Proceedings of the Steklov Institute of Mathematics, Vol. 218, 1997, pp. 1–14.
Translated from Trudy Matematicheskogo Instituta imeni V. A. Steklova, Vol. 218, 1997, pp. 7–19.

On Professor A. A. Karatsuba's Works in Mathematics

G. I. Arkhipov and V. N. Chubarikov

Received February, 1997

We dedicate this article to the mathematical works of Anatoly Alekseevich Karatsuba who was sixty on January 31, 1997.

When writing about the works of an eminent scientist, it is natural to define some characteristic and impressive features of his creative activity. These distinguishing features of the scientific activity of Professor Karatsuba are the combinatorial inventiveness, thoroughness, and a definite completeness of the results. In our opinion, the future will give a true estimation of their significance. It stands to reason that it is impossible to mention all of them in one article. Here we shall only touch upon the most important aspects of his mathematical works.

Here are the main trends in the investigations carried out by A. A. Karatsuba. No doubt that our classification is conventional to a certain extent since it reflects the subjective viewpoint of the authors. A. A. Karatsuba published 115 research works in mathematics. He has made a fundamental contribution to the following divisions of number theory and the related fields of mathematics:

the theory of trigonometric sums and trigonometric integrals (Tarry's problem, the p-adic method, multiple trigonometric sums, estimate of Hardy's function in Waring problem, Artin's problem on the local representation of zero by a form, Hua Loo Keng problem on the index of convergence of a singular integral, estimates of incomplete short sums of Kloostermann,

the theory of Riemann zeta-function (Selberg's problem, distribution of zeros of the Davenport–Heilbronn function, distribution of zeros of Riemann zeta-function on short segments of the critical line),

the theory of Dirichlet characters (estimates of the sums of characters from "shifted" primes and the sums of characters of polynomials with a prime argument),

the theory of finite automata (the problem of exact estimation of the least length of an experiment for the recognition of the state of an automaton),

the theory of fast computations (the first general method and the algorithm of fast multiplication of multivalued numbers, a historical investigation of the appearance of arithmetic operations, the complexity of calculations of functions).

It must be added that A. A. Karatsuba is the author of the well-known textbook *Fundamentals of the Analytic Number Theory* (1975, 1983) and the author of the monograph *Multiple Trigonometric Sums* (1980, jointly with G. I. Arkhipov and V. N. Chubarikov), *Theory of Multiple Trigonometric Sums* (1987, jointly with G. I. Arkhipov and V. N. Chubarikov), *Riemann Zeta-Function* (1994, jointly with S. M. Voronin), *Complex Analysis in Number Theory* (1995). Karatsuba's investigations of the multidimensional problem of Dirichlet divisors were awarded the Chebyshev Prize of the Academy of Science of the USSR.

We shall not consider in detail the cycles of works connected with the theory of finite automata [1, 42] and with the theory of fast computations [4, 43, 111]. It should only be pointed out that their results can be found in textbooks and university courses and that they opened up new directions of investigations.

The initial period of Karatsuba's activity in the theory of trigonometric sums terminated with the elaboration of the p-adic method. In [3] he gave new estimates of sums of the form

$$S=\sum_{x=1}^{P} e^{2\pi i(a_1x/p^n+\cdots+a_nx^n/p)}, \qquad (a_s,p)=1, \quad 1\le s\le n,$$

the so-called L-sums, which appeared for the first time in connection with the estimation of the sum of Dirichlet characters in the investigations of A. G. Postnikov. These estimates led to new bounds of zeros of Dirichlet L-series modulo equal to the degree of a prime number and to the derivation of the asymptotic formula for the number of solutions of a Waring congruence of the form

$$x_1^n+\cdots+x_t^n\equiv N \,(\operatorname{mod} p^k), \qquad 1\le x_s\le P, \quad 1\le s\le n, \quad P<p^k,$$

to the solution of the problem of distribution of the fractional parts of the values of a polynomial with integer coefficients modulo p^k. In his work [7] Karatsuba realizes, for the first time, in the p-adic form, the Euler–Vinogradov "embedding principle" and constructs a p-adic analog of Vinogradov's u-numbers when estimating the number of solutions of a congruence of Waring type. Here is the formulation of the result.

Theorem. *Let*

$$x_1^n+\cdots+x_t^n\equiv N \,(\operatorname{mod} Q), \qquad 1\le x_s\le P, \quad 1\le s\le t, \tag{1}$$

with $P^r\le Q<P^{r+1}$, $1\le r\le \frac{1}{12}\sqrt{n}$, $Q=p^k$, $k\ge 4(r+1)n$, *and let* p *be a prime number. Then, for every positive integer* $n\ge 144$ *there exists* $p_0=p_0(n)$ *such that for any* $p>p_0$ *every positive integer* N *can be represented in form* (1) *for* $t\ge 20r+1$ *and for* $t<r$ *there exist* N *such that congruence* (1) *is not solvable.*

This new approach led to a new p-adic proof of Vinogradov's mean value theorem which plays the key part in Vinogradov's method of trigonometric sums. Karatsuba repeatedly returned to this theorem and improved its proof. Here is the result in the most general form [49, 50].

Let $J=J(P,n,k)$ be the number of solutions in integers of a system of equations of the form (Vinogradov's integral)

$$\begin{cases} x_1+\cdots+x_k=y_1+\cdots+y_k,\\ x_1^2+\cdots+x_k^2=y_1^2+\cdots+y_k^2,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1^n+\cdots+x_k^n=y_1^n+\cdots+y_k^n,\end{cases} \qquad 1\le x_1,\dots,x_k,y_1,\dots,y_k\le P. \tag{2}$$

Theorem. *Let* τ, $r_1,\dots,r_\tau$, n, k *be positive integers with* $\tau\ge 1$, $1=r_1\le r_2\le\cdots\le r_\tau\le n$. *We set*

$$\Delta(\tau)=\left(n-\frac{r_\tau-1}{2}\right)+\left(1-\frac{1}{r_\tau}\right)\left(n-\frac{r_{\tau-1}-1}{2}\right)+\cdots$$
$$+\left(1-\frac{1}{r_\tau}\right)\left(1-\frac{1}{r_{\tau-1}}\right)\cdots\left(1-\frac{1}{r_2}\right)\left(n-\frac{r_1-1}{2}\right),$$
$$\varkappa_\tau=\sum_{j=1}^{\tau}\bigl(r_j^2+\Delta(j)\bigr).$$

Then, for $k \geq n\tau$, $P \geq 1$, *the estimate*

$$J = J(P; n, k) \leq n^{2\Delta(\tau) r_\tau} 2^{\varkappa_\tau} (8k)^{2n\tau} P^{2k-\Delta(\tau)}$$

holds true.

If we set $r_2 = r_3 = \cdots = r_\tau = n$ in this theorem, then we obtain the formulation of Vinogradov's mean value theorem in a more traditional form, namely

$$J(P; n, k) \leq n^{2\Delta n} 2^{\varkappa} (8k)^{2n\tau} P^{2k-\Delta},$$

where

$$\Delta = \frac{n(n+1)}{2} - \frac{n^2}{2}\Big(1 - \frac{1}{n}\Big)^\tau, \qquad \varkappa = n^2\tau + \frac{n(n+1)}{2}\tau - \frac{n(n-1)}{2}\bigg(1 - \Big(1 - \frac{1}{n}\Big)^\tau\bigg) < \frac{3(n+1)^2\tau}{2}.$$

The system of equations (2) is said to be *complete*. If we omit some equations in it, then it becomes *incomplete* or "*dented*".

One more element of Karatsuba's p-adic method is the transition from "dented" systems of equations to complete systems at the expense of a local p-adic change of variables [13, 14]. Of importance in these investigations in the p-adic interpretation of Vinogradov's method of smoothing in the theory of dual trigonometric sums.

Let us consider the system of equations

$$\begin{cases} x_1^{m_1} + \cdots + x_k^{m_1} = y_1^{m_1} + \cdots + y_k^{m_1}, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & 1 \leq x_1, \dots, x_k, y_1, \dots, y_k \leq P, \\ x_1^{m_s} + \cdots + x_k^{m_s} = y_1^{m_s} + \cdots + y_k^{m_s}, & 1 \leq m_1 < m_2 < \cdots < m_s < m_{s+1} = n. \\ \quad x_1^{n} + \cdots + x_k^{n} = y_1^{n} + \cdots + y_k^{n}, & \end{cases} \tag{3}$$

We denote by I_k the number of solutions of this system.

We have [13] the following

Theorem. *Let* r *be an arbitrary positive integer*, $1 \leq r \leq n$, *and let the integer* t *be defined by the inequalities* $m_t \leq r \leq m_{t+1}$. *Then, for* $k \geq 6rn \ln n$, *the estimate*

$$I_k \ll P^{2k-\delta}, \qquad \text{where} \quad \delta = m_1 + \cdots + m_t + (s - t + 1)r$$

holds true.

For incomplete systems of equations, in which the variables run through numbers with small prime divisors, Karatsuba used a multiplicative shift of variables. This led to a qualitatively new estimate of trigonometric sums and to a new mean value theorem for these systems of equations.

Let $g(x)$ be a monotonically increasing function with $g(x) \geq \ln\ln x$ for sufficiently large x and

$$\lim_{x\to\infty} \frac{g(x)}{\ln x} = 0.$$

The positive integer m is called a *number with small prime divisors of the class* E_g if, for every prime divisor p of the number m, the inequality $\ln p \leq g(m)$ is satisfied.

Let $\mathbf{P}$ be the amount of numbers from the class E_g which do not exceed P. Then we have the asymptotics

$$\mathbf{P} \sim P e^{-\omega \ln \omega} \quad \text{as} \quad P \to \infty, \qquad \omega = \frac{\ln P}{g(P)}.$$

Theorem. *Let* $1 \le r < \cdots < m < \cdots < t \le n$,

$$f(x) = \alpha_r x^r + \cdots + \alpha_m x^m + \cdots + \alpha_t x^t,$$

with

$$\alpha_m = \frac{a}{q} + \frac{\theta}{q^2}, \qquad (a, q) = 1, \quad |\theta| \le 1,$$

and with

$$P^{\varepsilon m} \le q \le P^{(1-\varepsilon)m}$$

for a certain positive ε, $\varepsilon < 1/2$.

Suppose, furthermore, that

$$S = \sum_{\substack{x \le P \\ x \in E_g}} e^{2\pi i f(x)}.$$

Then

$$|S| \ll \mathbf{P}^{1-\gamma/(k \ln k)},$$

where $k = r + \cdots + m + \cdots + t$, $\gamma = \gamma(\varepsilon) > 0$ *and the constant under the sign* $\ll$ *is absolute.*

Note that Karatsuba's p-adic method includes the techniques of estimation of the measure of the set of points with small values of functions via the values of their parameters (coefficients etc.), and, conversely, the estimation of these parameters via the measure of a set in the real and the p-adic metric. This aspect of the p-adic method manifested itself most vividly in the estimation of trigonometric integrals, and this led to the solution of the Hua Loo Keng problem on the index of convergence of the improper integral in Tarry's problem [54, 55].

Theorem. *The integral*

$$\theta_0 = \theta(k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_0^1 e^{2\pi i(\alpha_1 x + \cdots + \alpha_n x^n)}\, dx \right|^{2k} d\alpha_1 \cdots d\alpha_n,$$

converges for $2k > (n^2+n)/2 + 1$ *and diverges for* $2k \le (n^2+n)/2 + 1$.

Theorem. *Suppose that the positive integers* $r, \dots, n$ *are such that*

$$1 \le r < \cdots < n, \qquad r + \cdots + n < \frac{n^2+n}{2},$$

$$\theta_0' = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_0^1 e^{2\pi i(\alpha_r x^r + \cdots + \alpha_n x^n)}\, dx \right|^{2k} d\alpha_r \cdots d\alpha_n.$$

Then the integral θ_0' *converges for* $2k > r + \cdots + n$ *and diverges for* $2k \le r + \cdots + n$.

All these investigations carried out by Karatsuba include the p-adic form of the Hardy–Littlewood–Ramanujan–Vinogradov circle method. The work [78] was a glaring example of the application of this method. Continuing here his investigations concerning the estimation of trigonometric sums, in which the summation is carried out over numbers with small prime divisors, Karatsuba obtained a new estimate of the well-known Hardy function $G(n)$ in the Waring problem (for $n \ge 400$):

$$G(n) < 2n \ln n + 2n \ln \ln n + 12n.$$

Developing Vinogradov's method of estimation of sums with prime numbers, Karatsuba obtained, in 1970–71, a power decrease of the length of the interval of variation of the summation variable on which the sum of the values of Dirichlet's nonprincipal character on a sequence of shifted prime numbers is estimated nontrivially. The statement of this problem and the best results obtained in it belong to Vinogradov. Here is the exact formulation of the result [26].

Theorem. *Suppose that q is a prime number, $(k,q)=1$, $\chi(a)$ is a nonprincipal character modulo q, ω is a arbitrary number from the interval $0<\omega\le 1/4$, and $q^{1/2+\omega}\le N<q$. Then we have the following estimate*:

$$S_N=\sum_{p\le N}\chi(p+k)\ll Nq^{-\rho},\qquad \rho=\frac{\omega^2}{1024},$$

where the constant under the sign $\ll$ depends only on ω.

It is appropriate here to cite what Linnik said in 1971 about this direction of investigations at the International Conference on Number Theory dedicated to the 80th anniversary of I. M. Vinogradov. "Of a considerable importance are Vinogradov's investigations in the field of the asymptotics of Dirichlet characters. Already in 1952 an estimate was obtained of the sum of Dirichlet characters of the shifted prime numbers $\sum_{p\le N}\chi(p+k)$ which gave a power lowering as compared to N already for $N>q^{0.75+\varepsilon}$, $\varepsilon>0$, where q is the absolute value of the character. This estimate is important in principle since in its depth it exceeds the result of the direct application of the extensions of Riemann's conjecture and, obviously, is the truth which, in this direction, is deeper than the indicated conjecture (provided that the conjecture is valid). Quite recently Karatsuba managed to improve this estimate."

In his work [48] Karatsuba obtained the law of distribution of pairs of quadratic residues and nonresidues in the form $p+k$ in the case, where the length of the interval was essentially smaller than the absolute value of the Legendre symbol.

Theorem. *Let $\tau_1=\pm1$, $\tau_2=\pm1$, q be a prime number, and k_1 and k_2 be integers such that $k_1k_2(k_1-k_2)$ is incomparable with zero modulo q, ω be a real number under the condition $0<\omega<0.5$, $N>q^{0.75+\omega}$. Then the number T of primes p, which do not exceed N and are such that*

$$\left(\frac{p+k_1}{q}\right)=\tau_1,\qquad \left(\frac{p+k_2}{q}\right)=\tau_2,$$

is equal to

$$T=\frac{1}{4}\pi(N)+O\big(Nq^{-0.01\omega^2}\big),$$

where the constant under the sign O depends only on ω.

In 1970 Karatsuba proposed a method for solving ternary multiplication problems [27]. At present this method is widely used by many specialists in analytic number theory. As an application of his method, Anatolii Alekseevich considered a problem on the distribution of the product of shifted prime numbers in an arithmetic progression.

Suppose that $\omega\in(0,1/4]$, n is a sufficiently large positive number, D is a prime number, $D\le n^{\varkappa_0}$, where $\varkappa_0=1/(4.6+\omega)$,

$$(a,D)=(l,D)=1,\qquad \alpha\in\left[\Big(\frac{1}{2}+\omega\Big)\frac{\ln D}{\ln n},1-4.1\frac{\ln D}{\ln n}\right],$$

$n_1 \geq n^{1-\alpha}$, $n_2 \geq n^{\alpha}$, and p, p' are prime numbers:

$$\pi(x) = \sum_{p \leq x} 1, \qquad \pi_2 = \pi_2(n_1, n_2, a, l) = \sum_{\substack{p(p'+a) \equiv l \pmod{D} \\ p \leq n_1,\ p' \leq n_2}} 1.$$

Theorem. *There exists an absolute constant $\gamma > 0$ such that*

$$\pi_2 = \frac{1}{\varphi(D)} \pi(n_1)\pi(n_2) + O\big((n_1 n_2)^{1+\varepsilon} D^{-1-\gamma\omega^2}\big),$$

where the constant under the sign O depends only on ω and $\varepsilon > 0$ is an arbitrarily small number.

In 1971–72 Anatolii Alekseevich carried out investigations concerning the multidimensional problem of Dirichlet divisors. It is well known that at present the best estimates of the order of growth of Riemann zeta-function in the critical strip near the straigth line $\operatorname{Re} s = 1$ are obtained with the use of Vinogradov's method. These estimates allowed Karatsuba to obtain an essentially better estimate for the remainder in the problem of Dirichlet divisors for the k-dimensional function of divisors in the case, where the number k is large [31, 33]. The most remarkable result here is the estimate of the remainder $R_k(x)$ in this problem which is uniform relative to all parameters:

$$R_k(x) \leq x^{1-ck^{-2/3}} (c_1 \ln x)^k,$$

where $c > 0$, $c_1 > 0$ are absolute constants.

It should also be pointed out that in his work [31] Karatsuba proved one general result which yields the following assertion.

Theorem. *Let $c_0 > 0$ be an arbitrary fixed number, γ be a constant, $1 < \gamma < 3/2$, and $F(x) = e^{c_0 (\ln x)^{\gamma}}$. For the integer $m \neq 0$ we consider the sum*

$$T_m = \sum_{x=1}^{P} e^{2\pi i m F(x)}.$$

If $|m| \leq e^{(\ln P)^{3-2\gamma-\varepsilon_0}}$, where ε_0 is a fixed number from the interval $(0, 3-2\gamma)$, then we have the following estimate:

$$|T_m| \leq A P e^{-\alpha (\ln P)^{3-2\gamma}},$$

where A and α are some absolute positive constants.

This gives an example of a function which grows faster than the polynomial and for which the sequence $\{F(n)\}$ of fractions parts is uniformly distributed modulo unity.

A somewhat separate place is occupied by Karatsuba's investigations concerning Artin's problem of the p-adic representation of zero by a form of an arbitrary degree. Here, instead of the supposed power growth of the number of variables for a nontrivial representation of zero by a form, it was shown that this number of variables must grow almost exponentially in accordance with the degree [62–64]. We have the following:

Theorem. *For any positive integer r there exists $n_0 = n_0(r)$ such that for any $n \geq n_0$ there exists a form $F(x_1, \ldots, x_k)$ of a degree smaller than n, with integer coefficients, the number of whose variables k is*

$$k \geq 2^u, \qquad u = \frac{n}{(\log_2 n)(\log_2 \log_2 n) \cdots \underbrace{(\log_2 \cdots \log_2 n)}_{r} \underbrace{(\log_2 \cdots \log_2 n)^3}_{r+1}},$$

and which has only a trivial representation of zero in 2-adic numbers.

A similar result was also proved there for an arbitrary odd prime modulus p.

In 1968 Karatsuba posed the problem of creating a theory of multiple trigonometric Weyl sums similar to Vinigradov's theory, namely, the sums

$$S = S(A) = \sum_{x_1=1}^{P_1} \cdots \sum_{x_r=1}^{P_r} e^{2\pi i F(x_1,\dots,x_r)},$$

where

$$F(x_1,\dots,x_r) = \sum_{t_1=1}^{n_1} \cdots \sum_{t_r=1}^{n_r} \alpha(t_1,\dots,t_r) x_1^{t_1} \cdots x_r^{t_r},$$

and A is a collection of real coefficients $\alpha(t_1,\dots,t_r)$.

We shall consider here only some main results of the theory. Its central moment is the mean value theorem similar to Vinogradov's theorem in the case of simple sums. We shall give the most perfect formulation of this theorem [59].

Theorem. *Let* $n_1,\dots,n_r$, $P_1,\dots,P_r$ *be positive integers*, $P_1 = \min(P_1,\dots,P_r)$, $m = (n_1+1) \times \cdots (n_r+1)$. *Suppose, furthermore, that* Ω *is an* m*-dimensional cube in a Euclidean space of the form*

$$0 \le \alpha(t_1,\dots,t_r) < 1, \qquad 0 \le t_1 \le n_1, \quad \dots, \quad 0 \le t_r \le n_r,$$

and

$$J = J(P_1,\dots,P_r; n_1,\dots,n_r; K, r) = \int_{\Omega}\!\cdots\!\int |S(A)|^{2K}\, dA.$$

Then, for any $\tau \ge 0$ *and* $K \ge K_\tau = m\tau$, *for the quantity* J *we have the estimate*

$$J \le K_\tau^{2m\tau} \varkappa^{4\varkappa^2 \Delta(\tau)} 2^{8m\varkappa\tau} (P_1 \cdots P_r)^{2K} P^{-\varkappa\Delta(\tau)},$$

where $\varkappa = n_1\nu_1 + \cdots + n_r\nu_r$, $\gamma\varkappa = 1$,

$$\Delta(\tau) = \frac{m}{2}\bigl(1 - (1-\gamma)^\tau\bigr), \qquad P = (P_1^{n_1} \cdots P_r^{n_r})^\gamma,$$

and the positive integers $\nu_1,\dots,\nu_r$ *are such that*

$$-1 < \frac{\ln P_s}{\ln P_1} - \nu_s \le 0, \qquad s = 1,\dots,r.$$

The estimation of the multiple trigonometric sum is based on the mean value theorem and the lemma on the multiplicity of the intersection of special multidimensional parallelepipeds [57].

Theorem. *We divide the points of the cube* Ω *into two classes* Ω_1 *and* Ω_2. *To the class* Ω_1 *we refer the points* A *with coordinates* $\alpha(t_1,\dots,t_r)$ *for which*

$$\alpha(t_1,\dots,t_r) = \frac{a(t_1,\dots,t_r)}{q(t_1,\dots,t_r)} + \beta(t_1,\dots,t_r),$$

where $0 \le a(t_1,\dots,t_r) < q(t_1,\dots,t_r)$, $\bigl(a(t_1,\dots,t_r), q(t_1,\dots,t_r)\bigr) = 1$,

$$|\beta(t_1,\dots,t_r)| \le P_1^{-t_1} \cdots P_r^{-t_r} P^{0.1}, \qquad 0 \le t_1 \le n_1, \quad \dots, \quad 0 \le t_r \le n_r, \quad t_1 + \cdots + t_r \ge 1,$$

and the least common multiple Q of the numbers $q(t_1,\dots,t_r)$, $t_1+\dots+t_r\ge 1$ does not exceed $P^{0.1}$. To the class Ω_2 we refer all the other points of the cube Ω. Let the point $A\in\Omega_1$. Then we have the estimate

$$|S(A)|\le 2^{6r}(5n^{2n})^{r\nu(Q)}\big(\tau(Q)\big)^{r-1}P_1\cdots P_rQ^{-\mu},\qquad \mu\max(n_1,\dots,n_r)=1.$$

If, in addition, we set

$$\delta(t_1,\dots,t_r)=P_1^{t_1}\cdots P_r^{t_r}\beta(t_1,\dots,t_r),\qquad \delta=\max_{t_1,\dots,t_r}|\delta(t_1,\dots,t_r)|,$$

then the estimate

$$|S(A)|\le 2^{6r}(5n^{2n})^{r\nu(Q)}\big(\tau(Q)\big)^{r-1}P_1\cdots P_r(\delta Q)^{-\mu}\big(\ln(\delta+2)\big)^{r-1}$$

is also valid for $\delta>1$. Let the point $A\in\Omega_2$. We set $\tau(t_1,\dots,t_r)=P_1^{t_1}\cdots P_r^{t_r}P^{-1/6}$ and represent the coordinates $\alpha(t_1,\dots,t_r)$ of the point A in the form

$$\alpha(t_1,\dots,t_r)=\frac{a(t_1,\dots,t_r)}{q(t_1,\dots,t_r)}+\frac{\theta(t_1,\dots,t_r)}{q(t_1,\dots,t_r)\tau(t_1,\dots,t_r)},$$
$$\big(a(t_1,\dots,t_r),q(t_1,\dots,t_r)\big)=1,\quad 1\le q(t_1,\dots,t_r)\le\tau(t_1,\dots,t_r),\quad |\theta(t_1,\dots,t_r)|\le 1,$$
$$0\le t_1\le n_1,\quad\dots,\quad 0\le t_r\le n_r,\quad t_1+\dots+t_r\ge 1.$$

We denote by Q_0 the least common multiple of the numbers $q(t_1,\dots,t_r)$ under the condition that $t_1+\dots+t_r\ge 2$. Then the estimate

$$|S(A)|\le(5n^{2n})^{r\nu(Q_0)}\big(\tau(Q_0)\big)^{r-1}P_1\cdots P_rQ^{-0.1\mu}+2^{8r}(r\mu^{-1})^{r-1}P_1\cdots P_rP^{-0.05\mu}$$

is valid for $Q_0\ge P^{1/6}$. Here $\tau(Q)$ is the amount of divisors of the number Q and $\nu(Q)$ is the amount of different prime divisors of the number Q.

Completing the theme connected with trigonometric sums, we shall speak about Karatsuba's recent investigations concerning Kloostermann's sums. The following two theorems are proved in [109].

Theorem. *Let $m>1$ be an integer, k be a positive integer, and let X, X_1 be such that $X\ge 3$, $k<X<X_1\le 2X$, $k2^{2k-1}X^{2k-1}<m$. Consider the congruence*

$$x_1^*+\dots+x_k^*\equiv x_{k+1}^*+\dots+x_{2k}^*\pmod m,$$

where $X<x_j\le X_1$, $(x_j,m)=1$, $x_jx_j^*\equiv 1\pmod m$, $j=1,\dots,2k$. Then the estimate

$$I\le(2k)^{80k^3}X^k(\log X)^{4k^2}$$

is valid for the number $I=I_k(X)$ of its solutions.

This theorem actually asserts that the upper bound of the number I of solutions of the congruence exceeds only by the factor $(\log X)^{4k^2}$ the upper bound of all the collections $(x_1,\dots,x_{2k})$ for which $(x_1,\dots,x_k)$ is the permutation of the collection $(x_{k+1},\dots,x_{2k})$.

Theorem. *Let $m > 1$ be an integer, k and s be positive integers, and let the real numbers X, X_1, Y, Y_1 satisfy the inequalities*

$$3 \le X, \quad k < X < X_1 \le 2X, \quad k2^{2k-1}X^{2k-1} < m,$$
$$3 \le Y, \quad s < Y < Y_1 \le 2Y, \quad s2^{2s-1}Y^{2s-1} < m.$$

Let a and b be integers with $(a, m) = d \ge 1$. We denote by A the set of positive integers n of the form $n = xy$, where $X < x \le X_1$, $Y < y \le Y_1$, $(xy, m) = 1$. Then, for the trigonometric sum

$$S = \sum_{n \in A} \exp\left(2\pi i \frac{an^* + bn}{m}\right)$$

the estimate $|S| \le XY\Delta$ is valid, where

$$\Delta = (2s)^{40s^2/k}(2k)^{40k^2/s}(smdY)^{1/2ks}(\log Y)^{2s/k}(\log X)^{2k/s}X^{-1/2s}Y^{-1/2k}.$$

In conclusion, we shall touch upon Karatsuba's works on the theory of Riemann zeta-function and the works containing the investigations of other Dirichlet arithmetic series.

It is well known that the central problem of the contemporary theory of Riemann zeta-function is Riemann's conjecture. Note that the zeta-function possesses a number of remarkable properties, namely, Euler's product, fuctional equation. If we consider Dirichlet series, then much depends on the arithmetic nature of the sequence of coefficients of these series. In his works Karatsuba investigates the Riemann zeta-function itself as well as Dirichlet series which can either possess or do not possess the characteristic properties indicated above.

Let $N(T)$ denote the number of zeros of the zeta-function in the critical strip for $0 \le \operatorname{Re} s \le 1$, $0 < \operatorname{Im} s \le T$, and let $N_0(T)$ be the number of zeros of $\zeta(1/2 + it)$ for $0 < t \le T$. The well-known Riemann–Mangoldt theorem gives an asymptotic formula of the quantity $N(T)$ as $T \to +\infty$:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Note that Riemann's conjecture asserts the validity of the relation $N_0(T) = N(T)$ for all $T > 0$.

In 1914 Hardy proved the infinity of the number of zeros of Riemann's zeta-function on the critical line and in 1921 Hardy and Littlewood obtained a more exact result, namely,

a) for $H = T^a$, $a > 1/2$, and for a certain positive constant c we have

$$N_0(T + H) - N_0(T) \ge cH; \tag{4}$$

b) for $H = T^{1/4}\log^2 T$ the inequality

$$N_0(T + H) - N_0(T) > 0 \tag{5}$$

holds true.

In 1942 Selberg proved that for $H \ge T^{0.5+\varepsilon}$ the estimate

$$N_0(T + H) - N_0(T) \ge cH\log T, \tag{6}$$

was valid, where $c = c(\varepsilon) > 0$ is a constant.

At the same time, Selberg stated a conjecture that inequality (6) was valid for $H > T^a$, where $a < 1/2$.

In 1974–76 Y. Moser showed that

$$N_0(T+H) - N_0(T) > 0$$

for $T \geq T_0 > 0$, $H \geq T^{1/6}\log^2 T$, and the inequality

$$N_0(T+H) - N_0(T) \geq cH$$

was valid for $H = T^{5/12}\log T$.

In 1983 Karatsuba proved Selberg's conjecture [68, 70; 105, pp. 38–39]. Moreover, he found out that inequality (6) was valid for $H \geq T^a$, $a > 27/82$. Note that in 1981, at the expense of a principally new method of estimation of trigonometric sums, Karatsuba proved that the segment of the critical line of length $H \geq T^{5/32}\log^2 T$ contained at least one odd-order zero of Riemann zeta-function, i.e., the validity of inequality (5) was proved for these H.

Let $\chi_1(n)$ be a complex character modulo 5 such that $\chi_1(2) = i$. The Davenport–Heilbronn function is the function

$$\frac{1-i\varkappa}{2}L(s,\chi_1) + \frac{1+i\varkappa}{2}L(s,\overline{\chi}_1),$$

where

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \qquad \varkappa = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}.$$

This function satisfies the functional equation of a Riemannian type but does not have the Euler product. There is no analog of Riemann's conjecture for $f(s)$, i.e., the statement that all complex zeros of $f(s)$ lie on the straight line $\operatorname{Re} s = 1/2$ is incorrect. Nevertheless, in 1980 Voronin proved that the straight line $\operatorname{Re} s = 1/2$ was an exceptional set of zeros of $f(s)$, i.e., $N_0(T)$ is the number of zeros of odd order of the function $f(s)$ under the conditions $\operatorname{Re} s = 1/2$, $0 < \operatorname{Im} s \leq T$, then, for $T > T_0$, we have

$$N_0(T) \geq T\exp\Big(\frac{1}{20}\sqrt{\log\log\log\log T}\Big).$$

Karatsuba obtained in [89] a considerably stronger result.

Theorem. *Let ε and ε_1 be arbitrarily small fixed positive numbers not exceeding* 0.01. *Then, for $H = T^{27/82+\varepsilon}$, $T \geq T_0(\varepsilon,\varepsilon_1) > 0$, the inequality*

$$N_0(T+H) - N_0(T) \geq H(\log T)^{1/2-\varepsilon_1}$$

holds true.

It should be pointed out that the works considered in this article do not completely cover all the directions of the mathematical activity of A. A. Karatsuba. In this connection, we present here the complete list of his works in mathematics.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Fundamental Research (project no. 97-01-00721).

LIST OF RESEARCH WORKS BY A. A. KARATSUBA

1. Solution of One Problem from the Theory of Finite Automata, *Usp. Mat. Nauk*, 1960, vol. 15, no. 3, pp. 157–159.
2. Tarry Problem for a System of Congruences, *Mat. Sb.*, 1961, vol. 55, no. 2, pp. 209–220.
3. Estimates of Trigonometric Sums of a Special Form and their Applications, *Dokl. Akad. Nauk SSSR*, 1961, vol. 137, no. 3, pp. 513–514.
4. Multiplication of Multivalued Numbers of Automata, *Dokl. Akad. Nauk SSSR*, 1962, vol. 145, no. 2, pp. 293–294 (jointly with Yu. Ofman).
5. Analog of Waring Problem, *Vestn. Mosk. Univ., Ser.* 1, 1962, no. 1, pp. 38–46.
6. Distribution of Fractional Parts of Polynomials of a Special Form, *Vestn. Mosk. Univ., Ser.* 1, 1962, no. 3, pp. 34–38.
7. Waring Problem for Congruence Modulo Equal to the Power of a Prime Number, *Vestn. Mosk. Univ., Ser.* 1, 1962, no. 4, pp. 28–38.
8. *Trigonometricheskie summy spetsial'nogo vida i ikh prilozheniya* (Trigonometric Sums of a Special Form and their Applications), Cand. Sci. Thesis, Moscow: MSU, 1962.
9. On the Mean Value Theorem, *Dokl. Akad. Nauk SSSR*, 1963, vol. 149, no. 2, pp. 245–248 (jointly with N. M. Korobov).
10. Trigonometric Sums of a Special Form and their Applications, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1964, vol. 28, no. 1, pp. 237–248.
11. Supplement no. 4, Hua Loo Keng's book, in: *Abschätzungen von Exponentia und ihre Anwendung in der Zahlentheorie*, Leipzig, 1959.
12. On systems of Congruences, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1965, vol. 29, no. 4, pp. 935–944.
13. On the Estimation of the Number of Solutions of Certain Equations, *Dokl. Akad. Nauk SSSR*, 1965, vol. 165, no. 1, pp. 31–32.
14. Systems of Congruences and Equations of Waring Type, *Dokl. Akad. Nauk SSSR*, 1965, vol. 165, no. 2, pp. 274–276.
15. The Mean Value Theorems and Complete Trigonometric Sums, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1966, vol. 30, no. 1, pp. 183–206.
16. *Metod trigonometric summ i teoremy o srednem* (Method of Trigonometric Sums and Mean Value Theorems), Ph. D. Thesis, Moscow: MSU, 1966.
17. Asymptotic Formulas for a Certain Class of Trigonometric Sums, *Dokl. Akad. Nauk SSSR*, 1966, vol. 169, no. 1, pp. 9–11.
18. Method of Trigonometric Sums and Mean Value Theorems, *Mat. Zametki*, 1967, vol. 1, no. 1, pp. 99–110.
19. On the Estimates of Comlete Trigonometric Sums, *Mat. Zametki*, 1967, vol. 1, no. 2, pp. 199–208.
20. On One System of Indeterminate Equations, *Mat. Zametki*, 1968, vol. 4, no. 2, pp. 125–128.
21. On One Asymptotic Formula, *Trudy Mosk. Mat. Obshch.*, 1968, vol. 18, no. 1, pp. 77–82.
22. The Sum of Characters and Primitive Roots in Finite Fields, *Dokl. Akad. Nauk SSSR*, 1968, vol. 180, no. 6, pp. 1287–1289.
23. On Trigonometric Sums, *Dokl. Akad. Nauk SSSR*, 1969, vol. 189, no. 1, pp. 31–34.
24. On Sums of Characters with Prime Numbers, *Dokl. Akad. Nauk SSSR*, 1970, vol. 190, no. 3, pp. 517–518.
25. On the Estimates of Sums of Characters, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1970, vol. 34, no. 1, pp. 20–30.
26. Sums of Characters with Prime Numbers, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1970, vol. 34, no. 2, pp. 299–321.
27. Distribution of Products of Shifted Prime Numbers in Arithmetic Progressions, *Dokl. Akad. Nauk SSSR*, 1970, vol. 192, no. 4, pp. 724–727.
28. Distribution of Power Residues and Nonresidues in Additive Sequences, *Dokl. Akad. Nauk SSSR*, 1970, vol. 196, no. 4, pp. 759–760.
29. Sums of Characters with Primes Belonging to an Arithmetic Progression, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1971, vol. 35, no. 3, pp. 469–484.
30. On One Arithmetic Sum, *Dokl. Akad. Nauk SSSR*, 1971, vol. 199, no. 4, pp. 770–772.
31. Estimates of Trigonometric Sums Obtained by Vinogradov's Method and their Applications, *Trudy MIAN SSSR*, 1971, vol. 112, pp. 241–255.
32. Problem of Dirichlet Divisors in Number Fields, *Dokl. Akad. Nauk SSSR*, 1972, vol. 204, no. 3, pp. 540–541.
33. Uniform Estimate of the Remainder in the Problem of Dirichlet Divisors, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1972, vol. 36, no. 3, pp. 475–483.
34. On Certain Problems of the Theory of Prime Numbers Connected with Vinogradov's Method, *Trudy MIAN SSSR*, 1973, vol. 132, pp. 257–262.
35. On the Lower Bounds of the Sums of Characters of Polynomials, *Mat. Zametki*, 1973, vol. 14, no. 1, pp. 67–72.
36. The Mean Value of the Absolute Value of Trigonometric Sums, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1973, vol. 37, no. 6, pp. 1203–1227.

37. Sums of Characters from a Sequence of Shifted Prime Numbers and their Applications, *Mat. Zametki*, 1975, vol. 17, no. 1, pp. 155–159.
38. On Certain Problems of Contemporary Analytic Number Theory, *Mat. Zametki*, 1975, vol. 17, no. 2, pp. 341–350.
39. On Certain Arithmetic Problems with Numbers Which Have Small Prime Divisors, *Acta Arith.*, 1975, vol. 27, pp. 489–492.
40. Trigonometric Sums and their Applications, in: *Proc. Internat. Congr. Math.*, Vancouver, 1975, vol. 1, pp. 365–368.
41. *Osnovy analiticheskoi teorii chisel* (Fundamentals of Analytic Number Theory), Moscow: Nauka, 1975.
42. Experimente mit Automaten, *Elektronische Information Verarbeitung und Kybernetic*, 1975, vol. 11, p. 611.
43. Berechnungen und die Kompliziertheit von Beziehungen, *Elektronische Information Verarbeitung und Kybernetic*, 1975, vol. 11, pp. 603–606.
44. On One System of Congruences, *Mat. Zametki*, 1976, vol. 19, no. 3, pp. 389–392.
45. On the Distribution of the Values of Nonprincipal Characters, *Trudy MIAN SSSR*, 1976, vol. 142, pp. 156–164.
46. The Upper Bound of the Absolute Value of a Multiple Trigonometric Sum, *Trudy MIAN SSSR*, 1977, vol. 143, pp. 3–31 (jointly with G. I. Arkhipov and V. N. Chubarikov).
47. Distribution of the Values of Legendre Symbols of Polynomials with Primes, *Dokl. Akad. Nauk SSSR*, 1978, vol. 238, no. 3, pp. 524–526.
48. Sums of Legendre Symbols of Second-Order Polynomials with Primes, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1978, vol. 42, no. 2, pp. 315–324.
49. On Vinogradov's Integral, *Dokl. Akad. Nauk SSSR*, 1978, vol. 239, no. 4, pp. 764–765 (jointly with G. I. Arkhipov).
50. A New Estimate of Vinogradov's Integral, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1978, vol. 42, no. 4, pp. 751–762 (jointly with G. I. Arkhipov).
51. The Exact Estimate of the Number of Solutions of One System of Diophantine Equations, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1978, vol. 42, no. 6, pp. 1187–1226 (jointly with G. I. Arkhipov and V. N. Chubarikov).
52. Asymptotics of One Arithmetic Sum, *Mat. Zametki*, 1978, vol. 24, no. 6, pp. 737–740.
53. Distribution of Fractions Parts of Polynomials in Several Variables, *Mat. Zametki*, 1979, vol. 25, no. 1, pp. 3–14 (jointly with G. I. Arkhipov and V. N. Chubarikov).
54. The Index of Convergence of the Singular Integral in Tarry's Problem, *Dokl. Akad. Nauk SSSR*, 1979, vol. 248, no. 2, pp. 268–272 (jointly with G. I. Arkhipov and V. N. Chubarikov).
55. Trigonometric Integrals, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1979, vol. 43, no. 5, pp. 971–1003 (jointly with G. I. Arkhipov and V. N. Chubarikov).
56. On One System of Diophantine Equations, *Dokl. Akad. Nauk SSSR*, 1980, vol. 252, no. 2, pp. 275–276 (jointly with G. I. Arkhipov and V. N. Chubarikov).
57. *Monograph Multiple Trigonometric Sums*, Trudy MIAN SSSR, vol. 151, Moscow: Nauka, 1980 (jointly with G. I. Arkhipov and V. N. Chubarikov).
58. Uniform Estimates of Multiple Trigonometric Sums, *Dokl. Akad. Nauk SSSR*, 1980, vol. 252, no. 6, pp. 1289–1291 (jointly with G. I. Arkhipov and V. N. Chubarikov).
59. Multiple Trigonometric Sums and their Applications, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1980, vol. 44, no. 4, pp. 723–781 (jointly with G. I. Arkhipov and V. N. Chubarikov).
60. On the Distance Between Adjacent Zeros of Riemann's Zeta-Function Which Lie on the Critical Line, *Trudy MIAN SSSR*, 1981, vol. 157, pp. 49–63.
61. On One Problem with Prime Numbers, *Dokl. Akad. Nauk SSSR*, 1981, vol. 259, no. 6, pp. 1291–1293.
62. On a Local Representation of Zero by a Form, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1981, vol. 45, no. 5, pp. 948–961 (jointly with G. I. Arkhipov).
63. On the Representation of Zero by a Form in the Field of p-adic Numbers, *Dokl. Akad. Nauk SSSR*, 1982, vol. 262, no. 1, pp. 11–13 (jointly with G. I. Arkhipov).
64. On One Problem of the Congruence Theory, *Usp. Mat. Nauk*, 1982, vol. 37, no. 5, pp. 161–162 (jointly with G. I. Arkhipov).
65. Special Cases of the Theory of Multiple Trigonometric Sums, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1983, vol. 47, no. 4, pp. 707–784 (jointly with G. I. Arkhipov and V. N. Chubarikov).
66. New Uniform Estimates of Multiple Trigonometric Sums, *Dokl. Akad. Nauk SSSR*, 1983, vol. 272, no. 1, pp. 11–12 (jointly with G. I. Arkhipov and V. N. Chubarikov).
67. *Osnovy analiticheskoi teorii chisel* (Fundamentals of Analytic Number Theory), 2nd edition, Moscow: Nauka, 1983.
68. On the Zeros of Riemann's Zeta-Function on Short Segments of the Critical Line, *Dokl. Akad. Nauk SSSR*, 1983, vol. 272, no. 6, pp. 1312–1314.
69. Method of Trigonometric Sums in Number Theory, *Trudy MIAN SSSR*, 1984, vol. 168, pp. 4–30 (jointly with G. I. Arkhipov).

70. On the Zeros of the Function $\zeta(s)$ on Short Segments of the Critical Line, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1984, vol. 48, no. 3, pp. 569–584.
71. On the Zeros of Riemann's Zeta-Function, *Dokl. Akad. Nauk SSSR*, 1984, vol. 276, no. 3, pp. 535–539.
72. Quadratic Residues and Nonresidues in Sparse Sequences, *Dokl. Akad. Nauk SSSR*, 1984, vol. 274, no. 5, pp. 1044–1046.
73. Vinogradov's Method of Trigonometric Sums, *Trudy MIAN SSSR*, 1984, vol. 163, pp. 97–103.
74. Distribution of the Zeros of the Function $\zeta(\frac{1}{2}+it)$, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1984, vol. 48, no. 6, pp. 1214–1224.
75. On the Zeros of Riemann's Zeta-Function on the Critical Line, *Trudy MIAN SSSR*, 1985, vol. 167, pp. 167–178.
76. On the Zeros of the Function $\zeta(\frac{1}{2}+it)$, *Dokl. Akad. Nauk SSSR*, 1985, vol. 281, no. 5, pp. 1038–1041.
77. On The Zeros of the Function $\zeta(s)$ in the Neighborhood of the Critical Line, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1985, vol. 49, no. 2, pp. 326–333.
78. On the Function $G(n)$ in Waring Problem, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1985, vol. 49, no. 5, pp. 935–947.
79. On Real Zeros of the Function $\zeta(\frac{1}{2}+it)$, *Usp. Mat. Nauk*, 1985, vol. 40, no. 4, pp. 171–172.
80. Riemann's Zeta-Function and its Zeros, *Usp. Mat. Nauk*, 1985, vol. 40, no. 5, pp. 19–70.

80a. On Zeros of the Riemann Zeta Function on the Critical Line, in: *Diophantische Approximationen*, Tagungsbericht, Oberwolfach: Math. Forsch. Inst. Oberwolfach, 1986, vol. 42.

81. Hilbert–Kamke Problem in the Analytic Number Theory, *Mat. Zametki*, 1987, vol. 41, no. 2, pp. 272–284.
82. Multidimensional Analog of the Waring Problem, *Dokl. Akad. Nauk SSSR*, 1987, vol. 295, no. 3, pp. 521–523 (jointly with G. I. Arkhipov).
83. Distribution of Pairs of Quadratic Residues and Nonresidues of a Special Kind, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1987, vol. 51, no. 5, pp. 994–1009.
84. *Teoriya kratnykh trigonometricheskikh summ* (Theory of Multiple Trigonometric Sums), Moscow: Nauka, 1987 (jointly with G. I. Arkhipov and V. N. Chubarikov).
85. Approximation of Exponential Sums by Shorter Ones, *Proc. Indian Acad. Sci. Math. Sci.*, 1987, vol. 97, no. 1–3, pp. 167–178.
86. Waring's Problem in Several Dimensions, in: *Diophantische Approximationen*, Tagungsbericht, Oberwolfach: Math. Forsch. Inst. Oberwolfach, 1988, vol. 42.
87. In One Diophantine Inequality, *Acta Arith.*, 1989, vol. 53, pp. 309–324.
88. On the Zeros of Certain Dirichlet Series, *Usp. Mat. Nauk*, 1990, vol. 45, no. 1, pp. 175–176.
89. On the Zeros of the Davenport–Heilbronn Function Lying on the Critical Line, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1990, vol. 54, no. 2, pp. 303–315.
90. Distribution of Prime Numbers, *Usp. Mat. Nauk*, 1990, vol. 45, no. 5, pp. 81–140.
91. On the Zeros of a Special Kind of Functions Connected with Dirichlet Series, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1991, vol. 55, no. 3, pp. 483–514.
92. On the Density Theorem, *Trudy MIAN SSSR*, 1991, vol. 200, pp. 185–196.
93. Zeros of Linear Combinations of Z-functions Corresponding to Dirichlet Series, *Dokl. Akad. Nauk SSSR*, 1991, vol. 317, no. 5, pp. 1053–1054.
94. On the Zeros of Riemann's Zeta-Function on the Critical Line, *Internat. Sympos. in Memory of Hua Loo Keng*, 1991, vol. 1, pp. 117–192.
95. Distribution of the Values of Dirichlet Characters on Additive Sequences, *Dokl. Akad. Nauk SSSR*, 1991, vol. 319, no. 3, pp. 543–544.
96. On the Distribution of Real Zeros of the Function $\zeta(\frac{1}{2}+it)$, *Dokl. Akad. Nauk SSSR*, 1992, vol. 322, no. 2, pp. 217–218.
97. On the Number of Zeros of Riemann's Zeta-Function Lying on Almost All Short Segments of the Criticle Line, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1992, vol. 56, no. 2, pp. 372–397.
98. *The Riemann Zeta-Function*, Berlin, New York: W. de Gruyter, 1992 (jointly with S. M. Voronin).
99. Revision of the Theorems on the Number of Zeros, which Lie on the Criticle Line, of Certain Dirichlet Series, *Usp. Mat. Nauk*, 1992, vol. 47, no. 2 (284), pp. 193–194.
100. On Zeros of the Davenport–Heilbronn Function, *Proc. Amalfi Conf. Analytic Number Theory*, 1992, pp. 271–293.
101. On the Zeros of Dirichlet Arithmetic Series Which Do Not Have the Euler Product, *Izv. Rus. Akad. Nauk, Ser. Mat.*, 1993, vol. 57, no. 5, pp. 3–14.
102. On the Zeros of One Class of Functions Generated by Hurwitz Function, *Usp. Mat. Nauk*, 1993, vol. 48, no. 5, pp. 175–176.
103. Distribution of Inverse Values in a Ring of Residues from the Given Absolute Value, *Dokl. Rus. Akad. Nauk*, 1993, vol. 333, no. 2, pp. 138–139.
104. A New Approach to the Problem of Zeros of Certain Dirichlet Series, *Trudy MIAN*, 1994, vol. 207, pp. 180–196.
105. *Complex Analysis in Number Theory*, Stud. Adv. Math, vol. 187, London, Tokyo, Boca Raton, Ann. Arbor: CRC Press, 1994.

106. Zeros of Dirichlet Arithmetic Series, *Math. Slovaca*, 1994, vol. 44, no. 5, pp. 633–649.
107. *Dzeta Funktsiya Riemanna* (Riemann's Zeta-Function), Moscow: Fiz.-Mat. Lit., 1994 (jointly with S. M. Voronin).
108. Fractional Parts of a Special Kind of Functions, *Izv. Rus. Akad. Nauk, Ser. Mat.*, 1995, vol. 59, no. 4, pp. 61–80.
109. Analogs of Kloostermann's Sums, *Izv. Rus. Akad. Nauk, Ser. Mat.*, 1995, vol. 59, no. 5, pp. 93–102.
110. On One Arithmetic Function, *Usp. Mat. Nauk*, 1995, vol. 50, no. 5, pp. 247–248.
111. Complexity of Computations, *Trudy MIAN*, 1995, vol. 211, pp. 186–202.
112. Sums of Fractional Parts of a Special Kind of Functions, *Dokl. Rus. Akad. Nauk*, 1996, vol. 149, no. 3, p. 302.
113. On the $S(t)$ Function, *Izv. Rus. Akad. Nauk, Ser. Mat.*, 1996, vol. 60, no. 5, pp. 27–56.
114. The Density Theorem and the Behavior of the Argument of Riemann's Zeta-Function, *Mat. Zametki*, 1996, vol. 60, no. 3, pp. 448–449.
115. Additive Congruence, *Izv. Rus. Akad. Nauk, Ser. Mat.*, 1997, vol. 61, no. 2, pp. 119–132.

Translated by I. Aleksandrova