

On one problem of Gowers.

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In 1927 B.L. van der Waerden published his famous theorem on arithmetic progressions (see [1]) :

Theorem 1.1 *Let h and k be positive numbers. There exists a positive integer $N = N(h, k)$ such that, however the set $\{1, 2, \dots, N\}$ is partitioned into h subsets, at least one of the subsets contains an arithmetic progression of length k .*

Let N be a natural number and

$$a_k(N) = \frac{1}{N} \max\{|A| : A \subseteq [1, N],$$

A — does not contain an arithmetic progression of length $k\}$,

where $|A|$ denotes the cardinality of a set A . In [2] P. Erdos and P. Turan realised that it ought to be possible to find arithmetic progression of length k in any set with positive density. In other words they conjectured that for any $k \geq 3$

$$a_k(N) \rightarrow 0, \text{ as } N \rightarrow \infty \tag{1}$$

Clearly, this conjecture implies van der Waerden theorem.

In case $k = 3$ conjecture (1) was proved by K.F. Roth in [3]. In his paper Roth used the Hardy – Littlewood method to prove the inequality

$$a_3(N) \ll \frac{1}{\log \log N}.$$

At this moment the best result about a lower bound for $a_3(N)$ belongs to J. Bourgain He proved that

$$a_3(N) \ll \sqrt{\frac{\log \log N}{\log N}}. \tag{2}$$

For an arbitrary k conjecture (1) was proved by E. Szemerédi [5] in 1975.

The second proof of Szemerédi's theorem was given by H. Furstenberg in [18], using ergodic theory. Furstenberg showed that Szemerédi's theorem is equivalent to the multiple recurrence of almost every point in an arbitrary dynamical system. Here we formulate this theorem in the case of metric spaces :

Theorem 1.2 *Let X be a metric space with metric $d(\cdot, \cdot)$ and Borel sigma-algebra of measurable sets Φ . Let T be a measurable map of X into itself preserving the measure μ and let $k \geq 3$. Then*

$$\liminf_{n \rightarrow \infty} \max\{d(T^n x, x), d(T^{2n} x, x), \dots, d(T^{(k-1)n} x, x)\} = 0.$$

for almost all $x \in X$.

A. Behrend in [11] obtained a lower bound for $a_3(N)$

$$a_3(N) \gg \exp(-C(\log N)^{\frac{1}{2}}),$$

where C is an absolute constant. Lower bounds for $a_k(N)$ with an arbitrary k can be found in [8].

Unfortunately, Szemerédi's methods give very weak upper bound for $a_k(N)$. Furstenberg's proof gives no bound. Only in 2001 W.T. Gowers [6] obtained a quantitative result about the speed of tending to zero of $a_k(N)$ with $k \geq 4$. He proved the following theorem.

Theorem 1.3 *Let $\delta > 0$, $k \geq 4$ and $N \geq \exp \exp(C\delta^{-K})$, where $C, K > 0$ is absolute constants. Let $A \subseteq \{1, 2, \dots, N\}$ be a set of cardinality at least δN . Then A contains an arithmetic progression of length k .*

In other words, W.T. Gowers proved that for any $k \geq 4$, we have $a_k(N) \ll 1/(\log \log N)^{c_k}$, where constant c_k depends on k only.

In the present paper we shall deal with the following problem. Consider the two-dimensional lattice $[1, N]^2$ with basis $\{(1, 0), (0, 1)\}$. Define

$$L(N) = \frac{1}{N^2} \max\{|A| : A \subseteq [1, N]^2 \text{ and}$$

$$A \text{ — does not contain any triple } \{(k, m), (k + d, m), (k, m + d), d > 0\} \\ \text{with positive } d\}. \tag{3}$$

A triple from (3) will be called a "corner". In papers [9, 18] shown that $L(N)$ tends to 0 as N tends to infinity. W.T. Gowers (see [6]) set a question about the speed of convergence to 0 of $L(N)$.

In [12] V. Vu proposed the following solution. Let us define $\log_* N$ as the largest integer k such that $\log_{[k]} N \geq 2$, where $\log_{[1]} N = \log N$ and for $l \geq 2$ $\log_{[l]} = \log(\log_{[l-1]} N)$. V. Vu proved that

$$L(N) \leq \frac{100}{\log_*^{1/4} N}$$

The main result of our paper is

Theorem 1.4 *Let $\delta > 0$, $N \geq \exp \exp \exp(\delta^{-c})$, where $c > 0$ is an absolute constant. Let $A \subseteq \{1, \dots, N\}^2$ be a set of cardinality at least δN^2 . Then A contains a triple $(k, m), (k + d, m), (k, m + d)$, where $d > 0$.*

Thus, we obtain the bound $L(N) \ll 1/(\log \log \log N)^{C_1}$, where $C_1 > 0$ is an absolute constant.

Moreover, a simple lower bound for $L(N)$ will be obtained (see Proposition 3.1).

In our proof we develop the approach presented in [6, 10].

2. On α -uniformity.

Let $A \subset \mathbf{Z}_N$ be a set of size δN and let $\chi_A(s)$ be the characteristic function of A . Let us define the *balanced* function of A to be $f(s) = \chi_A(s) - \delta$.

Let D denote the closed unit disk in \mathbf{C} .

Definition. A function $f : \mathbf{Z}_N \rightarrow D$ will be called α -uniform, if

$$\sum_k \left| \sum_s f(s) \overline{f(s-k)} \right|^2 \leq \alpha N^3. \quad (4)$$

Let us now define a set A to be α -uniform if its balanced function is.

Let f be a function from \mathbf{Z}_N to \mathbf{C} . For $r \in \mathbf{Z}_N$ we set

$$\widehat{f}(r) = \sum_k f(k) e(-kr),$$

where $e(x) = e^{2\pi i x/N}$. The function \widehat{f} is the discrete Fourier transform of f . We need in some simple facts on Fourier transform

$$N \sum_s |f(s)|^2 = \sum_r |\widehat{f}(r)|^2 \quad (5)$$

$$N \sum_s f(s) \overline{g(s)} = \sum_r \widehat{f}(r) \overline{\widehat{g}(r)}. \quad (6)$$

$$N \sum_k \left| \sum_s f(s) \overline{g(s-k)} \right|^2 = \sum_r |\widehat{f}(r)|^2 |\widehat{g}(r)|^2 \quad (7)$$

We need in Lemma 2.2 from [6].

Lemma 2.1 *Let $f : \mathbf{Z}_N \rightarrow D$ be α -uniform. Then*

- 1) $\sum_r |\widehat{f}(r)|^4 \leq \alpha N^4$
- 2) $\max_r |\widehat{f}(r)| \leq \alpha^{\frac{1}{4}} N$
- 3) $\left| \sum_k \left| \sum_s f(s) \overline{g(s-k)} \right|^2 - \frac{1}{N} \left| \sum_s f(s) \right|^2 \left| \sum_s g(s) \right|^2 \right| \leq \alpha^{\frac{1}{2}} N^2 \|g\|_2^2$, for every function $g, g : \mathbf{Z}_N \rightarrow D$.

Otherwise, if for all r we have $|\widehat{f}(r)| \leq \alpha N$ for some $\alpha > 0$ then f is α^2 uniform.

Proof. Using (7), we get

$$N \sum_k \left| \sum_s f(s) \overline{f(s-k)} \right|^2 = \sum_r |\widehat{f}(r)|^4. \quad (8)$$

and we prove 1). Further, we have

$$\max_r |\widehat{f}(r)|^4 \leq \sum_r |\widehat{f}(r)|^4 \leq \alpha N^4$$

and we prove 2). Suppose for all r we have $|\widehat{f}(r)| \leq \alpha N$ for some $\alpha > 0$. Then

$$\sum_r |\widehat{f}(r)|^4 \leq \alpha^2 N^2 \sum_r |\widehat{f}(r)|^2 = \alpha^2 N^3 \sum_s |f(s)|^2 \leq \alpha^2 N^4$$

We see that function f is α^2 -uniform. Now we shall proof 3). Let $(f * g)(k) = \sum_s f(s) \overline{g(s-k)}$. Then $\sum_k \sum_s f(s) \overline{g(s-k)} = \sum_s f(s) \cdot \overline{\sum_s g(s)}$.

$$\begin{aligned} & \sum_k \left| (f * g)(k) - \frac{1}{N} \sum_s f(s) \cdot \overline{\sum_s g(s)} \right|^2 = \\ & = \sum_k \left| \sum_s f(s) \overline{g(s-k)} \right|^2 - \frac{1}{N} \left| \sum_s f(s) \right|^2 \left| \sum_s g(s) \right|^2 = \sigma \end{aligned} \quad (9)$$

Let $\phi(k) = (f * g)(k) - \frac{1}{N} \sum_s f(s) \cdot \overline{\sum_s g(s)}$. Then $\widehat{\phi}(0) = \sum_k \phi(k) = 0$. Clearly, $r \neq 0$ $\widehat{\phi}(r) = \widehat{f}(r) \widehat{g}(r)$. By (9), it follows that

$$\sum_k \left| \sum_s f(s) \overline{g(s-k)} \right|^2 - \frac{1}{N} \left| \sum_s f(s) \right|^2 \left| \sum_s g(s) \right|^2 = \sum_k |\phi(k)|^2$$

Using (5), we get

$$\sigma = \frac{1}{N} \sum_r |\widehat{\phi}(r)|^2 = \frac{1}{N} \sum_{r \neq 0} |\widehat{\phi}(r)|^2 = \frac{1}{N} \sum_{r \neq 0} |\widehat{f}(r)|^2 |\widehat{g}(r)|^2 \quad (10)$$

If f is α -uniform, then using 2), we have $\max_r |\widehat{f}(r)| \leq \alpha^{\frac{1}{4}}N$. Combining (10) and (5), we obtain

$$\sigma \leq \alpha^{\frac{1}{2}}N \cdot \sum_r |\widehat{g}(r)|^2 \leq \alpha^{\frac{1}{2}}N^2 \sum_s |g(s)|^2 = \alpha^{\frac{1}{2}}N^2 \|g\|_2^2$$

This proves the Lemma 2.1.

Given a function $f : \mathbf{Z}_N^2 \rightarrow \mathbf{C}$ and $\vec{r} = (r_1, r_2)$ we set

$$\widehat{f}(\vec{r}) = \widehat{f}(r_1, r_2) = \sum_{k,m} f(k, m)e(-(kr_1 + mr_2)).$$

Definition. A function $f : \mathbf{Z}_N^2 \rightarrow D$ will be called α -uniform, if

$$\sum_{\vec{t}} \left| \sum_{\vec{s}} f(\vec{s}) \overline{f(\vec{s} - \vec{t})} \right|^2 \leq \alpha N^6. \quad (11)$$

Let us now define a set $A \subseteq \mathbf{Z}_N^2$, $|A| = \delta N^2$ to be α -uniform if its balanced function $f(\vec{s}) = \chi_A(\vec{s}) - \delta$ is. Obviously, all statements of Lemma 2.1 is true for these functions.

Lemma 2.1' *Let $f : \mathbf{Z}_N^2 \rightarrow D$ be α -uniform. Then*

- 1) $\sum_{\vec{r}} |\widehat{f}(\vec{r})|^4 \leq \alpha N^8$
- 2) $\max_{\vec{r}} |\widehat{f}(\vec{r})| \leq \alpha^{\frac{1}{4}}N^2$
- 3) $\left| \sum_{\vec{k}} \left| \sum_{\vec{s}} f(\vec{s}) g(\vec{s} - \vec{k}) \right|^2 - \frac{1}{N^2} \left| \sum_{\vec{s}} f(\vec{s}) \right|^2 \left| \sum_{\vec{s}} g(\vec{s}) \right|^2 \right| \leq \alpha^{\frac{1}{2}}N^4 \|g\|_2^2$, for every function g , $g : \mathbf{Z}_N^2 \rightarrow D$.

Otherwise, if for all r we have $|\widehat{f}(r)| \leq \alpha N^2$ for some $\alpha > 0$ then f is α^2 uniform.

Let $P_1, P_2 \subseteq \mathbf{Z}_N$ be arithmetic progressions. A set $P \subseteq \mathbf{Z}_N^2$ is called two-dimensional arithmetic progression if $P = P_1 \times P_2$.

Lemma 2.2 *Let $A \subseteq \mathbf{Z}_N^2$ be α -uniform of cardinality δN^2 . Let $P_1, P_2 \subseteq \mathbf{Z}_N$ be arithmetic progressions with difference 1 and let $P = P_1 \times P_2$ be a two-dimensional arithmetic progression. Then $||A \cap P| - \delta|P|| \leq 16\alpha^{\frac{1}{4}}N^2$.*

Proof. Let $P = \{a, a+1, \dots, a+M_1-1\} \times \{b, b+1, \dots, b+M_2-1\}$. For any $\vec{r} = (r_1, r_2)$, $r_1 \neq 0$, $r_2 \neq 0$, we have

$$|\widehat{P}(\vec{r})| = \left| \frac{e(-r_1 M_1) - 1}{e(-r_1) - 1} \right| \cdot \left| \frac{e(-r_2 M_2) - 1}{e(-r_2) - 1} \right| \leq \frac{N^2}{r_1 r_2}$$

If $r_1 = 0, r_2 \neq 0$, then $|\widehat{P}(\vec{r})| \leq N^2/r_2$. In the same way, $|\widehat{P}(\vec{r})| \leq N^2/r_1$ for $r_2 = 0, r_1 \neq 0$. Hence $\sum_{\vec{r} \neq 0} |\widehat{P}(\vec{r})|^{4/3} \leq 16N^{8/3}$. Using (6), we get

$$\sigma = ||A \cap P| - \delta|P|| = \left| \sum_{\vec{r}} \chi_A(\vec{r}) \chi_P(\vec{r}) - \delta|P| \right| = \frac{1}{N^2} \sum_{\vec{r} \neq 0} \widehat{\chi}_A(\vec{r}) \overline{\widehat{\chi}_P(\vec{r})}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} \left| \sum_{\vec{r} \neq 0} \widehat{\chi}_A(\vec{r}) \overline{\widehat{\chi}_P(\vec{r})} \right| &\leq \left(\sum_{\vec{r} \neq 0} |\widehat{\chi}_A(\vec{r})|^4 \right)^{1/4} \left(\sum_{\vec{r} \neq 0} |\widehat{\chi}_P(\vec{r})|^{4/3} \right)^{3/4} \leq \\ &\leq 16N^2 \left(\sum_{\vec{r} \neq 0} |\widehat{\chi}_A(\vec{r})|^4 \right)^{1/4} \end{aligned}$$

Since A is α -uniform, it follows that

$$\sigma \leq \frac{1}{N^2} 16N^2 (\alpha N^8)^{1/4} \leq 16\alpha^{1/4} N^2$$

as required.

Let \vec{e}_1 and \vec{e}_2 be two vectors $(1, 0)$ and $(0, -1)$.

Let $E_1 \times E_2$ be a subset of \mathbf{Z}_N^2 and let f be a function from \mathbf{Z}_N^2 to D . We shall write that $f : E_1 \times E_2 \rightarrow D$ if $f(\vec{s}) = 0$ for any $\vec{s} \notin E_1 \times E_2$.

Definition 2.3 Let $\alpha \in [0, 1]$ and let $E_1 \times E_2 \subseteq \mathbf{Z}_N^2$. Function $f : E_1 \times E_2 \rightarrow D$ will be called α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) if the following condition hold

$$\sum_{\vec{s} \in \mathbf{Z}_N^2} \sum_{u \in \mathbf{Z}_N} \sum_{r \in \mathbf{Z}_N} f(\vec{s}) \overline{f(\vec{s} + u\vec{e}_2)} \overline{f(\vec{s} + r\vec{e}_1)} f(\vec{s} + u\vec{e}_2 + r\vec{e}_1) \leq \alpha |E_1|^2 |E_2|^2. \quad (12)$$

Let $f(k, m) = f(k\vec{e}_1 + m\vec{e}_2)$. Function f is α -uniform iff

$$\sum_{m, p \in \mathbf{Z}_N} \left| \sum_{k \in \mathbf{Z}_N} f(k, m) \overline{f(k, p)} \right|^2 \leq \alpha |E_1|^2 |E_2|^2. \quad (13)$$

A set $H \subseteq \mathbf{Z}_N^2$ is called *box* if $H = E_1 \times E_2$, where $E_1, E_2 \subseteq \mathbf{Z}_N$. If $|E_1| = |E_2|$ then H is called *square*.

Let $A \subseteq E_1 \times E_2$ and let $\chi(\vec{s}) = \chi_A(\vec{s})$ be the characteristic function of A . By $\delta_m = \delta_m^{\vec{e}_1}$ and $\gamma_k = \gamma_k^{\vec{e}_2}$ denote $\delta_m = 1/|E_1| \cdot \sum_{p=1}^N \chi(m\vec{e}_2 + p\vec{e}_1)$ and $\gamma_k = 1/|E_2| \cdot \sum_{p=1}^N \chi(k\vec{e}_1 + p\vec{e}_2)$. Let us define the *balanced* function of A to be $f(\vec{s}) = (\chi(\vec{s}) - \delta_m) \chi_{E_1 \times E_2}(\vec{s})$.

Let us now define a set $A \subseteq E_1 \times E_2$ to be α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) if its balanced function is.

By a *cube* we shall mean quadruple $(\vec{s}, \vec{s} + u\vec{e}_2, \vec{s} + r\vec{e}_1, \vec{s} + u\vec{e}_2 + r\vec{e}_1)$. We shall say that such a cube is contained in $A \subseteq \mathbf{Z}_N^2$ if quadruple $(\vec{s}, \vec{s} + u\vec{e}_2, \vec{s} + r\vec{e}_1, \vec{s} + u\vec{e}_2 + r\vec{e}_1)$ belongs to A .

Lemma 2.4 *Let A be a subset of \mathbf{Z}_N^2 of cardinality δN^2 . Then A contains at least $\delta^4 N^4$ cubes.*

Proof. Let $\chi(\vec{s})$ be the characteristic function of A and let $\vec{s} = k\vec{e}_1 + m\vec{e}_2$. The number of cubes in A is $\sum_{\vec{s}, u} \sum_r \chi(\vec{s}) \overline{\chi(\vec{s} + u\vec{e}_2)} \chi(\vec{s} + r\vec{e}_1) \chi(\vec{s} + u\vec{e}_2 + r\vec{e}_1) = \sum_{m,p} |\sum_k \chi(k, m) \overline{\chi(k, p)}|^2$. Using Cauchy–Bounyakovski inequality, we get

$$\begin{aligned} \sum_{m,p} |\sum_k \chi(k, m) \overline{\chi(k, p)}|^2 &\geq \frac{1}{N^2} (\sum_{m,p} \sum_k \chi(k, m) \overline{\chi(k, p)})^2 = \\ &= \frac{1}{N^2} (\sum_k |\sum_m \chi(k, m)|^2)^2 \geq \frac{1}{N^4} (\sum_{k,m} \chi(k, m))^4 = \delta^4 N^4. \end{aligned}$$

This completes the proof.

By \mathcal{C} denote the operator of complex conjugation. Let \vec{x} and \vec{y} be two vectors in \mathbf{C}^k . We shall write its inner product as $\vec{x} \cdot \vec{y}$ or (\vec{x}, \vec{y}) . Let $\varepsilon \in \{0, 1\}^k$. By $|\varepsilon|$ we denote $\sum_{i=1}^k \varepsilon_i$.

Lemma 2.5 *For every $\varepsilon \in \{0, 1\}^2$ let $f_\varepsilon(\vec{s})$ be a function from \mathbf{Z}_N^2 to D . Then*

$$\begin{aligned} &|\sum_{p,q} \sum_{\vec{s}} \prod_{\varepsilon \in \{0,1\}^2} \mathcal{C}^{|\varepsilon|} f_\varepsilon(\vec{s} + \varepsilon_1 p \vec{e}_1 + \varepsilon_2 q \vec{e}_2)| \leq \\ &\leq \prod_{\varepsilon \in \{0,1\}^2} |\sum_{p,q} \sum_{\vec{s}} \prod_{\eta \in \{0,1\}^2} \mathcal{C}^{|\eta|} f_\varepsilon(\vec{s} + \eta_1 p \vec{e}_1 + \eta_2 q \vec{e}_2)|^{\frac{1}{4}}. \end{aligned}$$

Proof. Let $\vec{s} = k\vec{e}_1 + m\vec{e}_2$. Then

$$\begin{aligned} &|\sum_{p,q} \sum_{\vec{s}} \prod_{\varepsilon \in \{0,1\}^2} \mathcal{C}^{|\varepsilon|} f_\varepsilon(\vec{s} + \varepsilon_1 p \vec{e}_1 + \varepsilon_2 q \vec{e}_2)| = \\ &= \sum_{k,p} \left(\sum_m f_{\{0,0\}}(k, m) \overline{f_{\{1,0\}}(k+p, m)} \right) \left(\sum_m \overline{f_{\{0,1\}}(k, m)} f_{\{1,1\}}(k+p, m) \right) \leq \\ &\leq \left(\sum_{k,p} |\sum_m f_{\{0,0\}}(k, m) \overline{f_{\{1,0\}}(k+p, m)}|^2 \right)^{\frac{1}{2}} \left(\sum_{k,p} |\sum_m \overline{f_{\{0,1\}}(k, m)} f_{\{1,1\}}(k+p, m)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The first bracket can be transformed as follows :

$$\begin{aligned} &\sum_{k,p} |\sum_m f_{\{0,0\}}(k, m) \overline{f_{\{1,0\}}(k+p, m)}|^2 = \\ &= \sum_{k,p} \sum_{m,r} f_{\{0,0\}}(k, m) \overline{f_{\{1,0\}}(k+p, m)} \overline{f_{\{0,0\}}(k, r)} f_{\{1,0\}}(k+p, r) = \end{aligned}$$

$$= \sum_{m,r} \left(\sum_k f_{\{0,0\}}(k,r) \overline{f_{\{0,0\}}(k,m)} \right) \left(\sum_k f_{\{1,0\}}(k,r) \overline{f_{\{1,0\}}(k,m)} \right) \quad (14)$$

The latter can be estimate with the help of the Cauchy–Bounyakovskiy inequality. Repeating this argument for the second bracket, we obtain the needed result.

Let f be a function from \mathbf{Z}_N^2 to \mathbf{C} . Define $\|f\|$ by the formula

$$\|f\| = \left| \sum_{\vec{s},u} \sum_r f(\vec{s}) \overline{f(\vec{s} + u\vec{e}_2)} \overline{f(\vec{s} + r\vec{e}_1)} f(\vec{s} + u\vec{e}_2 + r\vec{e}_1) \right|^{\frac{1}{4}} \quad (15)$$

Lemma 2.6 (15) is a norm.

Proof. Consider the sum

$$\|f + g\|^4 = \sum_{p,q} \sum_{\vec{s}} \prod_{\varepsilon \in \{0,1\}^2} \mathcal{C}^{|\varepsilon|} (f + g)(\vec{s} + \varepsilon_1 p \vec{e}_1 + \varepsilon_2 q \vec{e}_2) \quad (16)$$

If we expand the product (16) we obtain 16 terms of the form $\prod_{\varepsilon \in \{0,1\}^2} \mathcal{C}^{|\varepsilon|} f_{\varepsilon}(\vec{s} + \varepsilon_1 p \vec{e}_1 + \varepsilon_2 q \vec{e}_2)$, where f_{ε} is either f or g . For each one of these terms, if we apply Lemma 2.5, we have an upper estimate of $\|f\|^k \|g\|^l$, where k and l are the number of times that f_{ε} equals f and g respectively. Hence

$$\|f + g\|^4 \leq \sum_{k=0}^4 C_4^k \|f\|^k \|g\|^{4-k} = (\|f\| + \|g\|)^4.$$

as required.

Theorem 2.7 Let A be α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) and $\sum_p (\delta_p - \delta)^2 \leq \alpha N$. Then A contains at most $(\delta + 2\alpha^{1/4})^4 N^4$ cubes.

Proof. Let χ be the characteristic and let f be the balanced function of A . Then $\chi = f + \delta + (\delta_m - \delta)$. The statement that A is α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) is equivalent to the statement that $\|f\| \leq \alpha^{1/4} N$. We have $\|(\delta_m - \delta)\| = N^{\frac{1}{2}} (\sum_p (\delta_p - \delta)^2)^{\frac{1}{2}} \leq \alpha^{\frac{1}{2}} N$. The number of cubes in A is $\|\chi\|^4$. Using 2.6, we get $\|\chi\| \leq \|\delta\| + \|f\| + \|(\delta_m - \delta)\|$. Thus $\|\chi\|^4 \leq (\delta + 2\alpha^{1/4})^4 N^4$ as required.

Let Q_1 and Q_2 be subsets of $E_1 \times E_2 \subseteq \mathbf{Z}_N^2$ and h, g be its characteristic function respectively. Suppose $|E_1| = \beta_1 N$, $|E_2| = \beta_2 N$.

The next result is the main one of this section.

Theorem 2.8 Let function $f : E_1 \times E_2 \rightarrow D$ be α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) and sets E_1, E_2 be $\alpha_0 = 2^{-12} \alpha^3 \beta_1^{24} \beta_2^{24}$ -uniform. Then

$$\left| \sum_{\vec{s} \in \mathbf{Z}_N^2} \sum_{r \in \mathbf{Z}_N} h(\vec{s}) g(\vec{s} + r(\vec{e}_1 + \vec{e}_2)) f(\vec{s} + r\vec{e}_2) \right| \leq 2\alpha^{\frac{1}{4}} \beta_1^2 \beta_2^2 N^3.$$

Proof. Let $\vec{e} = \vec{e}_1 + \vec{e}_2$, $\vec{s} = k\vec{e}_1 + m\vec{e}_2$, $\chi(\vec{s}) = \chi_{E_1 \times E_2}(\vec{s})$ and $\chi_1(k) = \chi_{E_1}(k)$, $\chi_2(m) = \chi_{E_2}(m)$. By the Cauchy–Bounyakovski inequality

$$\sigma = \left| \sum_{\vec{s} \in \mathbf{Z}_N^2} \sum_{r \in \mathbf{Z}_N} h(\vec{s}) g(\vec{s} + r\vec{e}) f(\vec{s} + r\vec{e}_2) \right|^2 \leq \quad (17)$$

$$\left(\sum_{\vec{s}} |h(\vec{s})|^2 \right) \left(\sum_{\vec{s}} \chi(\vec{s}) \left| \sum_r g(\vec{s} + r\vec{e}) f(\vec{s} + r\vec{e}_2) \right|^2 \right) \quad (18)$$

$$= \left(\sum_{\vec{s}} |h(\vec{s})|^2 \right) \left(\sum_{\vec{s}} \chi(\vec{s}) \sum_{r,p} g(\vec{s} + r\vec{e}) \overline{g(\vec{s} + p\vec{e})} f(\vec{s} + r\vec{e}_2) \overline{f(\vec{s} + p\vec{e}_2)} \right) \quad (19)$$

$$= \|h\|_2^2 \sum_{\vec{s}} \sum_{r,u} \chi(\vec{s} - r\vec{e}) g(\vec{s}) \overline{g(\vec{s} + u\vec{e})} f(\vec{s} - r\vec{e}_1) \overline{f(\vec{s} + u\vec{e}_2 - r\vec{e}_1)} = \sigma_1 \quad (20)$$

Since $\chi(\vec{s} - r\vec{e}) f(\vec{s} - r\vec{e}_1) = \chi_1(k - r) \chi_2(m - r) f(\vec{s} - r\vec{e}_1) = \chi_2(m - r) f(\vec{s} - r\vec{e}_1)$, it follows that

$$\sigma_1 = \|h\|_2^2 \sum_{\vec{s}} \sum_{u,r} \chi_2(m - r) g(\vec{s}) \overline{g(\vec{s} + u\vec{e})} f(\vec{s} - r\vec{e}_1) \overline{f(\vec{s} + u\vec{e}_2 - r\vec{e}_1)} \quad (21)$$

$$= \|h\|_2^2 \sum_{\vec{s}} \sum_u g(\vec{s}) \overline{g(\vec{s} + u\vec{e})} \sum_r \chi_2(m - r) f(k - r, m) \overline{f(k - r, m + u)} \quad (22)$$

$$= \|h\|_2^2 \sum_{\vec{s}} \sum_u g(\vec{s}) \overline{g(\vec{s} + u\vec{e})} \sum_r \chi_2(r + m - k) f(r, m) \overline{f(r, m + u)} \quad (23)$$

$$= \|h\|_2^2 \sum_{m,u} \sum_r f(r, m) \overline{f(r, m + u)} \sum_k \chi_2(r + m - k) g(k, m) \overline{g(k + u, m + u)}$$

By Lemma 2.1 for all but $\alpha_0^{1/6} N$ choices of r the following inequality holds

$$\begin{aligned} & \left| \sum_k \chi_2(r + m - k) g(k, m) \overline{g(k + u, m + u)} - \right. \\ & \left. - \beta_2 \sum_k g(k, m) \overline{g(k + u, m + u)} \right| \leq \alpha_0^{1/6} N. \end{aligned}$$

We have $\alpha_0 = 2^{-12} \alpha^3 \beta_1^{24} \beta_2^{24}$. Using this, we get

$$\begin{aligned} \sigma_1 & \leq \beta \|h\|_2^2 \sum_{m,u} \sum_k g(k, m) \overline{g(k + u, m + u)} \left| \sum_r f(r, m) \overline{f(r, m + u)} \right| + 2\alpha_0^{1/6} N^6 \\ & \leq \beta \|h\|_2^2 \sum_{m,u} \sum_k \chi(k, m) \overline{\chi(k + u, m + u)} \left| \sum_r f(r, m) \overline{f(r, m + u)} \right| + 2\alpha_0^{1/6} N^6 \end{aligned}$$

We have

$$\sum_k \chi(k, m) \overline{\chi(k+u, m+u)} = \chi_2(m) \overline{\chi_2(m+u)} \sum_k \chi_1(k) \overline{\chi_1(k+u)}. \quad (24)$$

Since set E_1 be α_0 -uniform, it follows that

$$|\sigma_1|^2 \leq 2\beta_2^2 \|h\|_2^4 \beta_1^4 N^2 \left| \sum_{m,u} \chi_2(m) \overline{\chi_2(m+u)} \right| \left| \sum_r f(r, m) \overline{f(r, m+u)} \right|^2 + 2^5 \alpha_0^{1/3} N^{12}$$

Using the Cauchy–Bounyakovskiy inequality, we get

$$|\sigma_1|^2 \leq 2\beta_1^4 \beta_2^2 \|h\|_2^4 N^2 \left(\sum_{m,u} \chi_2(m) \chi_2(m+u) \right) \|f\|^4 + 2^5 \alpha_0^{1/3} N^{12} \quad (25)$$

$$\leq 4\beta_1^4 \beta_2^2 N^2 (\beta_1 \beta_2 N^2)^2 \beta_2^2 N^2 \alpha (\beta_1^2 \beta_2^2 N^4) = 4\alpha \beta_1^8 \beta_2^8 N^{12}. \quad (26)$$

Thus, we have $\sigma \leq 2\alpha^{1/4} \beta_1^2 \beta_2^2 N^3$ as required.

The next result is the main of this section.

Theorem 2.9 *Let $A \subseteq E_1 \times E_2$ be a set and have cardinality $|A| = \delta |E_1| |E_2|$. Let $|E_1| = \beta_1 N$, $|E_2| = \beta_2 N$ and sets E_1, E_2 be $10^{-330} \beta_1^{24} \beta_2^{24} \delta^{132}$ -uniform. Let A be α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) , $\alpha = 10^{-108} \delta^{44}$, $N \geq 10^{10} (\delta^4 \beta_1 \beta_2)^{-1}$ and $\sum_m |\delta_m - \delta|^2 \leq \alpha \beta_2 N$. Then A contains an corner.*

Proof. Let $\varepsilon = 2^{-20} \delta^2$ and $c = N/[\varepsilon N]$. We can find a partition of \mathbf{Z}_N^2 into two-dimensional arithmetic progressions with step 1 and cardinality $[\varepsilon N]^2$. Let us numerate the squares from left to right starting with the left upper corner. We shall not numerate the squares in the last column and the last string. The set of squares without numbers consists of two stripes. The width of each stripe is not greater then εN and the length equals N . By Lemma 2.2 these stripes contain not more then $8\varepsilon \beta_1 \beta_2 N^2$ points from $E_1 \times E_2$. Let A_i be the intersection the A with i -th square. Let the number of the enumerated squares be t and let the i -th square be $P_i \times S_i$, where $P_i, S_i \subseteq \mathbf{Z}_N$. Let $\vec{e} = \vec{e}_1 + \vec{e}_2$, $\vec{s} = k\vec{e}_1 + m\vec{e}$ and χ_i be the characteristic function of A_i . It follows from $\varepsilon = 2^{-20} \delta^2$ that $\sum_{i=1}^t \sum_{k,m} \chi_i(k, m) \geq (1 - 2^{-5}) \delta \beta_1 \beta_2 N^2$. Let us split $\{1, \dots, c\}$ into three arithmetic progressions K_1, K_2, K_3 with step 1 such that the length of any two progressions differ by at most 1. Then all numerated squares get separated into nine subsets. Among these subsets there exists such one, say V , that $\sum_{i \in V} \sum_{k,m} \chi_i(k, m) \geq 10^{-1} \delta \beta_1 \beta_2 N^2$. Using the Cauchy–Bounyakovskiy inequality, we get

$$\frac{1}{100} \delta^2 \beta_1^2 \beta_2^2 N^3 \leq \sum_k \left(\sum_{m,i} \chi_i(k, m) \right)^2 = \sum_{i,j=1}^t \sum_k \left(\sum_m \chi_i(k, m) \right) \left(\sum_m \chi_j(k, m) \right)$$

$$= \sum_{i \neq j} \sum_k \left(\sum_m \chi_i(k, m) \right) \left(\sum_m \chi_j(k, m) \right) + \sum_i \sum_k \left(\sum_m \chi_i(k, m) \right)^2 \quad (27)$$

Let us estimate the second term in (27). Let $\zeta = 10^{-330} \beta_1^{24} \beta_2^{24} \delta^{132}$. By Lemma 2.1 there is all but $(1 - \zeta^{1/6})N$ choices of k such that $|\sum_m \chi_{E_1}(k+m) \chi_{E_2 \cap S_i}(m) - \beta_1 |E_2 \cap S_i|| \leq \zeta^{1/6} N$. Using Lemma 2.2, we get $|E_2 \cap S_i| \leq 4\varepsilon \beta_2 N$. Hence there is all but $(1 - \zeta^{1/6})N$ choices of k such that

$$\left| \sum_m \chi_{E_1}(k+m) \chi_{E_2 \cap S_i}(m) \right| \leq 8\varepsilon \beta_1 \beta_2 N \quad (28)$$

Let B be the set such k 's that do not satisfy (28). Then $|B| \leq \zeta^{1/6} N$. Since $\sum_m \chi_{E_1 \cap P_i}(k+m) \chi_{E_2 \cap S_i}(m) \leq \sum_m \chi_{E_1}(k+m) \chi_{E_2 \cap S_i}(m)$, it follows that

$$\begin{aligned} \sum_i \sum_k \left(\sum_m \chi_i(k, m) \right)^2 &= \sum_i \sum_{k \in B} \left(\sum_m \chi_i(k, m) \right)^2 + \sum_i \sum_{k \notin B} \left(\sum_m \chi_i(k, m) \right)^2 \leq \\ &8\varepsilon \beta_1 \beta_2 N \sum_i \sum_k \sum_m \chi_i(k, m) + \zeta^{1/6} N \sum_i \left| \sum_m \chi_{E_2 \cap S_i}(m) \right|^2 = \sigma \end{aligned}$$

Using Lemma 2.2, we get

$$\sigma \leq 8\varepsilon \beta_1 \beta_2 N \beta_1 \beta_2 N^2 + 16\zeta^{1/6} N (\varepsilon \beta_2 N)^2 t \leq 10\varepsilon \beta_1^2 \beta_2^2 N^3$$

This yields that

$$\sum_{i \neq j} \sum_k \left(\sum_m \chi_i(k, m) \right) \left(\sum_m \chi_j(k, m) \right) \geq \frac{1}{200} \delta^2 \beta_1^2 \beta_2^2 N^3 \quad (29)$$

Then the inequality above imply that there exists two indexes $i_0, j_0, i_0 \neq j_0$ such that

$$\sum_k \left(\sum_m \chi_{i_0}(k, m) \right) \left(\sum_m \chi_{j_0}(k, m) \right) \geq 10^{-26} \delta^{10} \beta_1^2 \beta_2^2 N^3. \quad (30)$$

Let $i_0 < j_0$ and let $Q_1 = A_{i_0}, Q_2 = A_{j_0}$. Recall that $i_0, j_0 \in V$. Consider the sum

$$\sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) \chi_A(\vec{s} + r\vec{e}_2) \quad (31)$$

Split the sum (31) as

$$\sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) (\delta_{m+r} \chi_{E_1 \times E_2}(\vec{s} + r\vec{e}_2) + f_A(\vec{s} + r\vec{e}_2)) =$$

$$\begin{aligned}
&= \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) \chi_{E_1 \times E_2}(\vec{s} + r\vec{e}_2) \delta_{m+r} + \\
&\quad \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) f_A(\vec{s} + r\vec{e}_2) = \\
&\quad = \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) \delta_{m+r} + \\
&\quad \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) f_A(\vec{s} + r\vec{e}_2) = \\
&= \delta \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) + \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) (\delta_{m+r} - \delta) + \\
&\quad \sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) f_A(\vec{s} + r\vec{e}_2) \tag{32}
\end{aligned}$$

By Theorem 2.8 the third term in (32) does not exceed $2\alpha^{1/4}\beta_1^2\beta_2^2N^3$. By inequality (30) we have the first term in (32) is at least $10^{-26}\delta^{11}\beta_1^2\beta_2^2N^3$. Let us estimate the second term in (32). Let H be the set of m such that the following inequality holds $|\delta_m - \delta| > \alpha^{1/3}$. We have $\sum_m |\delta_m - \delta|^2 \leq \alpha\beta_2N$, then $|H| \geq \alpha^{1/3}N$. Now, let $\vec{s} = k\vec{e}_1 + m\vec{e}_2$. We get

$$\begin{aligned}
&|\sum_{\vec{s}} \sum_r \chi_{Q_1}(\vec{s}) \chi_{Q_2}(\vec{s} + r\vec{e}) (\delta_{m+r} - \delta)| = |\sum_{k,m,r} \chi_{Q_1}(k, m) \chi_{Q_2}(k+r, m+r) (\delta_{m+r} - \delta)| \\
&= |\sum_{k,r} \sum_m \chi_{Q_1}(k, m-r) \chi_{Q_2}(k+r, m) (\delta_m - \delta)| = \sigma_1 = \sigma_2 + \sigma_3,
\end{aligned}$$

where by σ_2, σ_3 we define the sums over $m \notin H$ and $m \in H$ respectively. The sets E_1, E_2 is $10^{-330}\beta_1^{24}\beta_2^{24}\delta^{132}$ -uniform. By Lemma 2.1, we get

$$\begin{aligned}
\sigma_2 &= |\sum_{k,r} \sum_{m \notin H} \chi_{Q_1}(k, m-r) \chi_{Q_2}(k+r, m) (\delta_m - \delta)| \leq \\
&\leq \alpha^{1/3} \sum_{k,r} \sum_m \chi_{E_1}(k) \chi_{E_1}(k+r) \chi_{E_2}(m) \chi_{E_2}(m-r) \leq 2\alpha^{1/4}\beta_1^2\beta_2^2N^3 \tag{33}
\end{aligned}$$

Further

$$\begin{aligned}
\sigma_3 &= |\sum_{k,r} \sum_{m \in H} \chi_{Q_1}(k, m-r) \chi_{Q_2}(k+r, m) (\delta_m - \delta)| \leq \\
&\leq \sum_{k,r} \sum_m \chi_{E_1}(k) \chi_{E_1}(k+r) \chi_H(m) \chi_{E_2}(m-r) \tag{34}
\end{aligned}$$

Let T_1 be the set of r such that $|\sum_k \chi_{E_1}(k)\chi_{E_1}(k+r) - \beta_1^2 N| > \zeta^{1/6} N$, and T_2 be the set of r such that $|\sum_k \chi_H(k)\chi_{E_2}(k+r) - \beta_2 |H|| > \zeta^{1/6} N$. By Lemma 2.1, we have $|T_1|, |T_2| \leq \zeta^{1/6} N$. It follows that

$$\begin{aligned} \sigma_3 &\leq \sum_{r \in (T_1 \cup T_2)} \left(\sum_k \chi_{E_1}(k)\chi_{E_1}(k+r) \right) \left(\sum_m \chi_H(m)\chi_{E_2}(m-r) \right) + \\ &+ \sum_{r \notin (T_1 \cup T_2)} \left(\sum_k \chi_{E_1}(k)\chi_{E_1}(k+r) \right) \left(\sum_m \chi_H(m)\chi_{E_2}(m-r) \right) \leq \\ &\leq 2\zeta^{1/6} N^3 + \beta_1^2 N |H| \beta_2 N + 2\zeta^{1/6} N^3 \leq 2\alpha^{1/4} \beta_1^2 \beta_2^2 N^3. \end{aligned} \quad (35)$$

Combining (33) and (35), we obtain $\sigma_1 \leq 4\alpha^{1/4} \beta_1^2 \beta_2^2 N^3$.

Since $\alpha = 10^{-108} \delta^{44}$, it follows that (31) $\geq 10^{-27} \delta^{11} \beta_1^2 \beta_2^2 N^3$. By the construction of Q_1 and Q_2 the sum equals the number of triples $\{(k, m), (k+d, m-d), (k, m-d)\}$ in $Q_1 \times Q_2 \times A$. Since $N \geq 10^{10} (\delta^4 \beta_1 \beta_2)^{-1}$, it follows that $10^{-27} \delta^{11} \beta_1^2 \beta_2^2 N^3 > 1$ and A contains a corner. This completes the proof.

3. Graphs.

Let us consider the set \mathbf{Z}_N^2 as two-dimensional lattice with the basis (\vec{e}_1, \vec{e}_2) .

Let a set A belong to some square $E_1 \times E_2$ of the two-dimensional lattice \mathbf{Z}_N^2 . Let the cardinality of both E_1 and E_2 be n . We shall associate with A some bipartite graph G_A (see [12]). Let ψ and ρ be two bijective maps from E_1 and E_2 to U and V respectively and assume $U \cap V = \emptyset$. Let $U = \{w_1, \dots, w_n\}$ and $V = \{v_1, \dots, v_n\}$. The set of vertices of the bipartite graph G_A is $U \sqcup V$. We shall connect a vertex v_j with a vertex w_i iff $(\psi^{-1}(i), \rho^{-1}(j)) \in A$. The set $A_v = \{w_i \in U \mid (v, w_i) \text{ is a vertex in } G_A\}$ is called the *neighbourhood* of a given vertex $v \in V$.

Let A be an arbitrary subset of \mathbf{Z}_N . Let A^* denote the embedding of A in \mathbf{Z}_N^2 by the rule $(x, y) \in A^*$ iff $x = a - y, a \in A$. It shall be used in the following proposition.

Proposition 3.1 *For any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbf{N}$ such that for an arbitrarily $N \geq N_\varepsilon$ one can find a set $Q \subseteq \{1, \dots, N\}^2$, $|Q| \geq N^{2 - \frac{\log 2 + \varepsilon}{\log \log N}}$ without corners.*

Proof. Behrend's theorem [11] states : for any $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbf{N}$ such that for any $K \geq K_\varepsilon$ there exists $A \subseteq \{1, \dots, K\}$, $|A| > K^{1 - \frac{\log 2 + \varepsilon}{\log \log K}}$ without arithmetic progressions of length 3. Let $K = N/3 > K_\varepsilon$. Using this Theorem we can find a set $A \subset [1, \dots, N/3]$ without progressions. Let us consider the set \mathbf{Z}_N^2 and let us enumerate its horizontal lines from down up.

Let $A \subseteq [N/3, \dots, 2N/3]$. Consider translations of the set $A : \{(A+i-1) \times \{i\}\}_{i=1}^{N/3} \subseteq \mathbf{Z}_N^2$. By \tilde{A} denote the union of all these translations. It is not hard to prove that \tilde{A} does not contain a corner. Moreover, \tilde{A} has cardinality at least $N^{2-\frac{\log 2+\varepsilon}{\log \log N}}/9$. This completes the proof.

Square matrix M is called *nonnegative* if its entries are nonnegative. The following theorem about such matrices is well-known (see, for example, [16]).

Theorem 3.2 *Let M be a nonnegative matrix and r be its spectral radius. Then*

- 1) r is an eigenvalue of M .
- 2) There exists a nonnegative eigenvector corresponding to the eigenvalue r .

Let $M = (m_{ij})$ be the adjacency matrix of the graph G_A and $T = MM'$, where M' is the conjugate matrix. Enumerate the eigenvalues μ_i of T so that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$. Let $\vec{u}_1, \dots, \vec{u}_n$ be the set of orthogonal eigenvectors corresponding to the eigenvalues μ_i . Let $\|\vec{u}_i\|_2^2 = n$, $i = 1, \dots, n$. Define $\vec{u} = (1, \dots, 1)$.

Let α_1 be a real number, $0 \leq \alpha_1 \leq 1$. Suppose $A \subseteq E_1 \times E_2$, $|A| = \delta|E_1||E_2|$ and the following condition holds

$$\sum_{m \in E_2} (\delta_m - \delta)^2 \leq \alpha_1^2 |E_2|. \quad (36)$$

Suppose in addition

$$\sum_{k \in E_1} (\gamma_k - \delta)^2 \leq \alpha_1^2 |E_1|. \quad (37)$$

In the rest of this section, conditions (36) and (37) shall be assumed to hold.

Lemma 3.3 *Let \vec{a} be a vector in \mathbf{C}^n and $C = (c_{ij})$ be a real matrix ($n \times n$). Then $(C\vec{a}, C\vec{a}) \leq \|\vec{a}\|^2 \cdot \sum_i \sum_j c_{ij}^2$*

The proof is trivial.

Lemma 3.4 *Let $\varepsilon \in (0, 1)$. Let $A \subseteq E_1 \times E_2$ be a set, $|E_1| = |E_2| = n$ and let $A = \delta n^2$. Then $\mu_1 \geq \delta^2 n^2$. Furthermore, if $\|\vec{u}_1 - \vec{u}\|^2 \leq \varepsilon^2 n$ then $\mu_1 < \delta^2 n^2 + (2\varepsilon + \alpha_1^2)n^2$.*

Proof. Let $M = (m'_{ij})$. We have (see [16])

$$\mu_1 n \geq (M' \vec{u}, M' \vec{u}) = \sum_i (\sum_j m'_{ij})^2 \quad (38)$$

Combining (38), the Cauchy–Bounyakovskiy inequality and the fact that $\sum_{i,j} m'_{ij} = |A| = \delta n$, we obtain

$$\mu_1 \geq \frac{1}{n} \sum_i (\sum_j m'_{ij})^2 \geq \frac{1}{n^2} (\sum_{i,j} m'_{ij})^2 = \delta^2 n^2.$$

Further, if $\|\vec{u}_1 - \vec{u}\|^2 \leq \varepsilon^2 n$, then $(M' \vec{u}_1, M' \vec{u}_1) = (M' \vec{u}, M' \vec{u}) + (M'(\vec{u}_1 - \vec{u}), M' \vec{u}) + (M' \vec{u}_1, M'(\vec{u}_1 - \vec{u}))$. Let us estimate the term $(M' \vec{u}_1, M'(\vec{u}_1 - \vec{u})) = \sigma$. By the Cauchy–Bounyakovskiy inequality, we get

$$(M' \vec{u}_1, M'(\vec{u}_1 - \vec{u}))^2 \leq (M' \vec{u}_1, M' \vec{u}_1) \cdot (M'(\vec{u}_1 - \vec{u}), M'(\vec{u}_1 - \vec{u}))$$

Using 3.3, we obtain

$$|\sigma|^2 \leq \|\vec{u}_1\|^2 \cdot n^2 \cdot \|\vec{u}_1 - \vec{u}\|^2 \cdot n^2 \leq \varepsilon^2 n^6.$$

It now follows that $|(M' \vec{u}_1, M'(\vec{u}_1 - \vec{u}))| \leq \varepsilon n^3$. In the same way $|(M'(\vec{u}_1 - \vec{u}), M' \vec{u})| \leq \varepsilon n^3$. Hence

$$\mu_1 n = (M' \vec{u}_1, M' \vec{u}_1) \leq (M' \vec{u}, M' \vec{u}) + 2\varepsilon n^3.$$

We have $(M' \vec{u}, M' \vec{u}) = \sum_i (\sum_j m'_{ij})^2 = n^2 \sum_i \delta_i^2$. Now, by (36) $\sum_m \delta_m^2 \leq (\alpha_1^2 + \delta^2)n$, so that

$$\mu_1 n \leq \delta^2 n^3 + 2\varepsilon n^3 + \alpha_1^2 n^3$$

as required.

We shall prove that a set A is α -uniform iff the graph G_A is quasi-random (see [10]).

Lemma 3.5 *Let $\varepsilon \in (0, 1)$. Let $A \subseteq E_1 \times E_2$ be a set, $|E_1| = |E_2| = n$ and let A have cardinality δn^2 . If A is α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) , and $\|\vec{u}_1 - \vec{u}\|^2 \leq \varepsilon^2 n$, then $\mu_2 < \alpha^{1/2} n^2 + 4\sqrt{\varepsilon} n^2 + 4\sqrt{\alpha_1} n^2$. Conversely, if $\mu_2 < \eta n^2$ and $\|\vec{u}_1 - \vec{u}\|^2 \leq \varepsilon^2 n$, then A is α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) , where $\alpha = \eta + 16\varepsilon + 16\alpha_1$.*

Proof. Since $\text{tr}(T) = \text{tr}(MM') = \sum_{i=1}^n \mu_i = \sum_{k,i} m_{ik}^2 = \delta n^2$, it follows that

$$\sum_{i=1}^n \mu_i = \delta n^2. \tag{39}$$

Denote the neighbourhood of vertex v_p in the graph G_A by A_p . Since $\text{tr}(T^2) = \sum_{i=1}^n \mu_i^2 = \sum_{p,q} |\sum_k m_{pk} m_{qk}|^2$, it follows that $\sum_{i=1}^n \mu_i^2 = \sum_{p,q} |A_p \cap A_q|^2$.

Using $\|\vec{u}_1 - \vec{u}\|^2 \leq \varepsilon^2 n$ and Lemma 3.4, we get $\delta^2 n^2 \leq \mu_1 \leq \delta^2 n^2 + 2\varepsilon n^2 + \alpha_1^2 n^2$. Hence

$$\sum_{p,q=1}^n |A_p \cap A_q|^2 - \delta^4 n^4 = \sum_{i=1}^n \mu_i^2 - \delta^4 n^4 = \sum_{i=2}^n \mu_i^2 + \theta(12\varepsilon + 3\alpha_1^2)n^4, \quad (40)$$

where $|\theta| \leq 1$.

Let $\vec{s} = k\vec{e}_1 + l\vec{e}_2$ and $f(\vec{s}) = f(k, l)$ be the balanced function of A . By B_l denote the restriction of A to l -th horizontal line. We have

$$\begin{aligned} \sigma &= \sum_{l,t=1}^N \left| \sum_{k=1}^N f(k, l) \overline{f(k, t)} \right|^2 = \\ &= \sum_{l,t=1}^N \left| \sum_{k=1}^N \left(\chi_A(k, l) - \delta_m \chi_{E_1 \times E_2}(k, l) \right) \left(\chi_A(k, t) - \delta_m \chi_{E_1 \times E_2}(k, t) \right) \right|^2 = \\ &\sum_{l,t \in E_2} \left| |B_l \cap B_t| - \delta_l \delta_t n \right|^2 = \sum_{l,t \in E_2} |B_l \cap B_t|^2 - 2n \sum_{l,t \in E_2} \delta_l \delta_t |B_l \cap B_t| + n^2 \sum_{l,t \in E_2} \delta_l^2 \delta_t^2 \\ &= \sum_{l,t \in E_2} |B_l \cap B_t|^2 - 2n \sum_{v \in E_1} \left(\sum_{l \in E_2} \delta_l \chi_A(v, l) \right)^2 + n^2 \left(\sum_{l \in E_2} \delta_l^2 \right)^2 \end{aligned} \quad (41)$$

Let us estimate the third term in (41). By the Cauchy–Bounyakovskiy inequality, we get

$$n^2 \left(\sum_{l \in E_2} \delta_l^2 \right)^2 \geq \delta^4 n^4. \quad (42)$$

On the other hand, we can rewrite inequality (36) as $\sum_{l \in E_2} \delta_l^2 \leq (\delta^2 + \alpha_1^2)n$, so that

$$n^2 \left(\sum_{l \in E_2} \delta_l^2 \right)^2 \leq \delta^4 n^4 + 3\alpha_1^2 n^4 \quad (43)$$

Combining (42) and (43), we obtain

$$n^2 \sum_{l,t \in E_2} \delta_l^2 \delta_t^2 = \delta^4 n^4 + 3\theta_1 \alpha_1^2 n^4, \quad (44)$$

where $|\theta_1| \leq 1$. Let us estimate the second term in (41). We have

$$\sum_{l \in E_2} \delta_l \chi_A(v, l) = \sum_{l \in E_2} (\delta_l - \delta) \chi_A(v, l) + \delta \sum_{l \in E_2} \chi_A(v, l) = \sum_{l \in E_2} (\delta_l - \delta) \chi_A(v, l) + n\delta \gamma_v$$

Combining (36) and the Cauchy–Bounyakovskiy inequality, we get

$$\left(\sum_{l \in E_2} (\delta_l - \delta) \chi_A(v, m) \right)^2 \leq \sum_{l \in E_2} (\delta_l - \delta)^2 \cdot \sum_{l \in E_2} \chi_A^2(v, l) \leq \alpha_1^2 \gamma_v n^2$$

Let $\rho(v) = \sum_{l \in E_2} (\delta_l - \delta) \chi_A(v, m)$. Then $|\rho(v)| \leq \alpha_1 \gamma_v n$. Let us write the second term in (41) as

$$\begin{aligned} \sigma_1 &= 2n \sum_{v \in E_1} \left(\sum_{l \in E_2} \delta_l \chi_A(v, l) \right)^2 = 2n \sum_{v \in E_1} \left(\rho(v) + n\delta\gamma_v \right)^2 = \\ &= 2n \left(\sum_{v \in E_1} \rho^2(v) + 2n\delta \sum_{v \in E_1} \rho(v)\gamma_v + n^2\delta^2 \sum_{v \in E_1} \gamma_v^2 \right) = 2n^3\delta^2 \sum_{v \in E_1} \gamma_v^2 + 6\theta_2\alpha_1 n^4, \end{aligned} \quad (45)$$

where $|\theta_2| \leq 1$. By the Cauchy–Bounyakovskiy inequality, we get $\sum_{v \in E_1} \gamma_v^2 \geq \delta^2 n$. By (37), it follows that $\sum_{v \in E_1} \gamma_v^2 \leq (\delta^2 + \alpha_1^2)n$. Hence

$$\sigma_1 = 2\delta^2 n^4 + 8\theta_3\alpha_1 n^4, \quad (46)$$

where $|\theta_3| \leq 1$. Substituting (44) and (46) in (41), we obtain

$$\sigma = \sum_{l, t \in E_2} |B_l \cap B_t|^2 - \delta^4 n^4 + 11\alpha_1 \theta_4 n^4, \quad (47)$$

where $|\theta_4| \leq 1$. Clearly, $\sum_{l, t \in E_2} |B_l \cap B_t|^2 = \sum_{p, q} |A_p \cap A_q|^2$. Substituting this equality and (47) in (40), we get

$$\sum_{i=2}^n \mu_i^2 = \sum_{l, t=1}^N \left| \sum_{k=1}^N f(k, l) \overline{f(k, t)} \right|^2 + \theta_5 n^4 (14\alpha_1 + 12\varepsilon), \quad (48)$$

where $|\theta_5| \leq 1$.

If A is α -uniform, then $\sum_{l, t=1}^N \left| \sum_{k=1}^N f(k, l) \overline{f(k, t)} \right|^2 \leq \alpha n^4$. It follows that $\mu_2^2 \leq \sum_{i=2}^n \mu_i^2 \leq \alpha n^4 + 16\varepsilon n^4 + 16\alpha_1 n^4$. Hence $\mu_2 < \alpha^{1/2} n^2 + 4\sqrt{\varepsilon} n^2 + 4\alpha_1^{1/2} n^2$. Conversely, if $\mu_2 < \eta n^2$, then by (39) and (48), we get

$$\begin{aligned} \sum_{p, q} \left| \sum_k f(k, p) \overline{f(k, q)} \right|^2 &\leq \sum_{i=2}^n \mu_i^2 + 16\varepsilon n^4 + 16\alpha_1 n^4 \leq \\ &\leq \mu_2 \sum_{i=2}^n \mu_i + 16\varepsilon n^4 + 16\alpha_1 n^4 < \eta \delta n^4 + 16\varepsilon n^4 + 16\alpha_1 n^4. \end{aligned}$$

This completes the proof.

Lemma 3.6 *Let $B = (b_{ij})$ be a matrix $(n \times n)$ such that $|b_{ij}| \leq D$. Let $\vec{v} = (v_1, \dots, v_n)$, $(\vec{v}, \vec{v}) = n$ be the eigenvector of matrix B , corresponding to the eigenvalue λ , $|\lambda| \geq \alpha n D$, $0 < \alpha < 1$. Then for any $\xi > 0$, $\xi < 1/2$ there exists a natural number m , $m \leq 4/(\alpha\xi)^2$, complex numbers c_1, \dots, c_m and disjoint sets $F_1, \dots, F_m \subseteq \{1, \dots, n\}$ such that*

$$1) \quad \{1, \dots, n\} = \bigsqcup_{i=1}^m F_i.$$

$$2) \quad \text{For any } i \in \{1, \dots, m\} \text{ and } j \in F_i \quad |v_j - c_i| \leq \xi$$

$$3) \quad \text{For any } i \in \{1, \dots, m\} \quad |c_i| \leq 1/\alpha.$$

Proof. For any $i \in \{1, \dots, m\}$ we have

$$|\lambda v_i| = \left| \sum_{k=1}^n b_{ik} v_k \right| \leq nD \quad (49)$$

It follows that, for any $i = 1, \dots, m$ $|v_i| \leq 1/\alpha$. By U denote the closed disk in \mathbf{C} with center in 0 and radius $1/\alpha$. Split the set U into m sets U_1, \dots, U_m , $m \leq 4/(\alpha\xi)^2$ such that diameter of an arbitrary set is at most ξ . Let c_1, \dots, c_m be an arbitrary points from U_1, \dots, U_k . Let us consider the sets

$$F_i = \{j : v_j \in U_i\}, \quad i = 1, \dots, m.$$

The sets F_1, \dots, F_m and the numbers c_1, \dots, c_m satisfies 1) – 3). This completes the proof of Lemma 3.6.

Lemma 3.7 *Let C be a set and $A \subseteq C$, $|A| = \delta|C|$. Let Q_1, \dots, Q_m be a partition of C . Let B be the set of i such that $|A \cap Q_i| < (\delta - \eta)|Q_i|$, $\eta > 0$. Then*

$$\sum_{i \notin B} |A \cap Q_i| \geq \delta \sum_{i \notin B} |Q_i| + \eta \sum_{i \in B} |Q_i|. \quad (50)$$

Proof. We have

$$\delta|C| = \sum_{i=1}^m |A \cap Q_i| = \sum_{i \in B} |A \cap Q_i| + \sum_{i \notin B} |A \cap Q_i| < (\delta - \eta) \sum_{i \in B} |Q_i| + \sum_{i \notin B} |A \cap Q_i| \quad (51)$$

Using (51), we get

$$\begin{aligned} \sum_{i \notin B} |A \cap Q_i| &> \delta|C| - \delta \sum_{i \in B} |Q_i| + \eta \sum_{i \in B} |Q_i| = \\ &= \delta \sum_{i=1}^m |Q_i| - \delta \sum_{i \in B} |Q_i| + \eta \sum_{i \in B} |Q_i| = \delta \sum_{i \notin B} |Q_i| + \eta \sum_{i \in B} |Q_i|. \end{aligned} \quad (52)$$

This completes the proof.

Lemma 3.8 *Let A be a set $A \subseteq E_1 \times E_2$, $|E_2| \leq |E_1|$ and A have cardinality $\delta|E_1||E_2|$. Then for any $\zeta > 0, \zeta < \delta^2$ either A satisfies (36), (37) with $\alpha_1 = \zeta$, or there exist sets G_1 and G_2 , $G_1 \subseteq E_1$, $G_2 \subseteq E_2$ such that*

$$|A \cap (G_1 \times G_2)| > (\delta + \zeta^3/8)|G_1||G_2| \text{ and} \quad (53)$$

$$|G_1|, |G_2| > \zeta^3|E_2|/8. \quad (54)$$

Proof. If both inequalities (36),(37) are true for $\alpha_1 = \zeta$, then we obtain the result. Suppose inequality (36) does not hold. In this case, there exists at most $\zeta^2|E_2|/2$ choices of m such that $|\delta_m - \delta| > \zeta/2$. Let $n = |E_2|$. By B^+ denote the set of m such that $\delta_m > \delta + \zeta/2$ and by B^- denote the set of m such that $\delta_m < \delta - \zeta/2$. Then either $|B^+| \geq \zeta^2 n/4$ or $|B^-| \geq \zeta^2 n/4$. If $|B^+| \geq \zeta n/2$, then put $G_1 = E_1$, $G_2 = B^+$. Clearly, G_1, G_2 satisfies condition (54). Let us check (53). We have $|A \cap (G_1 \times G_2)| = |G_1| \sum_{m \in G_2} \delta_m > (\delta + \zeta/2)|G_1||G_2|$, so that (53) is true. If $|B^-| \geq \zeta^2 n/4$, then put $G_1 = E_1$, $G_2 = E_2 \setminus B^-$. Let us consider the partition $E_1 \times E_2$ into the sets $Q_i = E_1 \times \{x\}$, $x \in E_2$. Obviously, for any $i \in E_2$ the cardinality of Q_i equals $|E_1|$. By Lemma 3.7, we get

$$|A \cap (G_1 \times G_2)| = \sum_{i \notin B^-} |A \cap Q_i| \geq \delta \sum_{i \notin B^-} |Q_i| + \frac{\zeta}{2} \sum_{i \in B^-} |Q_i|$$

Since $|B^-| \geq \zeta^2 n/4$, it follows that

$$|A \cap (G_1 \times G_2)| \geq \delta|G_1||G_2| + \frac{\zeta^3}{8}n|E_1| \geq (\delta + \frac{\zeta^3}{8})|G_1||G_2|. \quad (55)$$

This implies that G_1, G_2 satisfies condition (53). We shall show that condition (53) is also true. Using (55), we get

$$|G_1||G_2| \geq |A \cap (G_1 \times G_2)| \geq \frac{\zeta^3}{8}n|E_1|.$$

Hence $|G_1| \geq n\zeta^3/8$. This completes the proof.

Let \vec{u}_1 be the nonnegative eigenvector corresponding to the first eigenvalue μ_1 of matrix T , and \vec{u}_2 be the eigenvector corresponding to the second eigenvalue μ_2 . Vector \vec{u}_1 exists by Theorem 3.2. Let $(\vec{u}_1, \vec{u}_1) = (\vec{u}_2, \vec{u}_2) = n$ and $(\vec{u}_1, \vec{u}_2) = 0$. Define $\vec{u} = (1, \dots, 1)$.

Proposition 3.9 *Let $A \subseteq E_1 \times E_2$ be a set, $|E_1| = |E_2| = n$ and let A have cardinality δn^2 . Let $\alpha > 0$ be a real number and*

$$A \text{ is not } \alpha\text{-uniform with respect to the basis } (\vec{e}_1, \vec{e}_2). \quad (56)$$

If $\|\vec{u} - \vec{u}_1\|^2 \leq \alpha^2/36 \cdot n$ then one can find the sets $G_1 \subseteq E_1$ and $G_2 \subseteq E_2$ such that

$$|A \cap (G_1 \times G_2)| > (\delta + 2^{-200} \alpha^{60}) |G_1| |G_2| \text{ and } |G_1|, |G_2| > 2^{-200} \alpha^{60} n. \quad (57)$$

If $\|\vec{u} - \vec{u}_1\|^2 > \alpha^2/36 \cdot n$ then (57) takes place even if (56) is not suppose to be true.

Proof. We can assume that inequalities (36), (37) hold for $\alpha_1 = 2^{-56} \alpha^{20}$. If it is not true, then we can find G_1, G_2 by Lemma 3.8.

Case 1. $\|\vec{u} - \vec{u}_1\|^2 \leq \alpha^2/36 \cdot n$.

By assumption A is not α -uniform. By Lemma 3.5, it follows that $\mu_2 \geq \alpha n^2/2$. Let $\mathcal{E} = (e_{ij})$ be the matrix ($n \times n$) such that $e_{ij} = 1, i, j = 1, \dots, n$ and put $M_1 = M - \delta \mathcal{E}, T_1 = M_1 M_1'$. We have

$$(M_1' \vec{u}_2, M_1' \vec{u}_2) = (M_1' \vec{u}_2, M_1' \vec{u}_2) + (M_1' \vec{u}_2, \delta \mathcal{E} \vec{u}_2) + (\delta \mathcal{E} \vec{u}_2, M_1' \vec{u}_2) + (\delta \mathcal{E} \vec{u}_2, \delta \mathcal{E} \vec{u}_2) \quad (58)$$

Let us estimate the second term in (58). Since $(\vec{u}_1, \vec{u}_2) = 0$, it follows that $(\vec{u}, \vec{u}_2) = (\vec{u} - \vec{u}_1, \vec{u}_2)$. Combining the Cauchy–Bounyakovskiy inequality and the inequality $\|\vec{u} - \vec{u}_1\|^2 \leq \alpha^2/36n$, we obtain

$$|(\vec{u}, \vec{u}_2)|^2 \leq \|\vec{u} - \vec{u}_1\|^2 n \leq \alpha^2/36n^2. \quad (59)$$

We have $\mathcal{E} \vec{u}_2 = \vec{u}(\vec{u}, \vec{u}_2)$. Using 3.3, the Cauchy–Bounyakovskiy inequality and (59), we get

$$|(M_1' \vec{u}_2, \delta \mathcal{E} \vec{u}_2)|^2 \leq (M_1' \vec{u}_2, M_1' \vec{u}_2) \cdot (\mathcal{E} \vec{u}_2, \mathcal{E} \vec{u}_2) \leq n^3 (\vec{u}(\vec{u}, \vec{u}_2), \vec{u}(\vec{u}, \vec{u}_2)) \leq \alpha^2/36n^6 \quad (60)$$

Hence $|(M_1' \vec{u}_2, \delta \mathcal{E} \vec{u}_2)| \leq 2^{-6} \alpha n^3$. In the same way $|(\delta \mathcal{E} \vec{u}_2, M_1' \vec{u}_2)| \leq 2^{-6} \alpha n^3$ and $|(\delta \mathcal{E} \vec{u}_2, \delta \mathcal{E} \vec{u}_2)| \leq 2^{-6} \alpha n^3$. Finally, we obtain

$$\alpha n^3/2 \leq \mu_2 n = (T \vec{u}_2, \vec{u}_2) = (M_1' \vec{u}_2, M_1' \vec{u}_2) \leq (M_1' \vec{u}_2, M_1' \vec{u}_2) + \frac{\alpha}{4} n^3. \quad (61)$$

$$(M_1' \vec{u}_2, M_1' \vec{u}_2) \geq \alpha n^3/4. \quad (62)$$

By a_{ij} denote the entries of the matrix M'_1 and by x_i denote the entries of the vector \vec{u}_2 . By the Cauchy–Bounyakovski inequality, it follows that $|\sum_{k=1}^n a_{ik}x_k| \leq n$, for any $i = 1, \dots, n$. Using (62), we get

$$\frac{\alpha}{4}n^3 \leq \sum_{i=1}^n \left| \sum_{k=1}^n a_{ik}x_k \right|^2 \leq n \sum_{i=1}^n \left| \sum_{k=1}^n a_{ik}x_k \right| \quad (63)$$

Clearly, all entries of the matrix T are bounded by n . Let us apply Lemma 3.6 to the matrix T and its eigenvector \vec{u}_2 with parameters $D = n$ and $\xi = \alpha/16$. By this Lemma we can find the sets F_1, \dots, F_m , $m \leq 2^{11}\alpha^{-4}$ and the complex numbers c_1, \dots, c_m such that conditions 1)–3) are satisfied. Combining (63) and the triangle inequality, we obtain

$$\sum_{i=1}^n \sum_{j=1}^m \left| \sum_{k \in F_j} a_{ik}x_k \right| \geq \frac{\alpha}{4}n^2 \quad (64)$$

Define

$$B = \{j : |F_j| < 2^{-16}\alpha^8 n\} \quad (65)$$

By the Cauchy–Bounyakovski inequality, we get

$$\sum_{i=1}^n \sum_{j \in B} \left| \sum_{k \in F_j} a_{ik}x_k \right| < n^2 m \sqrt{2^{-16}\alpha^8} \leq \frac{\alpha}{8}n^2 \quad (66)$$

Hence

$$\sum_{i=1}^n \sum_{j \notin B} \left| \sum_{k \in F_j} a_{ik}x_k \right| \geq \frac{\alpha}{8}n^2 \quad (67)$$

Using properties 2), 3) of Lemma 3.6, we obtain

$$\frac{\alpha}{16}n^2 \leq \sum_{i=1}^n \sum_{j \notin B} \left| \sum_{k \in F_j} a_{ik}c_j \right| \leq \frac{2}{\alpha} \sum_{i=1}^n \sum_{j \notin B} \left| \sum_{k \in F_j} a_{ik} \right| \quad (68)$$

Let us consider the sets

$$J_j^+ = \{i \mid j \notin B, \sum_{k \in F_j} a_{ik} \geq 0\}, \text{ and } J_j^- = \{i \mid j \notin B, \sum_{k \in F_j} a_{ik} < 0\} \quad (69)$$

Let C be the set of k such that $|\gamma_k - \delta| > \sqrt{\alpha_1}$. By (37), we have $|C| \geq \alpha_1 n$. Let $\bar{C} = \{1, \dots, n\} \setminus C$. Note that for all $j \notin B$ the following inequality holds $|C| \leq \alpha_1 n \leq \sqrt{\alpha_1} |F_j|$. We have

$$\left| \sum_i \sum_{k \in F_j} a_{ik} \right| = \left| n \sum_{k \in F_j} (\gamma_k - \delta) \right| = n \left| \sum_{k \in F_j \cap C} (\gamma_k - \delta) + \sum_{k \in F_j \cap \bar{C}} (\gamma_k - \delta) \right| \leq$$

$$\leq |C|n + \sqrt{\alpha_1}|F_j|n \leq 2\sqrt{\alpha_1}|F_j|n \leq \frac{\alpha^2}{64}|F_j|n$$

It follows that

$$\left| \sum_{i \in J_j^+} \sum_{k \in F_j} a_{ik} + \sum_{i \in J_j^-} \sum_{k \in F_j} a_{ik} \right| = \left| \sum_i \sum_{k \in F_j} a_{ik} \right| \leq \frac{\alpha^2}{64}|F_j|n.$$

Hence $\left| \sum_{i \in J_j^-} \sum_{k \in F_j} a_{ik} \right| \leq \sum_{i \in J_j^+} \sum_{k \in F_j} a_{ik} + \alpha^2|F_j|n/16$. Using (68), we get

$$\frac{\alpha^2}{128}n^2 \leq \sum_{j \notin B} \sum_{i \in J_j^+} \sum_{k \in F_j} a_{ik}. \quad (70)$$

Let $j_0 \notin B$ be the index for which the sum $\sum_{i \in J_j^+} \sum_{k \in F_j} a_{ik}$ is maximal. Then

$$\sum_{i \in J_{j_0}^+} \sum_{k \in F_{j_0}} a_{ik} \geq \frac{\alpha^2}{128m}n^2 \geq 2^{-18}\alpha^6n^2. \quad (71)$$

By (71), it follows that $|J_{j_0}^+| > 2^{-18}\alpha^6 \cdot n$. Put $G_1 = F_{j_0}$ and $G_2 = J_{j_0}^+$. Since $j_0 \notin B$, so that $|G_1| \geq 2^{-16}\alpha^8n$. Using (71), we get

$$|A \cap (G_1 \times G_2)| > (\delta + 2^{-18}\alpha^6)|G_1||G_2|.$$

It is clear that the sets G_1, G_2 satisfies the conditions (57).

Case 2. $\|\vec{u} - \vec{u}_1\|^2 > \alpha^2/36 \cdot n$.

Since $(\vec{u}_1, \vec{u}) \geq 0$, it follows that

$$\alpha^2/36 \cdot n < \|\vec{u} - \vec{u}_1\|^2 = (\vec{u}, \vec{u}) - 2(\vec{u}, \vec{u}_1) + (\vec{u}_1, \vec{u}_1) = 2n - 2(\vec{u}, \vec{u}_1).$$

Hence

$$0 \leq (\vec{u}_1, \vec{u}) \leq n - 2^{-13}\alpha^2 \cdot n \quad (72)$$

We have $\mathcal{E}^2 = n\mathcal{E}$ and $\mathcal{E}\vec{u}_1 = \vec{u}(\vec{u}, \vec{u}_1)$. Then

$$\begin{aligned} (T_1\vec{u}_1, \vec{u}_1) &= (T\vec{u}_1, \vec{u}_1) - \delta(\mathcal{E}M'\vec{u}_1, \vec{u}_1) - \delta(M\mathcal{E}\vec{u}_1, \vec{u}_1) + \delta^2(\mathcal{E}^2\vec{u}_1, \vec{u}_1) = \\ &= (T\vec{u}_1, \vec{u}_1) - \delta(M'\vec{u}_1, \mathcal{E}\vec{u}_1) - \delta(M\mathcal{E}\vec{u}_1, \vec{u}_1) + \delta^2n|(\vec{u}_1, \vec{u})|^2 = \sigma \end{aligned} \quad (73)$$

Let us calculate the term $(M'\vec{u}_1, \mathcal{E}\vec{u}_1)$ in (73). $(M'\vec{u}_1, \mathcal{E}\vec{u}_1) = (\vec{u}, \vec{u}_1)(M'\vec{u}_1, \vec{u}_1) = (\vec{u}, \vec{u}_1)(\vec{u}_1, M\vec{u})$. Let \vec{v} equals $\{(\delta_m - \delta)\}_{m=1}^n$. Then $M\vec{u} = \delta n\vec{u} + n\vec{v}$. It follows

that $(\vec{u}_1, M\vec{u}) = \delta n(\vec{u}, \vec{u}_1) + n(\vec{u}_1, \vec{v})$. Combining the Cauchy–Bounyakovskiy inequality and (36), we obtain

$$(\vec{u}_1, \vec{v})^2 \leq \|\vec{u}_1\|_2^2 \|\vec{v}\|_2^2 = n \sum_m (\delta_m - \delta)^2 \leq \alpha_1^2 n^2$$

Hence $(M'\vec{u}_1, \mathcal{E}\vec{u}_1) = \delta n(\vec{u}_1, \vec{u})^2 + \theta \alpha_1 n^3$, where $|\theta| \leq 1$. In the same way $(M\mathcal{E}\vec{u}_1, \vec{u}_1) = \delta n(\vec{u}_1, \vec{u})^2 + \theta_1 \alpha_1 n^3$, where $|\theta_1| \leq 1$. It follows that

$$\sigma \geq (T\vec{u}_1, \vec{u}_1) - \delta^2 n(\vec{u}_1, \vec{u})^2 - 2\alpha_1 n^3. \quad (74)$$

By Lemma 3.4 $\mu_1 \geq \delta^2 n^2$, so that $(T\vec{u}_1, \vec{u}_1) = \mu_1(\vec{u}_1, \vec{u}_1) \geq \delta^2 n^3$. Using this and (72), we get

$$(T_1\vec{u}_1, \vec{u}_1) > 2^{-13} \alpha^2 n^3 \quad (75)$$

The only difference between (75) and (62) is the inequality (75) has the vector \vec{u}_2 instead of the vector \vec{u}_1 . Vector \vec{u}_1 as \vec{u}_2 is the eigenvector of matrix T . Moreover the eigenvalue μ_1 corresponding to the vector \vec{u}_2 more than eigenvalue μ_2 corresponding to the vector \vec{u}_1 . So, there exist sets G_1 and G_2 , $G_1 \subseteq E_1$, $G_2 \subseteq E_2$ such that $|G_1|, |G_2| > 2^{-200} \alpha^{60} n$ and $|A \cap (G_1 \times G_2)| > (\delta + 2^{-200} \alpha^{60}) |G_1| |G_2|$. This completes the proof.

In Proposition 3.9 the set $E_1 \times E_2$ is a square. Let us consider the case when $E_1 \times E_2$ is a box.

Proposition 3.10 *Let $A \subseteq E_1 \times E_2$ be a set of size $\delta |E_1| |E_2|$. Let $\alpha > 0$ be a real number and A is not α -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) . Then there exist two sets $G_1 \subseteq E_1$ and $G_2 \subseteq E_2$ such that*

$$|A \cap (G_1 \times G_2)| > (\delta + 2^{-500} \alpha^{70}) |G_1| |G_2| \text{ and} \quad (76)$$

$$|G_1|, |G_2| > 2^{-500} \alpha^{70} \min\{|E_1|, |E_2|\}. \quad (77)$$

Proof. We can assume that inequalities (36), (37) hold for $\alpha_1 = \alpha/10$. If these inequalities are not true, then we can find G_1, G_2 by Lemma 3.8. Let $K \subseteq E_1 \times E_2$ be an arbitrary set. Define $g_K(\vec{s}) = (\chi_A(\vec{s}) - \delta) \chi_K(\vec{s})$. Let $g = g_{E_1 \times E_2}$, $\varepsilon(\vec{s}) = (\delta - \delta_m) \chi_{E_1 \times E_2}(\vec{s})$ and $f(\vec{s})$ be the balanced function of A . Then $f(\vec{s}) = g(\vec{s}) + \varepsilon(\vec{s})$. Since the set A is not α -uniform, it follows that $\|f\| \geq \alpha |E_1|^2 |E_2|^2$. By Lemma 2.6, we have $\|f\| \leq \|g\| + \|\varepsilon\|$. Norm of the function $\varepsilon(\vec{s})$ equals $(|E_1|^2 \sum_{m,i} ((\delta - \delta_m)(\delta - \delta_i))^2)^{1/4}$. Using (36) with $\alpha_1 = \alpha/10$, we obtain $\|\varepsilon(\vec{s})\| \leq \alpha (|E_1| |E_2|)^{1/2} / 10$. It follows that $\|g\|^4 \geq \alpha |E_1|^2 |E_2|^2 / 2$.

Let us prove that there exist sets $W_1 \subseteq E_1$, $W_2 \subseteq E_2$ $|W_2| \leq |W_1| \leq 2|W_2|$ such that $\|g_{W_1 \times W_2}\|^4 \geq \alpha|W_1|^2|W_2|^2/16$. Without loss of generality it can be assumed that $|E_2| \leq |E_1|$. If $|E_2| \leq |E_1| \leq 2|E_2|$, then put $W_1 = E_1$, $W_2 = E_2$. In the converse case we have $|E_1| > 2|E_2|$. Let $E_1 = \sqcup_{i=1}^t R_i \sqcup W$, where $|R_1| = \dots = |R_t| = |E_2|$ and $|W| < |E_2|$. If $|W| \geq |E_2|/2$, then $|W| \leq |E_2| \leq 2|W|$. If $|W| < |E_2|/2$, then split $R_t \cup W$ into the sets Y_1, Y_2 of same size. In any case, we can find the partition of $E_1 \times E_2$ into m squares $Z_i \times E_2$ such that $|Z_i| \leq |E_2| \leq 2|Z_i|$. Note that $m \leq t + 2$.

Let $g_i = g_{Z_i \times E_2}$, $i = 1, \dots, m$. We have $g(\vec{s}) = \sum_i g_i(\vec{s})$. Let B be the set of i such that $\|g_i\|^4 \geq \alpha|Z_i|^2|E_2|^2/16$. Then we have $|B| \geq \alpha m/16$. Assume the converse. Then

$$\begin{aligned} \|g\|^4 &= \sum_{p,q} \left| \sum_k g(k,p)\overline{g(k,q)} \right|^2 = \sum_{p,q} \left| \sum_{i=1}^m \sum_k g_i(k,p)\overline{g_i(k,q)} \right|^2 \leq \\ &\leq m \sum_i \sum_{p,q} \left| \sum_k g_i(k,p)\overline{g_i(k,q)} \right|^2 = m \sum_{i \in B} \sum_{p,q} \left| \sum_k g_i(k,p)\overline{g_i(k,q)} \right|^2 + \\ &+ m \sum_{i \notin B} \sum_{p,q} \left| \sum_k g_i(k,p)\overline{g_i(k,q)} \right|^2 < \alpha m^2 |E_2|^4 / 8 < \alpha |E_1|^2 |E_2|^2 / 2. \end{aligned} \quad (78)$$

This contradicts the inequality $\|g\|^4 \geq \alpha|E_1|^2|E_2|^2/2$. Suppose for all $i \in B$ the following condition holds $\delta_{Z_i \times E_2}(A) < \delta - 2^{-450}\alpha^{64}$. Let $S = \cup_{i \notin B} Z_i$. Let us apply Lemma 3.6 for the matrix T and the eigenvector \vec{u}_2 of T with parameters $D = n$ and $\xi = \alpha/16$ and let us apply Lemma 3.7 to the set $C = E_1 \times E_2$ and its partition into the sets $Q_i = Z_i \times E_2$. By Lemma 3.7, we get

$$\sum_{i \notin B} |A \cap (Z_i \times E_2)| = |A \cap (S \times E_2)| \geq \delta |A \cap (S \times E_2)| + 2^{-450}\alpha^{64} \sum_{i \in B} |Z_i \times E_2| \quad (79)$$

For any Z_i, Z_j we have $|Z_i| \leq 2|Z_j|$. Using this fact and the inequality $|B| \geq \alpha m/16$, we obtain $\sum_{i \in B} |Z_i \times E_2| / \sum_{i \notin B} |Z_i \times E_2| \geq 2^{-5}\alpha$ and $\delta_{S \times E_2}(A) \geq \delta + 2^{-455}\alpha^{65}$. Put $G_1 = S$ and $G_2 = E_2$. Using (79), we get $|G_1||E_2| \geq 2^{-450}\alpha^{64}|Z_1||B||E_2|/2 \geq 2^{-455}\alpha^{65}|E_1|$. G_1 and G_2 satisfies the conditions (76), (77) which prove the Proposition. Thus there exists i_0 such that $\|g_{i_0}\|^4 \geq \alpha|Z_{i_0}|^2|E_2|^2/16$ and $\delta_{Z_{i_0} \times E_2}(A) \geq \delta - 2^{-450}\alpha^{64}$. Put $W_1 = E_2$, $W_2 = Z_{i_0}$. If $|W_1| \geq 3/2|W_2|$, then split $W_1 \times W_2$ into a square K and a rectangle P such that $|K| = |W_2|^2$, $P = P_1 \times P_2$, where $|P_2| = |W_2|$, $|P_1| \geq |W_1|/2$. If $|W_2| \leq |W_1| < 3/2|W_2|$, then split W_2 into two equal parts and split W_1

into two parts so that the length of the first part is $|W_2|/2$. Then $W_1 \times W_2$ partitioned into 2 squares and a rectangle.

In both cases the rectangles $P = P_1 \times P_2$ obtained under the described construction has the property that $|P_2| \leq |P_1| \leq 2|P_2|$. We can also split any of the obtained rectangle the way we have just done above. Let us iterate this procedure k times, $k = 2 \log_2(1/\alpha)$. We obtain at most 2^{k+1} squares K_i and at most 2^{k+1} boxes. The number of points in all boxes is at most $(2/3)^k |W_1| |W_2|$. By C denote the union of all these boxes. Then $\|g_C\| \leq (2/3)^{k/2} |W_1|^{1/2} |W_2|^{1/2} < (\alpha/16)^{1/4} |W_1|^{1/2} |W_2|^{1/2} / 2$. We have $\|g_{W_1 \times W_2}\| \geq (\alpha/16)^{1/4} |W_1|^{1/2} |W_2|^{1/2}$ and $g_{W_1 \times W_2} = \sum_i g_{K_i} + g_C$. By Lemma 2.6 there exists i_1 such that $\|g_{K_{i_1}}\| \geq (\alpha/16)^{1/4} |W_1|^{1/2} |W_2|^{1/2} / 2^{k+2}$. Let $F = K_{i_1}$ and $\bar{F} = \bigcup_{i \neq i_1} K_i \cup C$. The length of each K_i is at least $2^{-k} |W_2|$. The length of each rectangle from C is at least $2^{-(k+1)} |W_2|$. Hence $|F| \geq \alpha^4 |\bar{F}| / 10$.

If $\delta_{\bar{F}}(A) > \delta + 2^{-450} \alpha^{64}$, then the density of A in one of the squares K_i , $i \neq i_1$ or in one of the rectangles from C is at least $\delta + 2^{-450} \alpha^{64}$. Denote by G_1 and G_2 the sides of this square or rectangle. Then the sets G_1, G_2 satisfies the condition (76). The length of an arbitrary square K_i and any box from C is at least $2^{-(k+1)} |W_2|$. Hence the sets G_1, G_2 satisfies the condition (77).

Let us assume that $\delta_{\bar{F}}(A) \leq \delta + 2^{-210} \alpha^{64}$. Let $A_1 = A \cap F$, $A_2 = A \cap \bar{F}$. Then

$$(\delta - 2^{-450} \alpha^{64}) |W_1| |W_2| \leq |A| = |A_1| + |A_2| \leq \delta_F(A) |F| + (\delta + 2^{-450} \alpha^{64}) |\bar{F}|. \quad (80)$$

Combining (80) and the estimate $|F| \geq \alpha^4 |\bar{F}| / 10$, we get $\delta_F(A) \geq \delta - 2^{-445} \alpha^{60}$. Let us apply Proposition 3.9 to the square F . By this Proposition there exist the sets G_1, G_2 such that the conditions (76) and (77) is take place. This completes the proof of Proposition 3.10.

4. Proof of main result.

To prove Theorem 1.4 we need several lemmas.

Lemma 4.1 *Let $A \subseteq E_1 \times E_2$ be a set of cardinality $\delta |E_1| |E_2|$ and f_A be the balanced function of A . Then $\|f_A\| \rightarrow 0$ as $\delta \rightarrow 1$.*

Proof. By A_p denote the neighbourhood of vertex v_p in the graph G_A . Then

$$\begin{aligned} \sigma &= \|f_A\|^4 = \sum_{m,p \in E_2} \left| |A_m \cap A_p| - \delta_m \delta_p |E_1| \right|^2 = \\ &= \sum_{m,p \in E_2} |A_m \cap A_p|^2 - 2|E_1| \sum_{m,p \in E_2} |A_m \cap A_p| \delta_m \delta_p + |E_1|^2 \sum_{m,p \in E_2} \delta_m^2 \delta_p^2 \quad (81) \end{aligned}$$

Clearly, $|A_m \cap A_p| \geq (\delta_m + \delta_p - 1)|E_1|$. It follows that

$$\begin{aligned} 2|E_1| \sum_{m,p \in E_2} |A_m \cap A_p| \delta_m \delta_p &\geq 2|E_1|^2 \sum_{m,p \in E_2} (\delta_m + \delta_p - 1) \delta_m \delta_p = \\ 2|E_1|^2 \sum_{m,p \in E_2} (\delta_m^2 \delta_p + \delta_m \delta_p^2 - \delta_m \delta_p) &= 4\delta |E_1|^2 |E_2| \sum_{m \in E_2} \delta_m^2 - 2\delta^2 |E_1|^2 |E_2|^2 \quad (82) \end{aligned}$$

Note that $\delta_m \leq 1$. Then

$$|E_1|^2 \sum_{m,p \in E_2} \delta_m^2 \delta_p^2 = |E_1|^2 \left(\sum_{m \in E_2} \delta_m^2 \right)^2 \leq |E_1|^2 \left(\sum_{m \in E_2} \delta_m \right)^2 = \delta^2 |E_1|^2 |E_2|^2 \quad (83)$$

Combining (82), (83), the Cauchy–Bounyakovski inequality and (81), we obtain

$$\begin{aligned} \sigma &\leq \sum_{m,p \in E_2} \left| \sum_k \chi_A(k, m) \chi_A(k, p) \right|^2 - 4\delta |E_1|^2 |E_2| \sum_{m \in E_2} \delta_m^2 + 3\delta^2 |E_1|^2 |E_2|^2 \leq \\ &\leq \sum_{m,p \in E_2} \left(\sum_k \chi_A^2(k, m) \right) \left(\sum_k \chi_A^2(k, p) \right) - 4\delta |E_1|^2 \left(\sum_{m \in E_2} \delta_m \right)^2 + 3\delta^2 |E_1|^2 |E_2|^2 \\ &= 4\delta^2 |E_1|^2 |E_2|^2 - 4\delta^3 |E_1|^2 |E_2|^2 = 4|E_1|^2 |E_2|^2 \delta^2 (1 - \delta). \quad (84) \end{aligned}$$

By (84), it follows that $\|f_A\| \rightarrow 0$ as $\delta \rightarrow 1$. This completes the proof.

Two–dimensional arithmetic progression P is called *right square* if $P = P_1 \times P_2$, where P_1, P_2 are one–dimensional arithmetic progressions with equal differences and cardinalities.

Let $P = \{a, a + d, \dots, a + (t - 1)d\} \times \{b, b + d, \dots, b + (t - 1)d\}$ be a right square and let E be a set. The right square P is isomorphic to the square $\{1, \dots, t\}^2$. Let the isomorphism φ is given by $\varphi(a + kd, b + ld) = (k, l)$, where $k, l = 1, \dots, t$. The set E is called α –uniform in right square P , if the set $\varphi(E \cap P)$ is α –uniform in right square $\{1, \dots, t\}^2$. In other words E is α –uniform in right square P , if the balanced function f of the set $\varphi(E \cap P)$ satisfies (11).

One–dimensional case of Theorem 4.2 was actually proved in [6] (see also [14]).

Theorem 4.2 *Let ε, δ be a numbers, $0 < \varepsilon \leq \delta$ and let $\alpha(s) = Ks^\rho$, $K \in (0, 1]$, $\rho \geq 4$. Let $W \subseteq \{1, \dots, N\}^2$ be a set of size δN^2 and $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, where $C = 2^{1000\rho}$, $c_1 = 100\rho$, $c_2 = 2^{-128}$ and $\alpha = \alpha(\varepsilon)$. There exist right squares $P_1, \dots, P_M \subseteq \{1, \dots, N\}^2$ and the partitions of W into the sets $W \cap P_1, \dots, W \cap P_M$ and B such that*

- 1) For any i the set W is $\alpha(\delta_{P_i}(W))$ -uniform in progression P_i .
- 2) For any i we have $|W \cap P_i| \geq \varepsilon|P_i|$ and $|P_i| \geq N^{c_2^{1/\alpha}}$.
- 3) $|B| < 4\varepsilon N^2$.

Proof. To prove this Theorem, we need several lemmas.

Lemma 4.3 Let s and N be a natural numbers, $s \leq N$ and $\phi : \mathbf{Z}_N^2 \rightarrow \mathbf{Z}_N$ be a function such that $\phi(x, y) = r_1x + r_2y$, $(r_1, r_2) \neq \vec{0}$. Then \mathbf{Z}_N can be partitioned into arithmetic progressions P_1, \dots, P_M , $M \leq 8N^{4/3}/s^{2/3}$ with the same difference such that the diameter of $\phi(P_i \times P_j)$ is at most s for any $i, j \in \{1, \dots, M\}$ and the lengths of any two P_j differ by at most 1.

Note. The lengths of progressions P_1, \dots, P_M in Lemma 4.3 are at least $N/M \geq s^{2/3}/(16N^{1/3})$.

Proof of Lemma 4.3 Let $t = \lceil (N^2/s)^{1/3}/2 \rceil$. Let us consider $t^2 + 1$ vectors $\vec{0}, (r_1, r_2), 2(r_1, r_2), \dots, t^2(r_1, r_2) \in \mathbf{Z}_N^2$. Let us split the set $[1, N]^2$ into t^2 squares of same size. The side of any such square equals N/t . By the pigeonhole principal one of the squares will contain two vectors. Suppose that these vectors are $t_1(r_1, r_2)$ and $t_2(r_1, r_2)$ and let $t_2 > t_1$. Put $u = t_2 - t_1$. Then $0 < u \leq t^2$ and $|ur_1|, |ur_2| \leq N/t$. Split $\{1, \dots, N\}$ into congruence classes mod u . Each congruence class is an arithmetic progression of cardinality either $\lfloor N/u \rfloor$ or $\lceil N/u \rceil$. Let P and Q be arbitrary sets of at most $st/2N$ consecutive elements of two congruence classes. Then the diameter of $\phi(P \times Q)$ is at most

$$|ur_1||P| + |ur_2||Q| \leq N/t \cdot st/2N + N/t \cdot st/2N = s. \quad (85)$$

We have $st/2N \leq N/3t^2 \leq 1/2\lceil N/u \rceil$. Clearly, each congruence class can be divided into at most $4N^2/(ust)$ sub-progressions P_j , $|P_j| \leq st/2N$ such that the lengths of any two P_j differ by at most 1. Since the congruence classes themselves differ in size by at most 1, it is not hard to see that the whole of \mathbf{Z}_N can be thus partitioned. Hence, the number of sub-progressions is at most $8N^{4/3}/s^{2/3}$. Moreover, for any $i, j \in \{1, \dots, M\}$ the diameter $\phi(P_i \times P_j)$ is at most s . This completes the proof.

Lemma 4.4 Let $\alpha \in (0, 1)$ and $N \geq 2^{100}/\alpha^{10}$. Let $A \subseteq \mathbf{Z}_N^2$ be a set of size δN^2 and suppose that $|\widehat{\chi}_A(\vec{r})| \geq \alpha N^2$ for some $\vec{r} \neq \vec{0}$. Then the set \mathbf{Z}_N^2 can be partitioned into right squares S_1, \dots, S_r of same size and the set Ω such that $|S_i| \geq N^{1/2}$, $i = 1, \dots, r$, $|\Omega| < N^{11/6}$ and $1/r \cdot \sum_{j=1}^r |\delta_{S_j}(A) - \delta|^2 \geq \alpha^2/16$.

Proof of Lemma 4.4 Let $\vec{r} = (r_1, r_2)$ and $s = \lceil \alpha N/(4\pi) \rceil$. Since $N \geq 2^{100}/\alpha^{10}$, it follows that $s \neq 0$. Let us apply Lemma 4.3 with parameters s and N to the function $\phi(x, y) = r_1x + r_2y$. By this Lemma there exists a partition

of \mathbf{Z}_N into arithmetic progression with the same difference P_1, \dots, P_M , $M \leq 8N^{4/3}/s^{2/3}$ such that the diameter of $\phi(P_i \times P_j)$ is at most s for any $i, j \in \{1, \dots, M\}$ and the lengths of any two P_j differ by at most 1.

Let $\vec{s} = k\vec{e}_1 + m\vec{e}_2$ и $f(\vec{s}) = \chi_A(\vec{s}) - \delta$. Since $\vec{r} \neq \vec{0}$, it follows that $\widehat{f}(\vec{r}) = \widehat{\chi}_A(\vec{r})$. We have

$$\left| \sum_{k,m} f(k, m)e(-(r_1k + r_2m)) \right| \geq \alpha N^2 \quad (86)$$

Let us consider the partition \mathbf{Z}_N^2 into the sets $P_{ij} = P_i \times P_j$. Note that the number of these sets is M^2 . By the triangle inequality, it follows that

$$\sum_{i,j=1}^M \left| \sum_{(k,m) \in P_{ij}} f(k, m)e(-(r_1k + r_2m)) \right| \geq \alpha N^2 \quad (87)$$

Let i, j be any numbers, $i, j \in [1, M]$ and w_{ij} be any elements of P_{ij} . By Lemma 4.3, it follows that diameter $\phi(P_{ij})$ is at most $\alpha N/4\pi$. Hence for any $w \in P_{ij}$ we have $|e(-\phi(w)) - e(-\phi(w_{ij}))| \leq \alpha/2$. By (87), it follows that

$$\begin{aligned} \sum_{i,j=1}^M \left| \sum_{(k,m) \in P_{ij}} f(k, m) \right| &= \sum_{i,j=1}^M \left| \sum_{(k,m) \in P_{ij}} f(k, m)e(-\phi(w_{ij})) \right| \geq \\ &\geq \sum_{i,j=1}^M \left| \sum_{(k,m) \in P_{ij}} f(k, m)e(-\phi(k, m)) \right| - \\ &- \sum_{i,j=1}^M \left| \sum_{(k,m) \in P_{ij}} f(k, m)(e(-\phi(k, m)) - e(-\phi(w_{ij}))) \right| \geq \\ &\geq \alpha N^2 - \sum_{i,j=1}^M \alpha/2 |P_{ij}| = \alpha N^2/2 \end{aligned} \quad (88)$$

Let us show that for any P_{ij} we can find right square S such that $S \subseteq P_{ij}$ and $|S| \geq N^{1/2}$.

Recall that lengths of P_i and P_j differ by at most 1. If $|P_i| = |P_j|$, then put $S = P_{ij}$. Since $|P_i|, |P_j| \geq \alpha^{2/3}/(16(4\pi)^{2/3})N^{1/3}$, it follows that $|S| \geq \alpha^{4/3}/(2^8(4\pi)^{4/3})N^{2/3}$. By assumption $N \geq 2^{100}/\alpha^{10}$, so that $|S| \geq N^{1/2}$.

Suppose lengths of P_i and P_j are not equal. We can assume without loss of generality that $|P_i| = |P_j| + 1$. Let $P_i = \{x\} \sqcup Q$, where Q is an arithmetic

progression. Define $S = Q \times P_j$ and $\Omega_{ij} = A \cap (\{x\} \times P_j)$. Then $|S| \geq N^{1/2}$ and $|\Omega_{ij}| \leq |P_j|$.

After having repeated this procedure for all the sets P_{ij} we shall obtain a family of Λ right squares S_1, \dots, S_r , $r = M^2$ of same size. By Ω denote the union of all Ω_{ij} , $i, j = 1, \dots, M$. Then $|\Omega| \leq 2M^2N/M < N^{11/6}$. Using this and (88), we get

$$\sum_{S \in \Lambda} \left| \sum_{(k,m) \in S} f(k,m) \right| \geq \alpha N^2/4 \quad (89)$$

Let $\delta_j = \delta_{S_j}(A)$. By t denote the number of elements in each square from Λ . Using (89), we have

$$\sum_{j=1}^r |\delta_j - \delta| \geq \alpha N^2/4t \geq \alpha r/4 \quad (90)$$

By the Cauchy–Bounyakovski inequality, we get

$$\sum_{j=1}^r |\delta_j - \delta|^2 \geq \alpha^2 r/16 \quad (91)$$

as required.

Proof of Theorem 4.2 Let Σ be a family of disjoint sets C_1, \dots, C_m , $C_i \subseteq \mathbf{Z}_N^2$, $i = 1, \dots, m$. Define the function $E(\Sigma)(\vec{x})$ by the rule $E(\Sigma)(\vec{x}) = \sum_{j=1}^m \delta_{C_j}(W) \chi_{C_j}(\vec{x})$. It is clear that for an arbitrary family of sets Σ we have $|E(\Sigma)(\vec{x})| \leq 1$. It follows that $\|E(\vec{x})\|_2^2 = \sum_{\vec{x}} E(\Sigma)(\vec{x})^2 \leq N^2$.

The proof of Theorem 4.2 is a sort of an inductive process. At the i -th step of this process we shall construct a family $\Sigma^{(i)}$ of disjoint right squares C_1, \dots, C_{ν_i} and exceptional set $\Omega^{(i)} \supseteq \Omega^{(i-1)}$ such that

$$|C_i| \geq N^{(1/4)^i} \quad (92)$$

$$\|E(\Sigma^i)\|_2^2 \geq \|E(\Sigma^{i-1})\|_2^2 + 2^{-6} \alpha(\varepsilon) N^2 \quad (93)$$

and

$$|\Omega^{(i)} \setminus \Omega^{(i-1)}| < 2^{-6} \alpha(\varepsilon) \varepsilon N^2 \quad (94)$$

At the first step of our algorithm we put $\Sigma^{(1)} = \{\mathbf{Z}_N^2\}$ and $\Omega^{(1)} = \emptyset$. Then $\Sigma^{(1)}$ and $\Omega^{(1)}$ satisfies (92), (93) and (94). Let $E_1(\vec{x}) = E(\Sigma^{(1)})(\vec{x})$. Then $E_1(\vec{x}) = \delta$ and $\|E_1\|_2^2 = \delta^2 N^2$.

Let us make the second step of the inductive process. If set $W \subseteq \{1, \dots, N\}^2$ is $\alpha(\delta)$ -uniform then we obtain the result and terminate our

process. Suppose W is not $\alpha(\delta)$ -uniform. Let $f(\vec{s}) = \chi_W(\vec{s}) - \delta$. By Lemma 2.1' there exists vector $\vec{r} \neq \vec{0}$ such that $|\widehat{f}(\vec{r})| \geq \alpha(\delta)^{1/2}N^2$. Since $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, it follows that $N \geq 2^{100}/\alpha(\delta)^5$. By Lemma 4.4 the set \mathbf{Z}_N^2 can be partitioned into right squares S_1, \dots, S_r of same size and the set Ω such that $|S_i| \geq N^{1/2}$, $i = 1, \dots, r$, $|\Omega| < N^{11/6}$ and

$$\frac{1}{r} \sum_{j=1}^r |\delta_{S_j}(W) - \delta|^2 \geq \alpha(\delta)/16 \quad (95)$$

Put $\Sigma^2 = \{S_1, \dots, S_r\}$ and $\Omega^{(2)} = \Omega$. Since $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, it follows that $|\Omega^{(2)}| < 2^{-6}\alpha(\varepsilon)\varepsilon N^2$. Let f and g be arbitrary functions from \mathbf{Z}_N^2 to \mathbf{R} . By (f, g) denote its inner product : $(f, g) = \sum_{\vec{x}} f(\vec{x})g(\vec{x})$. Let $E_2(\vec{x}) = E(\Sigma^{(1)})(\vec{x})$. Then

$$\|E_2\|_2^2 = (E_2, E_2) = (E_1, E_1) + \|E_2 - E_1\|_2^2 + 2(E_1, E_2 - E_1) \quad (96)$$

Let us estimate the third term in (96). We have $\sum_{i=1}^r \sum_{\vec{x} \in S_i} E_2(\vec{x}) = |W \cap (\bigsqcup_{i=1}^r S_i)| = \delta N^2 - |\Omega|$. It follows that

$$\begin{aligned} |(E_1, E_2 - E_1)| &= \left| \sum_{\vec{x}} \delta(E_2(\vec{x}) - \delta) \right| = \left| \delta \sum_{\vec{x}} E_2(\vec{x}) - \delta^2 N^2 \right| = \\ &= \left| \delta \sum_{i=1}^r \sum_{\vec{x} \in S_i} E_2(\vec{x}) - \delta^2 N^2 \right| = \delta |\Omega| \leq N^{11/6} \end{aligned} \quad (97)$$

Let us calculate the second term in (96). The set \mathbf{Z}_N^2 partitioned be the sets S_1, \dots, S_r and Ω . Since $|\Omega| \leq N^{11/6}$, it follows that for any $t = 1, \dots, r$ we have $|S_t| \geq N^2/2r$. Using (95), we get

$$\begin{aligned} \|E_2 - E_1\|_2^2 &= \sum_{\vec{x}} |E_2(\vec{x}) - \delta|^2 = \sum_{t=1}^r \sum_{\vec{x} \in S_t} |E_2(\vec{x}) - \delta|^2 = \sum_{t=1}^r |S_t| |\delta_t - \delta|^2 \geq \\ &\geq \frac{N^2}{2} \frac{1}{r} \sum_{t=1}^r |\delta_t - \delta|^2 \geq 2^{-5}\alpha(\delta)N^2. \end{aligned} \quad (98)$$

Using (97), (98) and inequality $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, we obtain

$$\|E_2\|_2^2 \geq \|E_1\|_2^2 + 2^{-6}\alpha(\delta)N^2 \geq \|E_1\|_2^2 + 2^{-6}\alpha(\varepsilon)N^2 \quad (99)$$

$\Sigma^{(2)}$ and $\Omega^{(2)}$ satisfies (92), (93) and (94).

Suppose we have made i iterations, $i \leq 2^6/\alpha(\varepsilon)$. Suppose at i -th step of our inductive procedure we constructed the family $\Sigma^{(i)}$ of disjoint right squares C_1, \dots, C_{ν_i} and the exceptional $\Omega^{(i)} \supseteq \Omega^{(i-1)}$ satisfies (92), (93) and (94). Using (94), we obtain $|\Omega^{(i)}| < \varepsilon N^2$. Let $\delta_j = \delta_{C_j}(W)$. By H denote the set of right squares C_j of family $\Sigma^{(i)}$ such that $|C_j| < 2^{200}\alpha(\delta_j)^{-10}$. By U_1 denote the set of all squares $C_j \in (\Sigma^{(i)} \setminus H)$ such that W is $\delta_{C_j}(W)$ -uniform in C_j and by U_2 denote the set of all squares $C_j \in (\Sigma^{(i)} \setminus H)$ such that W is not $\delta_{C_j}(W)$ -uniform in C_j .

If $|W \cap \bigsqcup_{C_j \in U_2} C_j| < \varepsilon N^2$ then we obtain the result. Indeed, consider as a needed squares P_1, \dots, P_M the squares from $\Sigma^{(i)} \cap U_1$ such that W has density in each of them not less than ε . These squares satisfy conditions 1), 2) of the Theorem. Let $V = (U_1 \sqcup U_2) \setminus \bigsqcup_{i=1}^M P_i$. Clearly, $|V \cap W| \leq 2\varepsilon N^2$. Let us estimate cardinality of $H \cap W$. Let C_j be an arbitrary square from H . Then $|C_j| < 2^{200}\alpha(\delta_j)^{-10}$. On the other hand from (92) it follows that $|C_j| \geq N^{(1/2)^i} \geq N^{(c_2)^{1/\alpha}}$. Hence $\delta_j < 2^{20}/K \cdot N^{-(1/10\rho)c_2^{1/\alpha}}$. Since $\sum_{S_j \in H} |S_j| \leq N^2$, it follows that $|H \cap W| = \sum_{S_j \in H} \delta_j |S_j| < 2^{20}/K \cdot N^2 N^{-(1/10\rho)c_2^{1/\alpha}}$. Since $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, we get $|H \cap W| < 2^{-7}\varepsilon\alpha(\varepsilon)N^2 < \varepsilon N^2/2$. Put $B = (V \cap W) \sqcup (H \cap W) \sqcup \Omega^{(i)}$. We have $|B| < 4\varepsilon N^2$ which proves the Theorem.

Suppose $|W \cap \bigsqcup_{C_j \in U_2} C_j| \geq \varepsilon N^2$. Without loss of generality it can be assumed that $U_2 = \{C_1, \dots, C_l\}$. By (92), it follows that $l \leq N^2 N^{-(c_2)^{1/\alpha}}$. Let us consider an arbitrary right square C_j from U_2 . Let $f_j(\vec{s}) = \chi_{W \cap C_j}(\vec{s}) - \delta_j$. The application of Lemma 2.1' yields there exists a vector $\vec{p}_j \neq \vec{0}$ such that $|\widehat{f_j}(\vec{p}_j)| \geq \alpha(\delta_j)^{1/2}|C_j|$. Since $C_j \notin H$, it follows that $|C_j| \geq 2^{200}/\alpha(\delta_j)^{10}$. By Lemma 4.4 the set C_j can be partitioned into right squares $S_1^{(j)}, \dots, S_{r(j)}^{(j)}$ of same size and set Ω_j such that $|S_t^{(j)}| \geq |C_j|^{1/4}$, $t = 1, \dots, r(j)$, $|\Omega_j| < |C_j|^{11/12}$ and

$$\frac{1}{r(j)} \sum_{t=1}^{r(j)} |\delta_{S_t^{(j)}}(W) - \delta_j|^2 \geq \alpha(\delta_j)/16 \quad (100)$$

Put

$$\Sigma^{(i+1)} = \{C_1, \dots, C_b\}_{C_j \in U_1} \sqcup \left(\bigsqcup_{j=1}^l \bigsqcup_{t=1}^{r(j)} S_t^{(j)} \right)$$

and

$$\Omega^{(i+1)} = \Omega^{(i)} \sqcup (H \cap W) \sqcup \bigsqcup_{j=1}^l |\Omega_j|$$

Clearly, all sets from $\Sigma^{(i+1)}$ satisfies (92). Since squares $C_j \subseteq \mathbf{Z}_N^2$ are disjoint, it follows that $\sum_{j=1}^l |C_j| \leq N^2$. Let $Y = \bigsqcup_{j=1}^l \Omega_j$. By the Cauchy-Bounyakovskiy inequality, we get

$$|Y| \leq \sum_{j=1}^l |C_j|^{11/12} \leq \left(\sum_{j=1}^l |C_j| \right)^{11/12} \cdot l^{1/12} \leq N^2 N^{-1/12(c_2)^{1/\alpha}}$$

Since $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, it follows that $|Y| < 2^{-8}\alpha(\varepsilon)\varepsilon N^2$. Hence $|\Omega^{(i+1)} \setminus \Omega^{(i)}| = |H \cap W| + |Y| < 2^{-7}\alpha(\varepsilon)\varepsilon N^2 + 2^{-7}\alpha(\varepsilon)\varepsilon N^2 = 2^{-6}\alpha(\varepsilon)\varepsilon N^2$. We see that the set $\Omega^{(i+1)}$ satisfies (94).

Let us check inequality (93). Let $E_i(\vec{x}) = E(\Sigma^{(i)})(\vec{x})$ and $E_{i+1}(\vec{x}) = E(\Sigma^{(i+1)})(\vec{x})$, Then

$$\|E_{i+1}\|_2^2 = (E_{i+1}, E_{i+1}) = (E_i, E_i) + \|E_{i+1} - E_i\|_2^2 + 2(E_i, E_{i+1} - E_i) \quad (101)$$

Let us estimate the third term in (101). For any $C_j \in U_1$ we have $E_{i+1}(\vec{x}) = E_i(\vec{x})$. Hence $\sum_{\vec{x} \in C_j} (E_{i+1}(\vec{x}) - E_i(\vec{x})) = 0$. For any $C_j \in U_2$ we have $|\sum_{\vec{x} \in C_j} (E_{i+1}(\vec{x}) - E_i(\vec{x}))| \leq |\Omega_j|$. Hence

$$\begin{aligned} |(E_i, E_{i+1} - E_i)| &= \left| \sum_{\vec{x}} E_i(\vec{x})(E_{i+1}(\vec{x}) - E_i(\vec{x})) \right| = \\ &= \left| \sum_{C \in \Sigma^{(i)}} \sum_{\vec{x} \in C} E_i(\vec{x})(E_{i+1}(\vec{x}) - E_i(\vec{x})) \right| = \\ &= \left| \sum_{C \in \Sigma^{(i)}} \delta_C(W) \sum_{\vec{x} \in C} (E_{i+1}(\vec{x}) - E_i(\vec{x})) \right| \leq |Y| < 2^{-8}\alpha(\varepsilon)\varepsilon N^2 \quad (102) \end{aligned}$$

Let us calculate the second term in (101). Recall that $U_2 = \{C_1, \dots, C_l\}$. For any $j = 1, \dots, l$ the set C_j can be partitioned into the sets $S_1^{(j)}, \dots, S_{r(j)}^{(j)}$ and Ω_j . Since $|\Omega_j| \leq |C_j|^{11/12}$, we obtain for all $t = 1, \dots, r(j)$ the following inequality holds $|S_t^{(j)}| \geq |C_j|/2r(j)$. Let $\delta_{jt} = \delta_{S_t^{(j)}}(W)$. Using (100), we get

$$\begin{aligned} \|E_{i+1} - E_i\|_2^2 &= \sum_{\vec{x}} |E_{i+1}(\vec{x}) - E_i(\vec{x})|^2 = \sum_{j=1}^l \sum_{\vec{x} \in C_j} |E_{i+1}(\vec{x}) - E_i(\vec{x})|^2 = \\ &= \sum_{j=1}^l \sum_{t=1}^{r(j)} \sum_{\vec{x} \in S_t^{(j)}} |E_{i+1}(\vec{x}) - E_i(\vec{x})|^2 = \sum_{j=1}^l \sum_{t=1}^{r(j)} |S_t^{(j)}| |\delta_{jt} - \delta_j|^2 \geq \end{aligned}$$

$$\geq \frac{1}{2} \sum_{j=1}^l |C_j| \frac{1}{r(j)} \sum_{t=1}^{r(j)} |\delta_{jt} - \delta_j|^2 \geq 2^{-5} \sum_{j=1}^l |C_j| \alpha(\delta_j) \quad (103)$$

We have $|W \cap \bigsqcup_{C_j \in U_2} C_j| \geq \varepsilon N^2$. Whence $\sum_{j=1}^l \delta_j |C_j| \geq \varepsilon N^2$. By Hölder's inequality, it follows that

$$\begin{aligned} \varepsilon N^2 &\leq \sum_{j=1}^l \delta_j |C_j| = \sum_{j=1}^l (\delta_j |C_j|^{1/\rho}) |C_j|^{1-1/\rho} \leq \left(\sum_{j=1}^l \delta_j^\rho |C_j| \right)^{1/\rho} \left(\sum_{j=1}^l |C_j| \right)^{1-1/\rho} \\ &\leq N^{2(1-1/\rho)} \left(\sum_{j=1}^l \delta_j^\rho |C_j| \right)^{1/\rho} \end{aligned} \quad (104)$$

This yields that

$$\sum_{j=1}^l |C_j| \alpha(\delta_j) = K \sum_{j=1}^l \delta_j^\rho |C_j| \geq K \varepsilon^\rho N^2 = \alpha(\varepsilon) N^2 \quad (105)$$

Using (103), we have

$$\|E_{i+1} - E_i\|_2^2 \geq 2^{-5} \alpha(\varepsilon) N^2 \quad (106)$$

Combining (102), (106) and estimate $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, we obtain

$$\|E_{i+1}\|_2^2 \geq \|E_i\|_2^2 + 2^{-6} \alpha(\varepsilon) N^2 \quad (107)$$

We see that $\Sigma^{(i+1)}$ and $\Omega^{(i+1)}$ satisfies (92), (93) and (94).

We see that the condition $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$ allows to make $2^6/\alpha(\varepsilon)$ iterations. If the process stops before then we obtain the needed result. But (93) guaranties that the number of steps will be less $2^6/\alpha(\varepsilon)$. This completes the proof.

Corollary 4.5 *Let $W_1, W_2 \subseteq \mathbf{Z}_N$ be sets, $|W_1| = \beta_1 N$, $|W_2| = \beta_2 N$, $\zeta \in (0, 1)$ be a number, $\alpha(s) = Ks^\rho$, $K \in (0, 1]$, $\rho \geq 4$ and $a = \alpha(\zeta\beta_1\beta_2)$. Let $A \subseteq W_1 \times W_2$ be a set of cardinality $\delta|W_1||W_2|$ and $N \geq (C\alpha^{c_1})^{-(1/c_2)^{1/\alpha}}$, where $C = 2^{1000\rho}$, $c_1 = 100\rho$ and $c_2 = 2^{-128}$. There exists a right square $P = P_1 \times P_2$, $|P| \geq N^{c_2^{1/a}}$ and sets R_1, R_2 , $R_1 \subseteq W_1 \cap P_1$, $R_2 \subseteq W_2 \cap P_2$, $|R_1 \times R_2| \geq \zeta\beta_1\beta_2|P|$ such that R_1, R_2 is $\alpha(\delta_{P_1}(R_1))^{1/2}$, $\alpha(\delta_{P_2}(R_2))^{1/2}$ -uniform in P_1 and P_2 respectively and $\delta_{R_1 \times R_2}(A) \geq \delta - 4\zeta$.*

Proof. Let $\varepsilon = \zeta\beta_1\beta_2$. We apply Theorem 4.2 to the set $W = W_1 \times W_2$. Set $A_1 = A \setminus (A \cap B)$ has density in W at least $\delta - \zeta$. Hence there exists right

square $P = P_1 \times P_2$ such that W is $\alpha(\delta_P(E))$ -uniform in P , $|W \cap P| \geq \varepsilon|P|$, $|P| \geq N^{c_2^{1/\alpha}}$, $\alpha = \alpha(\varepsilon)$ and $\delta_{P \cap W}(A) \geq \delta - 4\zeta$. Let $W \cap P = R_1 \times R_2$ and $R_1 = \gamma_1|P_1|$, $R_2 = \gamma_2|P_2|$. Then $|W \cap P| = \gamma_1\gamma_2|P|$. By f , f_1 and f_2 denote the balanced functions of the sets W , R_1 and R_2 respectively. For any $r_1 \neq 0$, $r_2 \neq 0$, we get $|\hat{f}(r_1, r_2)| \leq (K\gamma_1^\rho\gamma_2^\rho)^{1/4}|P|$. If $r_1 \neq 0$, $r_2 = 0$, it follows that $\hat{f}(r_1, r_2) = \gamma_2|P_2|\hat{f}_1(r_1)$ and $|\hat{f}_1(r_1)| \leq K^{1/4}\gamma_1^{\rho/4}|P_1|$. For the same reason, $|\hat{f}_2(r)| \leq K^{1/4}\gamma_2^{\rho/4}|P_2|$ for all $r \in \mathbf{Z}_N \setminus \{0\}$. Using Lemma 2.1 to the sets R_1, R_2 , we obtain R_1, R_2 such that R_1 is $K^{1/2}\gamma_1^{\rho/2}$ -uniform in P_1 and R_2 is $K^{1/2}\gamma_2^{\rho/2}$ -uniform in P_2 . This completes the proof.

Proof of Theorem 1.4. Let $N_1 \in \mathbf{N}$ and $J_1, J_2 \subseteq \mathbf{Z}_{N_1}$ be sets, $|J_1| = \omega_1 N_1$, $|J_2| = \omega_2 N_1$. Let $A \subseteq J_1 \times J_2$ be a set of cardinality $\delta|J_1||J_2|$. Suppose A does not contain a corner. Let J_1, J_2 be $10^{-330}\omega_1^{24}\omega_2^{24}\delta^{132}$ -uniform and $N_1 \geq 10^{10}(\delta^4\omega_1\omega_2)^{-1}$. We shall prove that under this conditions there exist $J_1 \subseteq I_1$, $J_2 \subseteq I_2$ and $A' \subseteq A$ such that

- 1) $A' \subseteq I_1 \times I_2$.
- 2) $|A'| \geq (\delta + 10^{-10000}\delta^{3500})|I_1||I_2|$.
- 3) $|I_1|, |I_2| \geq 10^{-10000}\delta^{3500} \min(\omega_1 N_1, \omega_2 N_1)$.

Suppose (36) or (37) does not hold for $\alpha_1 = 10^{-108}\delta^{44}$. By Lemma 3.8 there exist sets $I_1 \subseteq J_1$, $I_2 \subseteq J_2$ satisfies (53). Let $A' = A \cap (I_1 \times I_2)$. The sets I_1, I_2 and A' satisfies 1) – 3).

So, we can assume that (36), (37) hold for $\alpha_1 = 10^{-108}\delta^{44}$. If set A is $10^{-108}\delta^{44}$ -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) , then by Theorem 2.9 there exists a corner in A . If set A is not $10^{-108}\delta^{44}$ -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) , then by Proposition 3.10 there exist sets $I_1 \subseteq J_1$, $I_2 \subseteq J_2$ and $A' = A \cap (I_1 \times I_2)$ satisfies 1) – 3). The condition 2) can be replaced even by a stronger one namely by $|A'| \geq (\delta + 10^{-9000}\delta^{3500})|I_1||I_2|$.

Let us come now to the proof itself.

Let $A \subseteq \{1, \dots, N\}^2$ be a set of size δN^2 and let A does not contain a corner. Let $E_1 = \{1, \dots, N\}$, $E_2 = \{1, \dots, N\}$. Then E_1 and E_2 is 0-uniform. By assumption $N \geq 10^{10}\delta^{-4}$. Repeating the argument used in the beginning of the proof, we can find a subset A' in A and $G_1 \subseteq E_1$, $G_2 \subseteq E_2$ satisfying 1) – 3). Let $|G_1| = \beta_1 N$, $|G_2| = \beta_2 N$. The set A' , same as A does not contain a corner and has density δ_1 in $G_1 \times G_2$ at least $\delta + 10^{-9000}\delta^{3500}$.

Let $\zeta = 10^{-10000}\delta^{3500}$. Let us consider the function $\alpha(s) = 10^{-660}(\zeta\beta_1\beta_2)^{48}\delta^{264}s^{48}$ and let $a = \alpha(\zeta\beta_1\beta_2)$. By assumption $N \geq (Ca^{c_1})^{-(1/c_2)^{1/a}}$. Hence we can apply Corollary 4.5 to the sets G_1, G_2 and A' . By this Corollary there exists a right square $P = P_1 \times P_2$, $|P| \geq N^{c_2^{1/a}}$ and sets $R_1, R_2, R_1 \subseteq (G_1 \cap P_1), R_2 \subseteq$

$(G_2 \cap P_2) |R_1| = \gamma_1 |P_1|$, $|R_2| = \gamma_2 |P_2|$, $|R_1 \times R_2| \geq \zeta \beta_1 \beta_2 |P|$ such that R_1, R_2 is $10^{-330} \gamma_1^{24} \gamma_2^{24} \delta^{132}$ -uniform in P_1, P_2 respectively and $\delta_{R_1 \times R_2}(A') \geq \delta_1 - 4\zeta$. Density A in $R_1 \times R_2$ is at least $\delta_1 - 4 \cdot 10^{-10000} \delta^{3500} \geq \delta + 10^{-10000} \delta^{3500}$.

Apply the same this argument to the right square P , $10^{-330} \gamma_1^{24} \gamma_2^{24} \delta^{132}$ -uniform sets R_1, R_2 , $R_1 \times R_2 \subseteq P$ and the set $A'' = A' \cap (R_1 \times R_2)$. Then we iterate the described construction.

Let at i -th step of our procedure we get right square $P^{(i)} = P_1^{(i)} \times P_2^{(i)}$, sets $R_1^{(i)}, R_2^{(i)}$, $R_1^{(i)} = \gamma_1^{(i)} |P_1^{(i)}|$, $R_2^{(i)} = \gamma_2^{(i)} |P_2^{(i)}|$ and set $A_i \subseteq R_1^{(i)} \times R_2^{(i)}$, $|A_i| = \delta_i |R_1^{(i)}| |R_2^{(i)}|$ such that sets $R_1^{(i)}, R_2^{(i)}$ is $10^{-330} (\gamma_1^{(i)})^{24} (\gamma_2^{(i)})^{24} \delta^{132}$ -uniform in $P_1^{(i)}, P_2^{(i)}$ respectively. If

$$|P_1^{(i)}| = |P_2^{(i)}| \geq (\delta^4 \gamma_1^{(i)} \gamma_2^{(i)})^{-1}, \quad (108)$$

then by the arguments used in the beginning of the proof we can find a subset A'_i in A_i and $G_1^{(i)} \subseteq R_1^{(i)}$, $G_2^{(i)} \subseteq R_2^{(i)}$ satisfies 1) – 3), $G_1^{(i)} = \beta_1^{(i)} |P_1^{(i)}|$, $G_2^{(i)} = \beta_2^{(i)} |P_2^{(i)}|$. In addition, A'_i same as A does not contain a corner and has density δ'_i in $G_1^{(i)} \times G_2^{(i)}$ is at least $\delta_i + 10^{-9000} \delta^{3500}$. Using 3), we get

$$\beta_1^{(i)}, \beta_2^{(i)} \geq 10^{-10000} \delta^{3500} \min(\gamma_1^{(i)}, \gamma_2^{(i)}) \quad (109)$$

Let $\zeta = 10^{-10000} \delta^{3500}$. Let us consider the function $\alpha_i(s) = 10^{-660} (\zeta \beta_1^{(i)} \beta_2^{(i)})^{48} \delta^{264} s^{48}$ and let $a_i = \alpha_i(\zeta \beta_1^{(i)} \beta_2^{(i)})$. If

$$|P_1^{(i)}| \geq (C a_i^{c_1})^{-(1/c_2)^{1/a_i}}, \quad (110)$$

then we can apply Corollary 4.5 to the sets $G_1^{(i)}, G_2^{(i)}$ and A'_i . By this Corollary there exists a right square $P^{(i+1)} = P_1^{(i+1)} \times P_2^{(i+1)}$ and sets $R_1^{(i+1)}, R_2^{(i+1)}$, $R_1^{(i+1)} \subseteq (G_1^{(i)} \cap P_1^{(i+1)})$, $R_2^{(i+1)} \subseteq (G_2^{(i)} \cap P_2^{(i+1)})$, $R_1^{(i+1)} = \gamma_1^{(i+1)} |P_1^{(i+1)}|$, $R_2^{(i+1)} = \gamma_2^{(i+1)} |P_2^{(i+1)}|$, $|R_1^{(i+1)} \times R_2^{(i+1)}| \geq \zeta \beta_1^{(i)} \beta_2^{(i)} |P^{(i+1)}|$, such that $R_1^{(i+1)}, R_2^{(i+1)}$ is $10^{-330} (\gamma_1^{(i+1)})^{24} (\gamma_2^{(i+1)})^{24} \delta^{132}$ -uniform in $P_1^{(i+1)}, P_2^{(i+1)}$ and $\delta_{R_1 \times R_2}(A'_i) \geq \delta'_i - 4\zeta$. Density A'_i in $R_1^{(i+1)} \times R_2^{(i+1)}$ is at least $\delta'_i - 4 \cdot 10^{-10000} \delta^{3500} \geq \delta'_i + 10^{-10000} \delta^{3500}$.

Combining inequalities $|R_1^{(i+1)} \times R_2^{(i+1)}| \geq \zeta \beta_1^{(i)} \beta_2^{(i)} |P^{(i+1)}|$ and (109), we obtain

$$\gamma_1^{(i+1)} \gamma_2^{(i+1)} \geq \zeta \beta_1^{(i)} \beta_2^{(i)} \geq 10^{-20000} \delta^{7000} \min(\gamma_1^{(i)}, \gamma_2^{(i)}). \quad (111)$$

Moreover,

$$|P_1^{(i+1)}| \geq |P^{(i)}|^{c_2^{1/a_i}} \quad (112)$$

Suppose that at each step of our algorithm the conditions (108) and (110) are satisfied. At each new iteration step the density of A in the sets $R_1^{(i)} \times R_2^{(i)}$ increases by at least $10^{-10000} \delta^{3500}$. This implies that the density of A in these sets tends to 1. Then by Lemma 4.1, $\|f_A\| \rightarrow 0$. Hence in a few steps $\|f_A\|$ will become smaller than $10^{-27} \delta^{11}$. In other words, in a few steps we can find a right square $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$ and $10^{-330} \gamma_1^{24} \gamma_2^{24} \delta^{132}$ -равномерные множества $\mathbf{R}_1, \mathbf{R}_2, |\mathbf{R}_1| = \gamma_1 |\mathbf{P}_1|, |\mathbf{R}_2| = \gamma_2 |\mathbf{P}_2|$ such that $A \cap (\mathbf{R}_1 \times \mathbf{R}_2)$ is $10^{-108} \delta^{44}$ -uniform with respect to the basis (\vec{e}_1, \vec{e}_2) , in $\mathbf{R}_1 \times \mathbf{R}_2$ and inequalities (36), (37) hold for $\alpha_1 = 10^{-108} \delta^{44}$. If $|\mathbf{P}_1| = |\mathbf{P}_2| \geq 10^{10} (\delta^4 \gamma_1 \gamma_2)^{-1}$, then by Theorem 2.9 A contains a corner.

We see that, if at each step of the algorithm the conditions (108) and (110) are satisfied then the proof is over. Let us check that the conditions (108), (110) hold.

Let us estimate the total number of steps of our procedure. By 2), it follows that the density of A reaches 2δ after at most $10^{10000} / \delta^{3499}$ further steps. It follows that, the total number of steps cannot be more than $10^{10000} / \delta^{3499} + 1/2 \cdot 10^{10000} / \delta^{3499} + 1/4 \cdot 10^{10000} / \delta^{3499} + \dots = 2 \cdot 10^{10000} / \delta^{3499} = O(\delta^{-\bar{c}})$, $\bar{c} > 0$ is an absolute constant.

At the first step densities of $R_1^{(1)}$ and $R_2^{(1)}$ in \mathbf{Z}_N equals 1. By (111), it follows that at i -th step, we have the inequality $\gamma_1^{(i)}, \gamma_2^{(i)} \geq (10^{-20000} \delta^{7000})^i$. Hence at the last step density $\mathbf{R}_1 \times \mathbf{R}_2$ in \mathbf{P} is at least $C_4 \delta^{C_5 \delta^{-\bar{c}}}$.

The total number of steps is at most $O(\delta^{-\bar{c}})$. At i -th step, we have $\beta_1^{(i)}, \beta_2^{(i)} \geq C_4 \delta^{C_5 \delta^{-\bar{c}}}$, $\zeta = O(\delta^g)$, $g > 0$ and $a_i = \delta^w (\beta_1^{(i)} \beta_2^{(i)})^q$, $w, q > 1$. Using this and inequality (112), we get $|P^{(i)}| \geq |P^{(i-1)}|^{\kappa_0^{(1/\delta)^{\delta^{-C_6}}}}$, where $0 < \kappa_0 < 1$, $C_6 > 0$. Hence at the last step, we have $|\mathbf{P}| \geq N^{\kappa^{(1/\delta)^{\delta^{-b}}}}$, where $0 < \kappa < 1$, $b > 1$. By assumption $N \geq \exp \exp \exp(\delta^{-c})$, $c > 0$. It follows that,

$$|\mathbf{P}| \geq N^{\kappa^{(1/\delta)^{\delta^{-b}}}} \geq \left(10^{10} \delta^4 (C_4 \delta^{C_5 \delta^{-\bar{c}}})\right)^{-2} \geq 10^{10} (\delta^4 \gamma_1 \gamma_2)^{-2} \quad (113)$$

This implies that at the last step of the iteration process the inequality (108) holds. Clearly, this condition was true at all the previous steps. Let us check if the condition (110) is satisfied at the last step. We need to check :

$$|\mathbf{P}| \geq N^{\kappa^{(1/\delta)^{\delta^{-b}}}} \geq ((1/\delta)^{\delta^{-C_7}})^{(1/c_2)^{(1/\delta)^{\delta^{-C_8}}}}, \quad (114)$$

where $C_7 > 0, C_8 > 0$ are absolute constants. By assumption $N \geq \exp \exp \exp(\delta^{-c})$. Using this, we get (114). This completes the proof of Theorem 1.4.

The author is grateful to Professor N.G. Moshchevitin for constant attention to this work.

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