Szemeredi in 1975 [4]. However, Szemeredi's proof is based on difficult combinatorial arguments. An alternative proof was proposed by Furstenberg in [9]. His approach uses the methods of ergodic theory.

Unfortunately, Szemeredi's methods give very approximate upper bounds for $a_k(N)$, while the ergodic approach yields no bounds at all. Only in 2001 did Gowers [5] obtain a quantitative result on the rate at which $a_k(N)$ approaches zero for $k \geq 4$. Specifically, he proved the following result.

Theorem 1. *Let $k \geq 4$. Then*

$$a_k(N) \ll \frac{1}{(\ln\ln N)^{c_k}},$$

where $c_k > 0$ is an effective constant depending only on k.

In this paper, we solve the following problem. Consider the two-dimensional lattice $[1, N]^2$ with the base $\{(1, 0), (0, 1)\}$. Let

$$L(N) = \frac{1}{N^2}\max\{|A|: A \subseteq [1, N]^2 \text{ and } A \text{ contains no triples of the form } (k, m), (k+d, m), (k, m+d), d > 0\}. \quad (3)$$

Triples from (3) will be referred to as corners. In [6, 9], it was proved that $L(N)$ tends to 0 as N approaches infinity. Gowers (see [5]) then asked what the rate of convergence of $L(N)$ to 0 is.

The following result was proved in [10] (see also [7, 8]).

Theorem 2. *Let $\delta > 0$; $N \gg \exp\exp\exp(\delta^{-c})$, where $c > 0$ is an effective constant; and $A \subseteq \{1, 2, \ldots, N\}^2$ be an arbitrary subset with a cardinality of, at least, δN^2.*

Then A contains a corner.

Below is the main result of this paper.

Theorem 3. *Let $\delta > 0$; $N \gg \exp\exp(\delta^{-c})$, where $c > 0$ is an effective constant; and $A \subseteq \{1, \ldots, N\}^2$ be an arbitrary subset with a cardinality of, al least, δN^2.*

Then A contains a corner.

Thus, we prove the bound $L(N) \ll \frac{1}{(\ln\ln N)^{C_1}}$, where $C_1 = \frac{1}{c}$.

Remark 1. The constant c in Theorem 3 can be set equal to 73.

The fundamental difference between this paper and [10] is that we now consider a new object—the so-called Bohr set. Note that today's best bound for $a_3(N)$ is proved using these sets (see [3]).

2. Proof sketch. Let Q be a subset of integers. The same letter Q will denote the characteristic function of this set.

A key point in [3] was the concept of Bohr sets.

Suppose that N and d are positive integers, $\varepsilon > 0$ is a real number, and $\theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbf{T}^d$.

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Definition 1. The Bohr set $\Lambda = \Lambda_{\theta, \varepsilon, N}$ is defined as

$$\Lambda_{\theta, \varepsilon, N} = \{n \in \mathbf{Z} | \ |n| \le N, \\ \|n\theta_j\| < \varepsilon \ \text{ for } j = 1, 2, \ldots, d\}.$$

Definition 2. Let $0 < \kappa < 1$ be a number. A Bohr set $\Lambda = \Lambda_{\theta, \varepsilon, N}$ is called regular if, for any ε' and N' such that

$$|\varepsilon - \varepsilon'| < \frac{\kappa}{100d}\varepsilon, \quad |N - N'| < \frac{\kappa}{100d}N,$$

we have

$$1 - \kappa < \frac{|\Lambda_{\theta, \varepsilon', N'}|}{|\Lambda_{\theta, \varepsilon, N}|} < 1 + \kappa.$$

The first lemma we need was proved in [3].

Lemma 1. *Let* $0 < \kappa < 1$ *be a number and* $\Lambda_{\theta, \varepsilon, N}$ *be a Bohr set.*

Then there is a pair (ε_1, N_1) *with the properties*

$$\frac{\varepsilon}{2} < \varepsilon_1 < \varepsilon, \quad \frac{N}{2} < N_1 < N$$

such that the $\Lambda_{\theta, \varepsilon_1, N_1}$ *is regular.*

Definition 3. Let $\varepsilon \in (0, 1]$ be a number and $\Lambda_{\theta, \varepsilon_0, N_0}$ be a Bohr set, where $\theta = (\theta_1, \theta_2, \ldots, \theta_d)$. A regular Bohr set $\Lambda' = \Lambda_{\theta', \varepsilon', N'}$ is said to be ε-accompanying for Λ if $\theta' = (\theta_1, \theta_2, \ldots, \theta_d, \theta_{d+1}, \ldots, \theta_{d+k})$ for $k \ge 0$, $\frac{\varepsilon\varepsilon_0}{2} \le \varepsilon' \le \varepsilon\varepsilon_0$, and $\frac{\varepsilon N_0}{2} \le N' \le \varepsilon N_0$. The existence of such a set follows from Lemma 1.

It is assumed that $k = 0$ unless otherwise stated.

Suppose that f is a function from $\mathbf{Z}$ in $\mathbf{C}$ that takes a final number of nonzero values. Let $\hat{f}(x)$ denote the Fourier coefficient of f:

$$\hat{f}(x) = \sum_{s \in \mathbf{Z}} f(s)e(-sx),$$

where $e(x) = e^{2\pi i x}$.

Definition 4. Let Λ be a Bohr set, $Q \subseteq \Lambda$ with $|Q| = \delta|\Lambda|$, α and ε be positive numbers, and Λ' be an ε-accompanying set for Λ. Consider the set

$$B = \{m \in \Lambda | \ \|(Q \cap (\Lambda' + m) - \delta(\Lambda' + m))^{\wedge}\|_\infty \ge \alpha|\Lambda'|\}.$$

The set Q is called (α, ε)-uniform if

$$|B| \le \alpha|\Lambda|, \tag{4}$$

$$\frac{1}{|\Lambda|}\sum_{m \in \Lambda} |\delta_{\Lambda' + m}(Q) - \delta|^2 \le \alpha^2, \tag{5}$$

$$\|(Q \cap \Lambda - \delta\Lambda)^{\wedge}\|_\infty \le \alpha|\Lambda|. \tag{6}$$

A similar definition of (α, ε)-uniformity was given in [5].

Let $\mathbf{D}$ denote the closed disk of radius 1 centered at 0 in the complex plane. For a set R, we write $f: R \to \mathbf{D}$ if f vanishes outside R.

Denote the vectors (1, 0) and (0, –1) by $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.

Let Λ_1 and Λ_2 be Bohr sets, $\varepsilon > 0$ be a number, and Λ' be an ε-accompanying set for Λ_1. Additionally, let E_1 and E_2 be subsets of Λ_1 and Λ_2, respectively, with $|E_1| = \beta_1|\Lambda_1|$ and $E_2 = \beta_2|\Lambda_2|$.

Definition 5. A function $f: \Lambda_1 \times \Lambda_2 \to \mathbf{D}$ is called (α, ε)-uniform with respect to the rectangular norm if

$$\|f\|^4_{\Lambda_1 \times \Lambda_2, \varepsilon} = \sum_{i \in \Lambda_1}\sum_{j \in \Lambda_2}\sum_k\sum_{m, u}\Lambda'(m - k - i)\Lambda'(u - k - i) \\ \times\left|\sum_r \Lambda'(k + r - j)f(r, m)f(r, u)\right|^2 \\ \le \alpha\beta_1^2\beta_2^2|\Lambda'|^4|\Lambda_1|^2|\Lambda_2|. \tag{7}$$

Denote the set Λ' in (7) by $\Lambda_1(\varepsilon)$.

Definition 6. Let $A \subseteq E_1 \times E_2$ with $|A| = \delta\beta_1\beta_2|\Lambda_1||\Lambda_2|$, $\delta > 0$, and $f(\mathbf{s}) = A(\mathbf{s}) - \delta(E_1 \times E_2)(\mathbf{s})$. Let $f_l(\mathbf{s}) = f(s_1 + l, s_2)\Lambda'(s_1)$, where $l \in \Lambda_1$. Consider the set

$$B = \{l \in \Lambda_1 | \ \|f_l\|^4_{\Lambda' \times \Lambda_2, \varepsilon} \\ > \alpha\beta_1^2\beta_2^2|\Lambda'(\varepsilon)|^4|\Lambda'|^2|\Lambda_2|\}.$$

The set A is said to be $(\alpha, \alpha_1, \varepsilon)$-uniform with respect to the rectangular norm if $|B| \le \alpha_1|\Lambda_1|$.

Our first result concerns the case where $A \subseteq E_1 \times E_2$ is a $(\alpha, \alpha_1, \varepsilon)$-uniform set with respect to the rectangular norm and the sets E_1 and E_2 are (α, ε)-uniform in the sense of Definition 4.

Let Λ be a Bohr set; $\Lambda = \Lambda_{\theta, \varepsilon_1, N_1}$ with $\theta \in \mathbf{T}^d$; and $E_1, E_2 \subseteq \Lambda$ with $|E_1| = \beta_1|\Lambda|$ and $|E_2| = \beta_2|\Lambda|$. Denote the Cartesian product $E_1 \times E_2$ by $\mathcal{P}$.

Theorem 4. *Suppose that set A belongs to* $E_1 \times E_2$ *and has the cardinality* $|A| = \delta|E_1||E_2|$. *Let sets* E_1 *and* E_2 *are* $(\alpha_0, 2^{-10}\varepsilon^2)$-*uniform, where* $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$ *and* $\varepsilon = \frac{2^{-100}\alpha_0^2}{100d}$. *Additionally, let A be* $(\alpha, \alpha_1, \varepsilon)$-*uniform with respect to the rectangular norm, where* $\alpha = 2^{-100}\delta^{12}$ *and* $\alpha_1 = 2^{-7}\delta$; *and let*

$$\ln N_1 \ge 2^{10}d\ln\frac{1}{\varepsilon_1\varepsilon}. \tag{8}$$

Then A contains a triple of the form $\{(k, m), (k + d, m), (k, m + d)\}$ *with* $d \ne 0$.

Suppose that $A \subseteq E_1 \times E_2$ contains no triples of the form $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \ne 0$. Applying Theorem 4, we obtain the following result.

Theorem 5. *Suppose that set A belongs to* $\mathcal{P}$, *has the cardinality* $|A| = \delta|E_1|\cdot|E_2|$, *and contains no triples of the form* $\{(k, m), (k + d, m), (k, m + d)\}$ *with* $d \neq 0$. *Let* E_1 *and* E_2 *be* $(\alpha_0, 2^{-10}\varepsilon^2)$-*uniform sets with* $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$ *and* $\varepsilon = \frac{2^{-100}\alpha_0^2}{100d}$, $\varepsilon' = 2^{-10}\varepsilon^2$, *and*

$$\ln N \geq 2^{10} d \ln\frac{1}{\varepsilon_0\varepsilon}.$$

Then there exists a Bohr set $\tilde{\Lambda}$ *and a vector* $\mathbf{y} = (y_1, y_2) \in \mathbf{Z}^2$ *such that, for* $F_1 \subseteq E_1 \cap (\tilde{\Lambda} + y_1)$ *and* $F_2 \subseteq E_2 \cap (\tilde{\Lambda} + y_2)$, *it is true that*

$$|F_1| \geq 2^{-125}\delta^{12}\beta_1|\tilde{\Lambda}|, \quad |F_2| \geq 2^{-125}\delta^{12}\beta_2|\tilde{\Lambda}|, \tag{9}$$

$$\delta_{F_1\times F_2}(A) \geq \delta + 2^{-500}\delta^{37}. \tag{10}$$

Moreover, for $\tilde{\Lambda} = \Lambda_{\tilde{\theta}, \tilde{\varepsilon}, \tilde{N}}$, *it holds that* $\tilde{\theta} = \theta$, $\tilde{\varepsilon} \geq 2^{-5}\varepsilon'\varepsilon_0$, *and* $N \geq 2^{-5}\varepsilon' N$.

Proposition 1. *Let* $\Lambda = \Lambda(\theta, \varepsilon_0, N)$ *be a Bohr set,* $\theta \in \mathbf{T}^d$, *and* $\mathbf{s} = (s_1, s_2)$ *be an integer vector. Suppose that* $\varepsilon, \sigma, \tau, \delta \in (0, 1)$ *be real numbers*; E_1 *and* E_2 *are sets such that* $E_i = \beta_i|\Lambda|$ *for* $i = 1, 2$; $\mathbf{E} = E_1 \times E_2$ *is a subset of* $(\Lambda + s_1) \times (\Lambda + s_2)$; $A \subset \mathbf{E}$; $\delta_{\mathbf{E}}(A) = \delta + \tau$; *and* $\varepsilon \leq \frac{\kappa}{100d}$, *where* $\kappa = 2^{-100}(\tau\beta_1\beta_2)^5\sigma^3$. *It is also supposed that*

$$N \geq (2^{-100}\varepsilon_0\varepsilon)^{-2^{100}((\tau\beta_1\beta_2)^{-5}\sigma^{-3} + d)^2} \tag{11}$$

and $\sigma \leq 2^{-100}\tau\beta_1\beta_2$.

Then there is a Bohr set $\Lambda' = \Lambda(\theta', \varepsilon', N')$, where $\theta' \in \mathbf{T}^D$, $D \leq 2^{30}(\tau\beta_1\beta_2)^{-5}\sigma^{-3} + d$, $\varepsilon' \geq (2^{-10}\varepsilon)^D\varepsilon_0$, and $N' \geq (2^{-10}\varepsilon)^D N$, and there is an integer vector $\mathbf{t} = (t_1, t_2)$ such that the following assertions hold for the sets $E_1' = (E_1 - t_1) \cap \Lambda'$, $E_2' = (E_2 - t_2) \cap \Lambda'$, and $\mathbf{E}' = E_1' \times E_2'$:

(i) $|\mathbf{E}'| \geq \frac{\beta_1\beta_2\tau|\Lambda'|}{16}$;

(ii) E_1' and E_2' are (σ, ε)-uniform subsets of Λ';

(iii) $\delta_{\mathbf{E}'}(A - \mathbf{t}) \geq \delta + \frac{\tau}{116}$.

3. Proof sketch of Theorem 3. The following result is due to B. Green.

Lemma 2. *Suppose that N is a positive integer and a set* $A \subseteq [-N, N]^2$ *with* $|A| = \delta(2N + 1)^2$ *does not contain triples of the form* $\{(k, m), (k + d, m), (k, m + d)\}$ *with* $d > 0$ *in the standard basis* $(1,0), (0, 1)$.

Then there exists a subset $A_1 \subseteq A$ *such that*

(i) $|A_1| \geq \frac{\delta^2(2N+1)^2}{4}$ *and*

(ii) A_1 *does not contain triples of the form* $\{(k, m), (k + d, m), (k, m + d)\}$ *with* $d \neq 0$.

For convenience, Theorem 5 and Proposition 1 can be combined in a single assertion.

Proposition 2. *Let* $\Lambda = \Lambda(\theta, \varepsilon_0, N)$ *be a Bohr set,* $\theta \in \mathbf{T}^d$, *and* $\mathbf{s} = (s_1, s_2)$ *be an integer vector. Suppose that sets* E_1 *and* E_2 *are such that* $E_i = \beta_i|\Lambda|$ *for* $i = 1, 2$, $\beta = \beta_1\beta_2$, $\mathbf{E} = E_1 \times E_2$ *is a subset of* $(\Lambda + s_1) \times (\Lambda + s_2)$; *and* E_1 *and* E_2 *are* $(\alpha_0, 2^{-10}\varepsilon^2)$-*uniform subsets of* $\Lambda + s_1$ *and* $\Lambda + s_2$, *respectively, where* $\alpha_0 = 2^{-2000}\delta^{96}\beta_1^{48}\beta_2^{48}$ *and* $\varepsilon = \frac{2^{-100}\alpha_0^2}{100d}$. *Suppose that a set* $A \subset \mathbf{E}$ *with* $\delta_{\mathbf{E}}(A) = \delta$ *does not contain triples of the form* $\{(k, m), (k + d, m), (k, m + d)\}$ *with* $d \neq 0$. *It is assumed that*

$$\ln N \geq 2^{1\,000\,000}(2^{250000}\delta^{-20\,000}\beta^{-200} + d)^3 \ln\frac{1}{\delta\beta\varepsilon_0}. \tag{12}$$

Then there is a Bohr set $\tilde{\Lambda}$ and a vector $\mathbf{y} = (y_1, y_2) \in \mathbf{Z}^2$ such that the following assertions hold for sets $E_1' \subseteq (E_1 - y_1 \cap \tilde{\Lambda})$ and $E_2' \subseteq (E_2 - y_2 \cap \tilde{\Lambda})$:

(i) Let $|E_1'| = \beta_1'|\tilde{\Lambda}|$, $|E_2'| = \beta_2'|\tilde{\Lambda}|$, and $\beta' = \beta_1'\beta_2'$. Then $\beta' \geq 2^{-1500}\delta^{100}\beta$.

(ii) E_1' and E_2' are $(\alpha_0', 2^{-10}\varepsilon'^2)$-uniform sets, where $\alpha_0' = 2^{-2000}\delta^{96}\beta'^{48}$, $\varepsilon' = \frac{2^{-100}\alpha_0'^2}{100D'}$, and $D \leq D' = 2^{250000} \times \delta^{-20000}\beta^{-200} + d$.

(iii) The Bohr set $\tilde{\Lambda} = \Lambda_{\tilde{\theta}, \tilde{\varepsilon}, \tilde{N}}$ has the properties $\tilde{\theta} \in \mathbf{T}^D$, $\tilde{\varepsilon} \geq (2^{-100}\varepsilon'^2)^D\varepsilon_0$, and $N \geq (2^{-100}\varepsilon'^2)^D N$.

(iv) $\delta_{E_1'\times E_2'}(A) \geq \delta + 2^{-600}\delta^{37}$.

Proof of Theorem 3. Suppose that $A \subseteq [-N, N]$ does not contain triples of the form $\{(k, m), (k + d, m), (k, m + d)\}$ with $d > 0$. Applying Lemma 2, we find a set A' with $A' \subseteq A$ and $|A'| \geq \frac{\delta^2}{4(2N+1)^2}$ that contains no triples of the form $\{(k, m), (k + d, m), (k, m + d)\}$ with $d \neq 0$. In what follows, we deal only with this set. Let $\delta' = \frac{\delta^2}{4}$.

The algorithm for proving Theorem 3 can be described as follows. At the ith step, we construct an integer vector $\mathbf{s}_i = (s_i^{(1)}, s_i^{(2)})$; a regular Bohr set $\Lambda_i = \Lambda_{\theta_i, \varepsilon_i, N_i}$; and the sets $E_i^{(1)} - s_i^{(1)} \subseteq \Lambda_i$ and $E_i^{(2)} - s_i^{(2)} \subseteq \Lambda_i$ with $|E_i^{(1)}| = \beta_i^{(1)}|\Lambda_i|$ and $|E_i^{(2)}| = \beta_i^{(2)}|\Lambda_i|$, where $\beta_i = \beta_i^{(1)}\beta_i^{(2)}$ and $\mathbf{E_i} = E_i^{(1)} \times E_i^{(2)}$. Moreover, the following conditions hold:

(i) $\beta_i \geq 2^{-1500}\delta'^{100}\beta_{i-1}$.

(ii) $E_i^{(1)}$ and $E_i^{(2)}$ are $(\alpha_0^{(i)}, 2^{-10}(\varepsilon_i')^2)$-uniform sets with $\alpha_0^{(i)} = 2^{-2000}\delta'^{96}\beta_i^{48}$ and $\varepsilon_i' = \frac{2^{-100}(\alpha_0^{(i)})^2}{100d_i}$.

(iii) $\Lambda_i = \Lambda_{\theta_i, \varepsilon_i, N_i}$, $\tilde{\theta} \in \mathbf{T}^{d_i}$, $d_i \le 2^{250000}\delta'^{-20000}\beta_{i-1}^{-200} + d_{i-1}$, $\varepsilon_i \ge (2^{-100}(\varepsilon_i')^2)^{d_i}\varepsilon_{i-1}$, $N_i \ge (2^{-100}(\varepsilon_i')^2)^{d_i}N_{i-1}$.

(iv) $\delta_{\mathbf{E_i}}(\mathbf{A}') \ge \delta_{\mathbf{E_{i-1}}}(A') + 2^{-600}\delta'^{37}$.

Next, Proposition 2 yields a new vector $\mathbf{s}_{i+1} = (s_{i+1}^{(1)}, s_{i+1}^{(2)}) \in \mathbf{Z}^2$; a regular Bohr set $\Lambda_{i+1} = \Lambda_{\theta_{i+1}, \varepsilon_{i+1}, N_{i+1}}$; and the sets $E_{i+1}^{(1)} - s_i^{(1)} \subseteq \Lambda_{i+1}$ and $E_{i+1}^{(2)} - s_{i+1}^{(2)} \subseteq \Lambda_{i+1}$ with $|E_{i+1}^{(1)}| = \beta_{i+1}^{(1)}|\Lambda_{i+1}|$ and $|E_{i+1}^{(2)}| = \beta_{i+1}^{(2)}|\Lambda_{i+1}|$, where $\beta_{i+1} = \beta_{i+1}^{(1)}\beta_{i+1}^{(2)}$, and $\mathbf{E_{i+1}} = E_{i+1}^{(1)} \times E_{i+1}^{(2)}$, which satisfy conditions (i)–(iv).

Let $\theta_0 = \{0\}$, $\Lambda_0 = \Lambda_{\theta_0, 1, N}$, $E_1 = E_2 = [-N, N]$, and $\beta_0 = 1$. It is easy to see that the Bohr set Λ_0 is regular and the sets E_1 and E_2 are $(2^{-2000}\delta'^{96}, 2^{-10000}\delta'^{400})$-uniform. Therefore, the zeroth step in the algorithm has been constructed.

Let us estimate the total number of steps in the algorithm. The density $\delta_{\mathbf{E_i}}(A')$ at the ith step is related to $\delta_{\mathbf{E_{i-1}}}(A')$ by condition (iv). Since $\delta_{\mathbf{E_i}}(A') \le 1$ for any i, which is easy to verify, the number of steps in the algorithm cannot be greater than $2^{700}\delta'^{-36} = K$.

Condition (iii) implies that $\beta_i \ge (2^{-1500}\delta'^{100})^i$, whence $d_i \le (C_1\delta)^{-C_1'i}$, where C_1, $C_1' > 0$ are absolute constants.

To complete the proof of the theorem, we need to verify inequality (12) at the last step of the algorithm. Using property (iii), we find

$$N_K \ge (C_2\delta)^{C_3\delta^{-C_4K}}N,$$

where C_2, C_3, $C_4 > 0$ are absolute constants. Condition (12) can be written in a similar fashion:

$$N_K \ge (C_2'\delta)^{-C_3'\delta^{-C_4'K}},$$

where C_2', C_3', $C_4' > 0$ are different absolute constants. Therefore, we have to check the inequality

$$N \ge (C_2''\delta)^{-C_3''\delta^{-C_4''K}} = \exp(\delta^{-C'\delta'^{-36}}), \qquad (13)$$

where C_2'', C_3'', C_4'', $C' > 0$ are absolute constants. By the theorem condition,

$$\delta \gg \frac{1}{(\ln\ln N)^{1/73}},$$

which yields

$$\delta' \gg \frac{1}{(\ln\ln N)^{2/73}},$$

and inequality (13) holds. Consequently, A' contains a triple $\{(k, m), (k+d, m), (k, m+d)\}$ with $d \ne 0$, which is a contradiction. Theorem 3 is proved.

Remark 2. Of course, the constant 73 in Theorem 3 can be somewhat reduced. Nevertheless, in my opinion, this constant cannot be made close to unity without applying a new idea.

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