

REMARKS ON ESTIMATING THE LEBESGUE FUNCTIONS OF AN  
ORTHONORMAL SYSTEM

UDC 517.5

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ABSTRACT. In this paper we clarify the relationship between lower bounds for Lebesgue functions and multiplicative inequalities of Gagliardo-Nirenberg type. We give simple proofs of some theorems on Lebesgue functions.

Bibliography: 10 titles.

Let  $\{\varphi_k(x)\}_1^\infty$  be an orthonormal system of functions (O.N.S.), defined on the interval  $[0, 1]$ . The Lebesgue functions  $L_n(t)$  of the system are defined by

$$L_n(t) = \int_0^1 \left| \sum_{k=1}^n \varphi_k(t) \varphi_k(x) \right| dx. \quad (1)$$

Estimates for the Lebesgue functions of various O.N.S. have long played an important role in the theory of orthogonal series (for details see [1] and [3]). In the present paper we are concerned only with lower bounds for the Lebesgue functions of collectively bounded O.N.S. The first such bound was obtained by Olevskii [4] in 1966; he proved the following theorem.

THEOREM A. For every collectively bounded O.N.S.  $\{\varphi_k(x)\}_1^\infty$  (i.e.,  $|\varphi_k(x)| \leq M$ ,  $1 \leq k < \infty$ ,  $x \in [0, 1]$ ) we have

$$\overline{\lim}_{n \rightarrow \infty} L_n(t) = \infty, \quad t \in E, \quad \text{mes } E > c > 0.$$

In 1967 Olevskii [5] announced, and in 1975 he published in the book [3], the following stronger result.

THEOREM B. Under the hypotheses of Theorem A, the inequality

$$\max_{1 \leq m \leq n} \ln^{-1} n \cdot L_m(t) > c' > 0, \quad t \in E_n, \quad \text{mes } E_n > c > 0, \quad (2)$$

holds for  $n = 1, 2, \dots$ .

Olevskii's method of proof can be said to be direct, since it depends only on consideration of the system  $\{\varphi_k(x)\}_1^\infty$  itself.

The problem of a nontrivial lower bound for the Lebesgue functions has a sort of dual

problem, namely to construct, for a given complete and collectively bounded O.N.S.  $\{\varphi_k(x)\}_1^\infty$ , a function  $f(x)$  of bounded variation such that

$$\sum_{k=1}^{\infty} \left| \int_0^1 \varphi_k(x) f(x) dx \right| = \infty.$$

This problem was considered in 1974 by Bočkarev (for details see [6] and [7]), who showed, in particular, that the following theorem holds.

**THEOREM C.** *Let  $\{\varphi_k(x)\}_1^\infty$  be a complete and collectively bounded O.N.S., and let functions  $\chi_j^N(x)$  ( $1 \leq j \leq N$ ,  $N = 2^r$ ,  $r > 0$  (an integer)) be defined by*

$$\chi_j^N(x) = \begin{cases} 0, & x \leq \frac{j-1}{N}, \\ 1, & x \geq \frac{j}{N}, \\ \text{linear on } \left[ \frac{j-1}{N}, \frac{j}{N} \right]. \end{cases}$$

Also let  $\chi_j^N(x) = \sum_{k=1}^{\infty} a_k^j \varphi_k(x)$ . Then

$$\frac{1}{N} \sum_{j=1}^N \sum_{k=1}^{\infty} |a_k^j| > c \ln N. \quad (3)$$

Bočkarev's proof was based on properties of the Haar system, and in particular it used the following inequality: given a series  $\sum_{k=1}^{\infty} a_k \chi_k(x)$  in the Haar system, with  $|\delta_n(x)| \leq 2^{-n}$ ,  $0 \leq n < \infty$ , where  $\delta_0(x) = a_1 \chi_1(x)$  and  $\delta_n(x) = \sum_{k=2^{n-1}+1}^{2^n} a_k \chi_k(x)$ ,  $n > 0$ ; then

$$\left\| \sum_{k=1}^{\infty} a_k \chi_k(x) \right\|_{L^1} \geq \frac{1}{8} \int_0^1 \sum_{n=0}^{\infty} 2^n \delta_n^2(x) dx. \quad (4)$$

In 1975 the author (see [8]) suggested a proof of Theorem C based on properties of the Hilbert matrix. Also in 1975 Bočkarev applied (4) to prove the following result (see [9]).

**THEOREM D.** *Let  $\{\varphi_k(x)\}_1^\infty$  be a collectively bounded O.N.S. Then for  $n = 1, 2, \dots$*

$$\frac{1}{n} \sum_{m=1}^n L_n(t) > c' \ln n, \quad t \in E_n, \quad \text{mes } E_n > c' > 0. \quad (5)$$

It is clear that (5) implies (2); the author has remarked that Theorem D also follows easily from Olevskii's inequality (see [3] and [5]): Under the hypotheses of Theorem A,

$$\max_{1 \leq m \leq n} \left\| \sum_{k=1}^m \varphi_k(x) \right\|_{L^1} \geq c \ln n, \quad n = 1, 2, \dots$$

In [10] Bočkarev showed that (5) is a simple corollary of the following numerical inequality:

For an arbitrary sequence  $\{a_k\}_1^\infty$  and for  $n = 1, 2, \dots$ ,

$$\max_{1 \leq k \leq n} |a_k| \left( \sum_{\nu=1}^n \left| \sum_{k=1}^{\nu} a_k \right| + n \left| \sum_{k=1}^n a_k \right| \right) \geq B \sum_{j=0}^N \sum_{q=2^j+1}^{2^{j+1}} \left[ \sum_{k=1}^n a_k \psi_q \left( \frac{2k-1}{2^{N+2}} \right) \right]^2, \quad (6)$$

where  $N$  is determined from  $2^{N-1} \leq n < 2^N$  and  $\{\psi_q(t)\}$  is the Schauder system (see [1], p. 50).

In proving (6), Bočkarev used inequalities like (4).

In concluding this survey we note that A. S. Krancberg recently suggested a proof of Theorem D based (like the discussion in [8]) on properties of the Hilbert matrix.

In the present paper we give proofs, essentially simpler than previous proofs, of the theorems quoted above on estimates for Lebesgue functions. The author has made the two following simple remarks:

a) Theorem D follows at once from the statement that for arbitrary numbers  $\{a_k\}$

$$\max_{1 \leq k \leq n} |a_k| \left( \sum_{p=1}^n \left| \sum_{k=1}^p a_k \right| \right) > c \sum_{v=0}^{n-1} \sum_{\mu=v+1}^n \frac{\left( \sum_{k=v+1}^{\mu} a_k \right)^2}{(v-\mu)^2} \quad (7)$$

(we note that (6) and (7), and likewise the methods of using them, have much in common).<sup>(1)</sup>

b) Inequality (7) is a discrete version of the following proposition which is related to well-known multiplicative inequalities for derivatives:

For every  $f(x)$ ,  $x \in (-\infty, \infty) = \mathbf{R}^1$ ,

$$N^2(f, W_2^{1/2}) \leq K \|f\|_{L^1(\mathbf{R}^1)} \|f'\|_{L^\infty(\mathbf{R}^1)}, \quad (8)$$

where  $W_2^{1/2}$  is a Sobolev space (with the norm  $\|f\|_{W_2^{1/2}} = \|f\|_{L^2(\mathbf{R}^1)} + N(f, W_2^{1/2})$ ), and the functional  $N(f, W_2^{1/2})$  is defined by

$$N(f, W_2^{1/2}) = \left( \int_0^\infty \|(\Delta_{\hbar} f)(x)\|_{L^2(\mathbf{R}^1)}^2 \hbar^{-2} d\hbar \right)^{1/2}, \quad (\Delta_{\hbar} f)(x) = f(x + \hbar) - f(x),$$

and  $\|f'\|_{L^\infty} = \sup_{x, \hbar > 0} \hbar^{-1} |(\Delta_{\hbar} f)(x)|$ .

Inequality (8) is similar to the Gagliardo-Nirenberg inequality (cf. [2], p. 237). The difference is that (8) involves a Sobolev space  $W_2^{1/2}$  of fractional order. The analogy suggests that it would be desirable to find a proof of (8) that would seem natural from the point of view of the theory of Sobolev classes. In response to this suggestion, O. V. Besov found a very simple proof. We shall give his argument below.

We now turn to the proof.

1) *Deduction of inequality (8) (Besov).* Without loss of generality we suppose that  $\|f'\|_{L^\infty} = 1$ . It is well known that, for  $1 \leq p < \infty$ ,

$$\|f(x)\|_{L^p(\mathbf{R}^1)}^p = p \int_0^\infty t^{p-1} \lambda(t) dt,$$

where  $\lambda(t) = \text{mes}\{x: |f(x)| > t\}$ . For every  $n > 0$  we define the sets

$$E(\hbar) = \{x: |f(x)| \leq \hbar\} \cap \{x: |f(x + \hbar)| \leq \hbar\}, \quad F(\hbar) = \mathbf{R}^1 / E(\hbar).$$

It is clear that  $\text{mes } F(\hbar) \leq 2\lambda(\hbar)$  and that for arbitrary  $x$  and  $\hbar > 0$  we have the following inequalities:

- a)  $|(\Delta_{\hbar} f)(x)| \leq |f(x)| + |f(x + \hbar)|$ ;
- b)  $|(\Delta_{\hbar} f)(x)| \leq \hbar$ .

<sup>(1)</sup>Inequality (6) can be derived from (7).

Now

$$\|(\Delta_{\hbar}f)(x)\|_{L^2(\mathbf{R}^1)}^2 = \|(\Delta_{\hbar}f)(x)\|_{L^2(E(\hbar))}^2 + \|(\Delta_{\hbar}f)(x)\|_{L^2(F(\hbar))}^2;$$

if we estimate the first term by using a) and the second by using b), we have

$$\begin{aligned} \|(\Delta_{\hbar}f)(x)\|_{L^2(\mathbf{R}^1)}^2 &\leq 2 \int_{E(\hbar)} f^2(x) + f^2(x + \hbar) dx + \hbar^2 \text{mes } F(\hbar) \\ &\leq 4 \int_{\{x: |f(x)| \leq \hbar\}} f^2(x) dx + 2\hbar^2 \lambda(\hbar) = 8 \int_0^{\hbar} t (\lambda(t) - \lambda(\hbar)) dt \\ &\quad + 2\hbar^2 \lambda(\hbar) \leq 8 \int_0^{\hbar} t \lambda(t) dt. \end{aligned}$$

Consequently

$$N^2(f, W_2^{1/2}) \leq 8 \int_0^{\infty} \hbar^{-2} \int_0^{\hbar} t \lambda(t) dt d\hbar = 8 \int_0^{\infty} \int_t^{\infty} \hbar^{-2} t \lambda(t) d\hbar dt = 8 \int_0^{\infty} \lambda(t) dt = 8 \|f\|_{L^1}.$$

2) *Derivation of inequality (7)*. For a given set of numbers  $\{a_k\}_1^n$  we define a continuous function  $f(x)$  on the real axis by putting

$$f(x) = \begin{cases} \sum_{k=1}^j a_k & \text{for } |x - j| \leq \frac{1}{3}, 1 \leq j \leq n, \\ 0 & \text{for } x \leq 1/3, \\ \text{linear on } \left[ j - \frac{2}{3}, j - \frac{1}{3} \right], & 1 \leq j \leq n, \\ f(x + n) = f(n - x). & \end{cases}$$

It is easy to see that

$$\|f'\|_{L^\infty} \leq 3 \max_{1 \leq k \leq n} |a_k|, \quad \|f\|_{L^1} \leq B \sum_{p=1}^n \left| \sum_{k=1}^p a_k \right|.$$

Applying (8) to  $f(x)$  yields

$$\begin{aligned} &\left( \max_{1 \leq k \leq n} |a_k| \right) \sum_{p=1}^n \left| \sum_{k=1}^p a_k \right| \geq c N^2(f, W_2^{1/2}) \\ &\equiv c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{f(y) - f(x)}{y - x} \right]^2 dy dx \geq c \cdot \sum_{v=0}^{n-1} \sum_{\mu=v+1}^n \int_{v-\frac{1}{3}}^{v+\frac{1}{3}} \int_{\mu-\frac{1}{3}}^{\mu+\frac{1}{3}} \left[ \frac{f(y) - f(x)}{y - x} \right]^2 dy dx \\ &\geq \frac{c}{9} \sum_{v=0}^{n-1} \sum_{\mu=v+1}^n \frac{\left( \sum_{k=v+1}^{\mu} a_k \right)^2}{(v - \mu)^2}. \end{aligned}$$

3) *Proof of Theorem D*. For a given  $n > 10$  let

$$F_n(t) = \sum_{k=[n/4]}^{[3n/4]} \Phi_k^2(t).$$

From the evident relations  $\|F_n(t)\|_{L^1} \geq n/3$  and  $\|F_n(t)\|_{L^\infty} \leq M \cdot n$ , it follows that  $\text{mes } E_n \equiv \text{mes}\{t: F_n(t) > c'n\} > c > 0$ . Applying (7) to the set of numbers  $\{\varphi_k(x)\varphi_k(t)\}_1^n$  and integrating with respect to  $x$  with  $t \in E_n$ , we obtain

$$\begin{aligned} \sum_{m=1}^n L_m(t) &= \int_0^1 \sum_{m=1}^n \left| \sum_{k=1}^m \varphi_k(x)\varphi_k(t) \right| dx \\ &\geq c' \int_0^1 \sum_{\nu=0}^{n-1} \sum_{\mu=\nu+1}^n |\nu - \mu|^{-2} \left[ \sum_{k=\nu+1}^{\mu} \varphi_k(x)\varphi_k(t) \right]^2 dx \\ &= \sum_{k=1}^n \varphi_k^2(t) \cdot \sum_{\nu, \mu: 0 \leq \nu < k \leq \mu \leq n} |\nu - \mu|^{-2} \geq c \ln n \sum_{k=\lfloor \frac{n}{4} \rfloor}^{\lfloor \frac{3n}{4} \rfloor} \varphi_k^2(t) \geq c' \ln n \cdot n, \end{aligned}$$

as required.

In conclusion, we show that results like Theorem C can be obtained from (8). We shall prove the following result of Bočkarëv:

Let  $\{\varphi_k(x)\}_1^\infty$  be a complete and collectively bounded O.N.S.; then

$$Q = \sum_{k=1}^\infty \int_0^1 \left| \int_0^t \varphi_k(x) dx \right| dt = \infty.$$

In fact, applying (8) to  $f_k(t)$ , where  $f_k(t) = \int_0^t \varphi_k(x) dx$  for  $0 \leq t \leq 1$ ;  $f_k(t) = 0$  for  $t < 0$ ; and  $f_k(1+t) = f_k(1-t)$ , and using Parseval's equation, we have

$$\begin{aligned} Q &= \frac{1}{2} \sum_{k=1}^\infty \|f_k(t)\|_{L^1(\mathbb{R}^1)} \geq c' \sum_{k=1}^\infty N^2(f_k, W_2^{1/2}) \\ &\geq c \sum_{k=1}^\infty \int_0^1 \int_z^1 (y-z)^{-2} \left( \int_z^y \varphi_k(x) dx \right)^2 dy dz = c \int_0^1 \int_z^1 \frac{1}{y-z} dy dz = \infty. \end{aligned}$$

Similarly we can derive the following theorem from inequalities closely related to (8):

**THEOREM 1.** Let  $\{\varphi_k(x)\}_1^\infty$  be a complete and collectively bounded O.N.S. For every function  $\Phi(t) \in C^2(0, \infty)$  such that  $\Phi(0) = \Phi'(0) = 0$ ,  $\Phi(t) > 0$  for  $t > 0$ ,  $\Phi'(z) \cdot z^{-1} \downarrow$  on  $(0, 1)$ , and  $\int_0^1 t^{-1} \Phi'(t) dt = \infty$ , we have the equation

$$\sum_{k=1}^\infty \int_0^1 \Phi \left( \left| \int_0^t \varphi_k(x) dx \right| \right) dt = \infty.$$

We also note a corollary of the inequality  $N^2(f, W_2^{1/2}) \leq C \|f\|_{L^2} \cdot \|f'\|_{L^2}$ .

**THEOREM 2.** For every complete O.N.S.  $\{\varphi_k(x)\}_1^\infty$  we have

$$\sum_{k=1}^\infty \left\| \int_0^t \varphi_k(x) dx \right\|_{L^2(0,1)} = \infty.$$

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Translated by R. P. BOAS