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On the mean value of certain functions connected with the convergence of orthogonal series

B. S. KAŠIN

Dedicated to Professor P. L. Ul'janov on his 50th birthday

In the study of the almost everywhere (a.e.) convergence of orthogonal series often arises the following question.

Let $\Phi=\Phi(n)=\{\varphi_i(x)\}_{i=1}^n$ be an orthonormal system of functions defined on the segment [0, 1]. Let us define the operator[1] $S_\Phi^*\colon l_2^n\to L^2(0,1)$ in the following way. If $y=\{y_i\}_{i=1}^n\in l_2^n$, then

(1) $$S_\Phi^*(y)=f(x)=\sup_{1\leqq r\leqq n}\left|\sum_{i=1}^r y_i\varphi_i(x)\right|.$$

Let $s(\Phi)$ denote the norm of the operator S_Φ^*, i.e.,

(2) $$s(\Phi)=\sup_{\|y\|_{l_2^n}\leqq 1}\|S_\Phi^*(y)\|_{L^2(0,1)}.$$

We would like to estimate the number $s(\Phi)$.

The classical result of D.E. Menšov and H. Rademacher (cf. [3, p. 188]) says that for every $\Phi=\Phi(n)$ [2] we have

(3) $$s(\Phi)\leqq C\ln n.$$

For every $n\geqq 1$ Menšov (cf. [3, p. 192]) exhibited an example of a system $\Phi_0=\Phi_0(n)$ such that

(4) $$s(\Phi_0)\geqq c\ln n.$$

In the present paper we shall study the "mean values" of the norms $s(\Phi)$. To illuminate the precise meaning of the term "mean value" we consider the following set of orthonormal systems $Q^n=\{\Phi\}$.

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[1] By l_2^n we denote the space R^n with Euclidean norm.

[2] In the sequel C, c, and B denote positive absolute constants.

If $\Phi \in Q^n$, then $\Phi = \{\varphi_i(x)\}_{i=1}^n$ and each function $\varphi_i(x)$ is piecewise constant on the n subintervals of [0, 1] of equal length. There is a one-to-one correspondance between the systems $\Phi \in Q^n$ and the elements of the group O^n of orthogonal matrices of order n. Namely, to the system $\Phi = \{\varphi_i(x)\}$ corresponds the matrix $A = \{a_{ij}\} \in O^n$ of the following form:

$$a_{ij} = \frac{1}{\sqrt{n}} \varphi_i\left(\frac{j-1/2}{n}\right) \quad (1 \leqq i,\ j \leqq n). \tag{5}$$

With the aid of this correspondance the Haar measure μ_n defined on O^n (cf. [6, p. 43]) may be carried over onto the set Q^n. In the sequel the measure of a closed set $E \in Q^n$ shall be denoted also by $\mu_n(E)$.

We have the following

Theorem. *There exist absolute constants C and $\gamma > 0$ such that for every $n \geqq 1$ and $t \geqq 0$ we have*

$$\mu_n\{\Phi \in Q^n : s(\Phi) \geqq t\} \leqq (Ce^{-\gamma t^2})^n. \tag{6}$$

The statement of this theorem is included in [4] without proof. From this theorem we immediately obtain the corollaries below.

Corollary 1. *There exists an absolute constant B such that*

$$\mu_n\{\Phi \in Q^n : s(\Phi) \geqq B\} < e^{-n}. \tag{6'}$$

Corollary 2. *Let $S(n)$ denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\sigma \in S(n)$ let Φ_σ denote the rearrangement of the terms of Φ $(\in Q^n)$ according to σ. Then*

$$\mu_n\{\Phi \in Q^n : \max_{\sigma \in S(n)} s(\Phi_\sigma) \geqq c\sqrt{\ln n}\} < n^{-n}.$$

From the estimation (6′) it follows that there are "very few" systems having the property discovered by Menšov (we emphasize that in (4) the system $\Phi_0(n)$ may be chosen from Q^n).

For the proof of Theorem 1 we need some lemmas. First we shall introduce some notations. The number of elements of any finite set E shall be denoted by $|E|$. Furthermore, if $x \in R^m$, then $(x)_j$ $(1 \leqq j \leqq m)$ denotes the jth coordinate of the vector x, and, as usually,

$$R_+^n = \{x = \{(x)_j\}_{j=1}^n : (x)_j \geqq 0,\ 1 \leqq j \leqq n\}; \quad S^n = \{x \in R^n : \|x\|_{l_2^n} = 1\}.$$

If L is a subspace in R^n, $\dim L = q$, then m_L denotes the normed Lebesgue measure on the q-dimensional sphere $S^n \cap L$ $(m_L(S^n \cap L) = 1)$. The measure $m_{R^n}(E)$ shall be denoted simply by $m(E)$.

Lemma 1. *For each* $m \geqq 1$ *one may choose a set of vectors* $\Omega_m = \{e\}$ *such that*
(i) $|\Omega_m| < C^m$;
(ii) *if* $e \in \Omega_m$, *then* $e \in R_+^m$ *and* $\|e\|_{l_2^m} \geqq 1/3$;
(iii) *for any vector* $y \in S^m \cap R_+^m$ *there is a vector* $e = e(y) \in \Omega_m$ *such that*

$$(e(y))_j \leqq (y)_j \quad (1 \leqq j \leqq m).$$

Proof. Suppose that $y \in S^m \cap R_+^m$. Let us set for $s = 0, 1, 2, \ldots$

$$E_s(y) = \{j : 2^{-s-1} < y_j \leqq 2^{-s},\ 1 \leqq j \leqq m\}.$$

It is easy to see that

$$\text{(7)} \quad \begin{aligned} &\text{a)} && |E_s(y)| < 2^{2(s+1)}, \\ &\text{b)} && \Big(\sum_{j : y_j > (2\sqrt{m})^{-1}} y_j^2\Big)^{1/2} \geqq \left(1 - \frac{1}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}. \end{aligned}$$

For each $y \in S^m \cap R_+^m$ let us define the vector $e(y) \in R^m$ in the following way:

$$\text{(8)} \qquad (e(y))_j = \begin{cases} 2^{-s-1} & \text{if } \ j \in E_s(y),\ 0 \leqq s \leqq \left[\frac{1}{2}\log_2 m\right] + 1, \\ 0 & \text{if } \ j \notin \bigcup E_s(y),\ 0 \leqq s \leqq \left[\frac{1}{2}\log_2 m\right]. \end{cases}$$

From the construction of $e(y)$ it is obvious that

$$\text{(9)} \qquad 0 \leqq (e(y))_j < (y)_j.$$

Besides, in virtue of the definition of $E_s(y)$ and part b) of relation (7) we have

$$\text{(9')} \qquad \left(\sum_{j=1}^{m} (e(y))_j^2\right)^{1/2} \geqq \frac{\sqrt{3}}{4} > \frac{1}{3}.$$

Let

$$\Omega_m = \bigcup_{y \in S^m \cap R_+^m} e(y).$$

On account of (9) and (9′) to conclude the proof of Lemma 1 it is enough to show that $|\Omega_m| \leqq C^m$. It is obvious that for any $y \in S^m \cap R_+^m$ and $s \geqq 0$ we have $|E_s(y)| \leqq m$. Consequently, the number of the different systems $\lambda(y)$ of the form

$$\lambda(y) = \{|E_0(y)|, \ldots, |E_{[(\log_2 m)/2]+1}(y)|\}$$

does not exceed $(m+1)^{2+(\log_2 m)/2}$.

From part a) of relation (7) it follows easily that the number of vectors $e(y)$ of the system Ω_m with the same system $\lambda(y)$ does not exceed

$$(c')^m \prod_{1 \leqq s \leqq (\log_2 m)/2} C_m^{2^{2s}}.$$

Finally, for the number $|\Omega_m|$ we obtain the estimate

$$|\Omega_m| \leqq (m+1)^{\log_2 m}(c')^m \prod_{1\leqq s\leqq(\log_2 m)/2} C_m^{2^{2s}}.$$

By using the elementary estimate $C_m^q \leqq C^q(m/q)^q$ we obtain that

$$|\Omega_m| \leqq C^m \prod_{s=1}^{[(\log_2 m)/2]} \left(\frac{m}{2^{2s}}\right)^{2^{2s}} \leqq C^m \prod_{q=1}^{[\log_2 m]} \left(\frac{m}{2^q}\right)^{2^q} =$$

$$= C^m \exp\left\{\ln \prod_{q=1}^{[\log_2 m]} \left(\frac{m}{2^q}\right)^{2^q}\right\} = C^m \exp\left\{\prod_{q=1}^{[\log_2 m]} 2^q \ln\frac{m}{2^q}\right\} \leqq$$

$$\leqq C^m \exp\left\{m\left(\sum_{q=1}^{[\log_2 m]} \frac{2^q}{m}\ln\frac{m}{2^q}\right)\right\} \leqq C^m \exp\{mc'\} \leqq C^m.$$

This proves Lemma 1.

We shall use the following well-known fact (cf., for example, [5, p. 335]). On the sphere S^n there exists a set $\Omega_n'=\{e'\}$ such that $|\Omega_n'|\leqq C^n$ and for any $y\in S^n$ there exists a vector $e'=e'(y)\in\Omega_n'$ for which

$$\|e'(y)-y\|_{l_2^n} \leqq \frac{1}{2}. \tag{10}$$

The estimate below is also well-known.

Lemma 2. *For* $t\geqq 0$, $1\leqq n<\infty$, *and* $x_0\in S^n$ *we have the inequality*

$$f(t) = m\{x\in S^n\colon |(x, x_0)| \geqq t\} \leqq e^{-t^2n/4}.$$

For $t>1$ the statement of Lemma 2 is obvious, as in this case $f(t)=0$. For $0\leqq t\leqq 1$ the set $\{x\in S^n\colon (x, x_0)\geqq t\}$ is contained in the hemisphere of radius $(1-t^2)^{1/2}$ and consequently,

$$f(t) \leqq (1-t^2)^{(n-1)/2} = [(1-t^2)^{1/t^2}]^{t^2 n(n-1)/2n} \leqq e^{-t^2n/4}$$

(in obtaining the last inequality we used the estimation $(1-x)^{1/x}\leqq e^{-1}$, $0<x<1$). Lemma 2 has been proved.

Lemma 2 immediately implies

Consequence. *Let* L *be a subspace of* R^n, $\dim L=m$, $0\neq x_0\in R^n$. *Then for* $t\geqq 0$ *we have*

$$m_L\{x\in S^n\cap L\colon |(x, x_0)| \geqq t\} \leqq \exp\left\{-\frac{1}{4}mt^2\|x_0\|_{l_2^n}^{-2}\right\}. \tag{11}$$

Lemma 3. *There are absolute constants* C *and* $\alpha>0$ *such that for every subspace* $L\subset R^n$ $(n>3$, $\dim L\geqq -1+n/2)$, *vector* $a=\{a_i\}\in S^n$, *and number* $t\geqq 0$ *we have the inequality*

$$m_L\left\{y\in S^n\cap L\colon \max_{1\leqq r\leqq n}\left|\sum_{i=1}^{r} a_i y_i\right| \geqq t\right\} \leqq Ce^{-\alpha t^2 n}. \tag{12}$$

Proof. It is enough to prove Lemma 3 for $t>20/n^{1/2}$, since for $t\leqq 20n^{-1/2}$ the statement of our lemma is obtained because of the choice of the constant C in (12). Without loss of generality we may suppose that $a_i\neq 0$ $(1\leqq i\leqq n)$.

For $s=0,1,2,\dots$ let us split the sum $\sum_{i=1}^{n} a_i y_i$ into the "pieces" $\{P_\nu^s\}_{\nu=0}^{2^s-1}$ in the following way:

$$P_0^0=\sum_{i=1}^{n} a_i y_i,\quad P_0^1=\sum_{i=1}^{n_1-1} a_i y_i+\tilde{a}_{n_1}y_{n_1},\quad P_1^1=(a_{n_1}-\tilde{a}_{n_1})y_{n_1}+\sum_{i=n_1+1}^{n} a_i y_i.$$

Analogously, the "piece" P_ν^{s-1} is represented in the form

$$P_\nu^{s-1}=P_{2\nu}^s+P_{2\nu+1}^s.$$

A similar division is used in the proof of Erdős' theorem on the a.e. convergence of lacunary trigonometric series (cf. [1, p. 705—708]).

It is not hard to see (cf. [1, p. 705, Lemma 2]) that the splitting $\{P_\nu^s\}_{\nu=0}^{2^s-1}$, $s=0,1,2,\dots$, of the above form may be chosen in such a way that for each "piece" $P_\nu^s=\sum_{i=1}^{n} a_i^{(\nu,s)} y_i$ $(0\leqq\nu<2^s)$ we have the estimation

$$\sum_{i=1}^{n}[a_i^{(\nu,s)}]^2\leqq 2^{-s}. \tag{13}$$

(13) implies that we can find an integer s_0 such that all "pieces" $P_\nu^{s_0}$ $(0\leqq\nu\leqq 2^s-1)$ contain at most two terms. Then we have the inequality

$$\max_{1\leqq r\leqq n}\left|\sum_{i=1}^{r} a_i y_i\right|\leqq\sum_{s=0}^{s_0}\max_{0\leqq\nu<2^s}|P_\nu^s|+\max_{1\leqq i\leqq n}|a_i y_i|.$$

Consequently, if for fixed vectors $\{a\}$ and $\{y\}$ and for $s=0,1,\dots,s_0$ we have the inequalities

$$\max_{0\leqq\nu<2^s}|P_\nu^s|\leqq\frac{t}{5(s+1)^2}\quad\text{and}\quad\max_{1\leqq i\leqq n}|a_i y_i|\leqq\frac{t}{3}, \tag{14}$$

then

$$\max_{1\leqq r\leqq n}\left|\sum_{i=1}^{r} a_i y_i\right|\leqq\frac{t}{3}+t\sum_{s=0}^{\infty}\frac{1}{5(s+1)^2}\leqq t.$$

Hence we have

$$m_L\left\{y\in S^n\cap L:\max_{1\leqq r\leqq n}\left|\sum_{i=1}^{r} a_i y_i\right|\geqq t\right\}\leqq\sum_{s=0}^{s_0}\sum_{\nu=0}^{2^s-1} m_L\left\{y\in S^n\cap L:|P_\nu^s|\geqq\frac{t}{5(s+1)^2}\right\}+ \tag{15}$$
$$+m_L\left\{y\in S^n\cap L:\max_{1\leqq i\leqq n}|a_i y_i|\geqq\frac{t}{3}\right\}.$$

Due to (13), the consequence of Lemma 2 (cf. (11)), and the inequality $\dim L \geqq -1+n/2$ we obtain that

$$\sum_{\nu=1}^{2^s-1} m_L\left\{y \in L \cap S^n : |P_\nu^s| \geqq \frac{t}{5(s+1)^2}\right\} \leqq 2^s \exp\left\{-\frac{2^s t^2 n}{100(s+1)^4}\right\}. \tag{16}$$

Besides, by using (11) we get that

$$m_L\left\{y \in S^n \cap L : \max_{1 \leqq i \leqq n} |a_i y_i| \geqq \frac{t}{3}\right\} \leqq \tag{16'}$$

$$\leqq \sum_{i=1}^{n} m_L\left\{y \in S^n \cap L : |y_i| > \frac{t}{3|a_i|}\right\} \leqq \sum_{i=1}^{n} \exp\left\{-\frac{t^2 n}{36 a_i^2}\right\}.$$

Comparing inequalities (15), (16), and (16′) we arrive at

$$m_L\left\{y \in S^n \cap L : \max_{1 \leqq r \leqq n} \left|\sum_{i=1}^{r} a_i y_i\right| \geqq t\right\} \leqq \tag{17}$$

$$\leqq \sum_{s=0}^{s_0} 2^s \exp\left\{-\frac{2^s t^2 n}{100(s+1)^4}\right\} + \sum_{i=1}^{n} \exp\left\{-\frac{t^2 n}{36 a_i^2}\right\}.$$

If $t^2 n > 200$, then it is easy to show that

$$\sum_{s=0}^{s_0} 2^s \exp\left\{-\frac{2^s t^2 n}{100(s+1)^4}\right\} < c' e^{-\gamma t^2 n} \quad (\gamma > 0),$$

and, moreover,

$$\sum_{i=1}^{n} \exp\left\{-\frac{t^2 n}{36 a_i^2}\right\} \leqq \sum_{q=0}^{\infty} \sum_{i:2^{-q-1} < a_i^2 \leqq 2^{-q}} \exp\left\{-\frac{t^2 n}{36 a_i^2}\right\} \leqq$$

$$\leqq \sum_{q=0}^{\infty} 2^q \exp\left\{-\frac{t^2 n}{36} 2^q\right\} \leqq c' e^{-\gamma t^2 n} \quad (\gamma > 0).$$

By putting the last inequalities into (17) we obtain the estimation (12). Lemma 3 is proved.

We are going to use the following property of the measure μ_n on the group O^n (cf. [6, p. 55]). Let us fix an arbitrary vector $e_0 \in S^n$ and denote by $O^{n-1} = \{\tilde{A}\}$ the subgroup of the group O^n whose elements are those matrices $\tilde{A}$ for which $\tilde{A} e_0 = e_0$. The symbol μ_{n-1}, as before, denotes Haar measure on O^{n-1}. For $e \in S^n$ let us choose an arbitrary matrix A_e from O^n with $A_e e_0 = e$. Then we have

$$\int f(A)\, d\mu_n = \int_{S^n} dm \int_{O^{n-1}} f(A_e \tilde{A})\, d\mu_{n-1} \tag{18}$$

for any measurable bounded function $f(A)$, $A \in O^n$.

Proof of Theorem. We may suppose that $n \geqq 10$, as for $n<10$ the statement of the theorem shall be satisfied if we choose the constant C in (6) large enough. Let a system $\Phi \in Q^n$ be given. Suppose that the matrix $A(\Phi)=A=\{a_{ij}\}\in O^n$ is defined by the equality (5). Then it is easy to see that

$$(19)\qquad s(\Phi) = \sup_{y=\{y_i\}\in S^n,\ \{N_j\}_{j=1}^n,\ 1\leqq N_j\leqq n} \left\{\sum_{j=1}^{n}\left(\sum_{i=1}^{N_j} y_i a_{ij}\right)^2\right\}^{1/2} =$$

$$= \sup_{y=\{y_i\}\in S^n,\ \{N_j\}} \left\{\sum_{j=1}^{[n/2]}\left(\sum_{i=1}^{N_j} y_i a_{ij}\right)^2 + \sum_{j=[n/2]+1}^{n}\left(\sum_{i=1}^{N_j} y_i a_{ij}\right)^2\right\}^{1/2}.$$

Consequently, it is enough to prove that for $m=[n/2]$ and $m=n-[n/2]$ and for all $t\geqq 0$ we have

$$(20)\qquad M(t) \equiv \mu\left\{A\in O^n:\ \sup_{y\in S^n,\ \{N_j\}}\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{n} y_i a_{ij}\right)^2\right]^{1/2} \geqq \frac{t}{\sqrt{2}}\right\} \leqq (Ce^{-\gamma t^2})^n.$$

Let the sequence $\{N_j\}_{j=1}^n$ of numbers be fixed. By using the properties of the system of vectors Ω_n' (cf. (10)) we obtain that

$$f(A,\{N_j\}) \equiv \sup_{y\in S^n}\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{N_j} a_{ij}y_i\right)^2\right]^{1/2} \leqq \sup_{y\in S^n}\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{N_j}(y-e'(y))_i a_{ij}\right\}^2\right]^{1/2} +$$

$$+\sup_{y\in S^n}\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{N_j}(e'(y))_i a_{ij}\right\}^2\right]^{1/2}.$$

Taking into account that $\left(\sum_{i=1}^{n}(y-e'(y))^2\right)^{1/2}\leqq 1/2$ we arrive at

$$f(A,\{N_j\}) \leqq \frac{1}{2}f(A,\{N_j\}) + \sup_{e'\in\Omega_n'}\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{N_j}(e'(y))_i a_{ij}\right\}^2\right]^{1/2}.$$

As $|\Omega_n'|\leqq C^n$ (cf. (10)), from the last inequality and from the definition of the function $M(t)$ (cf. (20)) we obtain that

$$(21)\qquad M(t) \leqq (C')^n \sup_{\{z_i\}=z\in S^n} \mu\left\{A\in O^n:\ \sup_{\{N_j\}}\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{N_j} z_i a_{ij}\right)^2\right]^{1/2} \geqq \frac{t}{\sqrt{2}}\right\}.$$

By virtue of Lemma 1 from inequality (21) it follows that

$$(22)\qquad M(t) \leqq C^n \sup_{z\in S^n,\ e\in\Omega_m} \mu\left\{A\in O^n:\ \sup_{\{N_j\}}\left|\sum_{i=1}^{N_j} z_i a_{ij}\right| \geqq \frac{t}{\sqrt{2}}(e)_j,\ 1\leqq j\leqq m\right\}.$$

Let the vectors $z\in S^n$, $e\in\Omega^m$, and the number $t\geqq 0$ be fixed. Then

$$(23)\qquad \mu\left\{A\in O^n:\ \sup_{\{N_j\}}\left|\sum_{i=1}^{N_j} z_i a_{ij}\right| \geqq \frac{t}{\sqrt{2}}(e)_j,\ 1\leqq j\leqq m\right\} = \int_{O^n}\prod_{j=1}^{m}\chi_j(A)\,d\mu_n,$$

where $A=\{a_{ij}\}\in O^n$, $1\leqq j\leqq m$, and

$$\chi_j(A)=\begin{cases}1 & \text{if } \sup\limits_{1\leqq N_j\leqq n}\left|\sum\limits_{i=1}^{N_j} z_i a_{ij}\right|\geqq \dfrac{t}{\sqrt{2}}(e)_j,\\ 0 & \text{otherwise.}\end{cases}$$

As $\|e\|_{l_2^m}\geqq 1/3$, $e\in\Omega_m$ (cf. Lemma 1), a multiple application of Lemma 3 and equality (18) show that

$$\int\limits_{O^n}\prod_{j=1}^{m}\chi_j(A)\,d\mu_n=\prod_{j=1}^{m}C\exp\left\{-\frac{\alpha}{2}t^2n(e_j)^2\right\}\leqq \tag{24}$$

$$\leqq C^m\exp\left\{-\frac{\alpha}{2}t^2n\sum_{j=1}^{m}(e)_j^2\right\}\leqq C^n\exp(-\gamma nt^2),$$

where $\gamma>0$ is an absolute constant.

From (22)—(24) we obtain that

$$M(t)\leqq C\exp\{-\gamma nt^2\}.$$

Thus inequality (20) is proved. As it was noted earlier, the statement of our theorem follows from (20).

Remark. We estimated the mean value of the function $s(\Phi)$ on the set Q^n consisting of orthonormal systems. The estimation of the mean value of the function $s(\Phi)$ on the set of all (not necessarily orthogonal) systems such that $\Phi=\{\varphi_i(x)\}_{i=1}^n$, $|\varphi_i(x)|\equiv 1$, the function $\varphi_i(x)$ being constant on the intervals $((j-1)/n,\ j/n)$, $1\leqq i,\ j\leqq n$, $x\in[0,1]$, may be carried out following the method of [2].

References

[1] Н. К. Бари, *Тригонометрические ряды,* Физматгиз (Москва, 1961).

[2] G. Bennett, V. Goodman, C. M. Newman, Norms of random matrices, *Pacific J. Math.,* **59** (1975), 359—365.

[3] S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen* (Warszawa—Lwów, 1935) — С. Качмаж и Г. Штейнгауз, *Теория ортогональных рядов,* Физматгиз (Москва, 1958).

[4] Б. С. Кашин, О некоторых свойствах ортогональных систем сходимости, *Труды матем. ин-та им. В. А. Стеклова АН СССР,* **143** (1977), 68—87.

[5] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum,* Springer (Berlin—Göttingen—Heidelberg, 1953) — Л. Фейеш Тот, *Расположения на плоскости, на сфере и в пространстве,* Физматгиз (Москва, 1958).

[6] A. Weil, *L'intégration dans les groupes topologiques et ses applications* (Paris, 1940) — А. Вейль, *Интегрирование в топологических группах и его применения,* Иностранная литература (Москва, 1950).

О среднем значении некоторых функций, связанных со сходимостью ортогональных рядов

Б. С. КАШИН

В статье рассматриваются множества Q^n, $1 \leqq n < \infty$, ортонормированных систем $\Phi = \{\varphi_i(x)\}_{i=1}^n$, состоящих из функций, постоянных на интервалах $\left(\frac{j-1}{n}, \frac{j}{n}\right)$, $1 \leqq j \leqq n$. На Q^n естественно переносится с группы ортогональных матриц порядка n мера Хаара. Изучается поведение на Q^n функции

$$S(\Phi) = \sup_{\sum_{i=1}^n y_i^2 = 1} \Big(\int_0^1 \sup_{1 \leqq r \leqq n} \big(\sum_{i=1}^r y_i \varphi_i(x)\big)^2 dx\Big)^{1/2}.$$

Доказывается, что при $t > 0$ и $n = 1, 2, \ldots$

$$\mu\{\Phi \in Q^n \colon s(\Phi) \geqq t\} \leqq (Ce^{-\gamma t^2})^n.$$

Б. С. КАШИН
СССР, МОСКВА 117 333
УЛ. ВАВИЛОВА 42
МАТЕМАТИЧЕСКИЙ ИНСТИТУТ
ИМ. В. А. СТЕКЛОВА АН СССР