DOCTORAL DISSERTATION

GENERAL ORTHONORMAL SYSTEMS AND CERTAIN PROBLEMS OF APPROXIMATION THEORY

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This is the author-review of a dissertation for the competition of the academic degree of the Doctor of Physicomathematical Sciences. The dissertation was defended on Dec. 8, 1977, in a meeting of the specialized academic council D.002.38.03 for the V. A. Steklov Mathematics Institute, Academy of Sciences of the USSR. The official opponents were: V. Ya. Kozlov, Corresponding Member of the Academy of Sciences of the USSR and Doctor of Physicomathematical Sciences; A. A. Talalyan, Corresponding Member of the Academy of Sciences of the Armenian SSR and Doctor of Physicomathematical Sciences; and Professor V. M. Tikhomirov, Doctor of Physicomathematical Sciences. The bibliography consists of 48 titles.

This dissertation is devoted to that part of theory of functions whose fundamental notion is the notion of orthogonality. In recent years the theory of orthogonal series has developed rapidly; one of the reasons for this is that a series of classical theorems about properties of the trigonometric system turn out to be valid for more general classes of orthonormal systems.

First of all, we will consider general orthonormal systems, because in many cases they are substantially better than the trigonometric and other classical systems.

The dissertation consists of an introduction and three chapters, each of which is divided into several sections.

In Chaps. II and III mainly almost everywhere (a.e.) convergence of orthogonal series is studied.

In Chap. I many problems of approximation theory are solved. Moreover, certain properties of orthonormal systems are essentially used.

All the chapters of this dissertation are interrelated; in particular, certain results of Chap. I are, roughly speaking, finite-dimensional analogs of theorems obtained in Chaps. II and III.

Let us describe the results obtained in the dissertation.

In Chap. I the diameters of classes of smooth functions and certain finite-dimensional sets are studied. As a preliminary, let us give some definitions.

Definition 1. Let X be a Banach space and K be a compactum in X. Then the quantity

$$d_n(K, X) = \inf \sup_{x \in K} \min_{y \in L_n} \| x - y \|_X,$$

where the infimum is taken over all possible n-dimensional planes $L_n \subset X$, is called the Kolmogorov n-diameter of the set K in the space X.

Further, if $\Phi = \{\varphi_i\}$ is a sequence of elements of X, then we set

$$E_n^{\Phi}(K, X) = \sup_{x \in K} \min_{\{\alpha_i\}_{i=1}^n \in R^n} \left\| x - \sum_{i=1}^n \alpha_i \varphi_i \right\|_X.$$

We denote the space R^n, equipped with the norm

$$\| x \|_{l_p^n} = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} & \text{for} \quad 1 \leqslant p < \infty, \\ \max\limits_{1 \leqslant i \leqslant n} |x_i| & \text{for} \quad p = \infty, \end{cases}$$

by l_p^n and the unit ball in l_p^n by B_p^n.

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Finally, let us recall that W_p^r ($r \geqslant 1$ is an integer) is the class of smooth functions that are defined on the segment [0, 1], whose (r − 1)-th derivatives are absolute continuous and for which

$$\| f(x) \|_{L^p} + \| f^{(r)}(x) \|_{L^p} \leqslant 1,$$

and $\tilde{W}_p^r$ is the analogous class of periodic functions.

Many authors have studied diameters of the classes W_p^r and $\tilde{W}_p^r$ in the Banach spaces $L^q(0, 1)$, $1 \leqslant q \leqslant \infty$. See [1-3] for a detailed survey of their results.

In Chap. I a number of final theorems on the behavior of the diameters $d_n(W_p^r, L^q)$ and $d_n(B_p^m, l_q^m)$ are proved. By finality we mean the exact determination of the order of diameters.*

Chapter I consists of three sections. The main theorems of the chapter — Theorems I.1 and I.2 — are contained in Sec. 1. In Sec. 2 two geometrical results are proved. By these results, partly of a model character, we can judge as to what type of problems arise in the estimation of the diameters $d_n(B_p^m, l_q^m)$. Finally, in Sec. 3 the estimates, obtained in Sec. 1 for $d_n(B_p^m, l_q^m)$ in the particular case p = 1, q = ∞, are sharpened.

The first estimates of the diameters $d_n(W_p^r, L^q)$ were obtained in 1936 by Kolmogorov [4] (for p = q = 2). In the 1950s Rudin [5] and Stechkin [6] determined the orders (as n → ∞) of the diameters $d_n(W_1^r, L^2)$ and $d_n(W_\infty^r, C)$. In 1960 Tikhomirov [7] for the first time calculated the exact value of the diameters $d_n(W_\infty^r, C)$, and then the orders of the diameters $d_n(W_p^r, L^q)$ for arbitrary $p \geqslant q$, and also for $1 \leqslant p < q \leqslant 2$, were determined in [2, 8-10]. Moreover, it was clarified that if $T = \{1, \sin 2\pi nx, \cos 2\pi nx\}_{n=1}^{\infty}$ is the trigonometric system, then

$$d_n(W_p^r, L^q) \asymp E_n^T(\tilde{W}_p^r, L^q) \asymp \begin{cases} n^{-r}, & \text{if} \quad p \geqslant q, \\ n^{-r-1/q+1/p}, & \text{if} \quad 1 \leqslant p < q \leqslant 2. \end{cases} \tag{1}$$

In the case p < q, q > 2 essentially little was known about the diameter $d_n(W_p^r, L^q)$. Certain authors assumed that in this case $d_n(W_p^r, L^q) \asymp E_n^T(\tilde{W}_p^r, L^q)$. However, Ismagilov [2] has now shown that for p = 1, q = ∞, and r = 2

$$d_n(W_1^2, C) \asymp d_n(\tilde{W}_1^2, C) < Cn^{-6/5} \ln n < C' n^{-1/5} \ln n E_n^T(\tilde{W}_1^2, C).$$

It follows from this result that the trigonometric system is not a good device for the approximation of functions of the class W_1^2 in the metric of C(0, 1).

The orders of the diameters $d_n(W_p^r, L^q)$ were not known for q > max (2, p), p > 1 (see below for the case p = 1). Moreover, the known estimates for $d_n(W_p^r, L^q)$ were not sharp even in the power scale.

The author has proved the following theorem.

<u>THEOREM I.1.</u> Let $1 < p < q \leqslant \infty$ and q > 2. Then the following relations are valid:

$$d_n(W_p^r, L^q) \asymp \begin{cases} n^{-r}, & \text{if} \quad p > 2, \\ n^{-r-1/2+1/p}, & \text{if} \quad p \leqslant 2. \end{cases}$$

Theorem I.1 completes the solution of the problem of the determination of the orders of the diameters $d_n(W_p^r, L^q)$, $r > 1$.

It is shown in the dissertation that the estimates for the diameter $d_n(W_p^r, L^q)$, obtained in Theorem I.1, are valid also for the classes W_p of a fractional order r > 0 such that rp > 1, $1 \leqslant p < q \leqslant \infty$, and q > 2.

The following corollary is also valid.

<u>COROLLARY I.2.</u> For arbitrary numbers p, q, r with q > max(p, 2), p > 1, there exists an orthonormal system $\Phi = \Phi_{p,r,q} = \{\varphi_i(x)\}_{i=1}^{\infty}$, $x \in [0, 1]$, $i = 1, 2, \ldots$, such that

$$d_n(W_p^r, L^q) \asymp E_n^{\Phi}(W_p^r, L^q) \quad (n \to \infty).$$

By virtue of what we have said earlier, the orthonormal system Φ, figuring in Corollary I.2, is substantially better (from the point of view of approximation of the classes $\tilde{W}_p^r$), than the trigonometric system.

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*In this connection we observe that by virtue of the relation $d_n(W_p^r, L^q) \asymp d_n(\tilde{W}_p^r, L^q)$ it is sufficient to determine the orders only for $d_n(W_p^r, L^q)$.

In [11], and also in [12], sufficiently precise methods to reduce the problem of the estimation of the diameters $d_n(W_p^r, L^q)$ to the problem of the estimation of the diameters $d_n(B_p^m, l_q^m)$ are suggested. This reduction of the problem to the finite-dimensional problem and then application of the author's results (see Theorem I.6) and Ismagilov's results [2] on the behavior of the diameters $d_n(B_1^m, l_\infty^m)$ enabled Gluskin and Maiorov (see [11, 13, 14]) to determine the orders of the diameters $d_n(W_1^r, L^q)$, $r>1$.

However, for p > 1 no good estimates for $d_n(B_p^m, l_q^m)$ $(q>\max(p,2))$ were known. Therefore, it was not possible to estimate the diameters $d_n(W_p^r, L^q)$ sufficiently accurately.

The study of behavior of the diameters $d_n(B_p^m, l_q^m)$ is of independent geometrical interest. Therefore a number of articles have been devoted to this problem (see [1] for details).

The diameter $d_n(B_p^m, l_q^m)$ has been determined exactly for $p \geqslant q$ (see [1, p. 232]).

In the case $1 \leqslant p < q \leqslant 2$, using the equality

$$d_n(B_1^m, l_2^m) = (1 - n/m)^{1/2},$$

which has been established by Stechkin [6], we easily obtain satisfactory estimates for the diameter $d_n(B_p^m, l_q^m)$.

The matter becomes complicated if the number q > 2. Even in the case m = 2n, the orders of the diameters $d_n(B_p^{2n}, l_q^{2n})$ were unknown for $1 < p < q$, $q > 2$. For example, in the most important case p = 2, q = ∞ we knew (see [2, 15]) only that $C_1 \cdot n^{-1/2} < d_n(B_2^{2n}, l_\infty^{2n}) < C \cdot n^{-1/4}$. The author has proved the following theorem.

THEOREM I.2. The following inequality holds for $m \geqslant n$:

$$d_n(B_2^m, l_\infty^m) \leqslant C \cdot n^{-1/2}(1 + \ln(m/n))^{3/2}.$$

At the end of Sec. 1 of Chap. I the orders of the diameters $d_n(B_p^{2n}, l_q^{2n})$ are determined in all those cases where they were unknown.

The following theorem has been proved.

THEOREM I.3. Let q > max (p, 2), p > 1. Then

$$d_n(B_p^{2n}, l_q^{2n}) \asymp \begin{cases} n^{-1/2+1/q}, & \text{if } p \leqslant 2, \\ n^{-1/p+1/q}, & \text{if } p > 2. \end{cases}$$

Let us observe that it is useful to know the behavior of the diameters $d_n(B_p^{2n}, l_q^{2n})$ for application to other problems of approximation theory.

The following theorem is proved in Sec. 2 of Chap. I.

THEOREM 1.4. For an arbitrary $n \geqslant 1$ there exists an orthogonal transformation T of the space R^n such that

$$C\|x\|_{l_2^n} \leqslant (n^{-1/2}/2)(\|Tx\|_{l_1^n} + \|x\|_{l_1^n}) \leqslant \|x\|_{l_2^n}, \quad x \in R^n.$$

At the end of Sec. 2 an estimate for the dimension of an almost spherical section of the n-dimensional octahedron B_1^n is given. The following theorem has been proved.

THEOREM I.5. For each positive number ε there exists a positive constant c_ε such that for an arbitrary $n \geqslant 1$ there exists a plane $L_{n,\varepsilon}$ in R^n with dim $L_{n,\varepsilon} > n(1-\varepsilon)$ such that

$$c_\varepsilon \|x\|_{l_2^n} \leqslant n^{-1/2}\|x\|_{l_1^n} \leqslant \|x\|_{l_2^n}$$

for arbitrary $x \in L_{n,\varepsilon}$.

Theorem 1.5 shows that the known results of Dvoretzky [16] about the existence of almost spherical sections of a centrally symmetric convex body can be substantially strengthened if this body is the octahedron B_1^n.

A result, similar to Theorem I.5, was obtained independently by Figiel, Lindenstrauss, and Milman [17]. Let us observe that whereas the statement of the problem itself, to be solved in Theorem I.5, is conventional, the formulation of Theorem I.4 is new and interesting.

In Sec. 3 estimates of the diameters of the octahedra B_1^m in the Banach space l_∞^m are given. As observed above, these estimates (obtained till the proof of Theorems I.1 and I.2) have been applied by other authors for the determination of the orders of the diameters

$d_n(W_1^r, L^q)$ and $d_n(H_1^r, L^q)$. It turns out that these estimates for $d_n(B_1^m, l_\infty^m)$ depend very weakly on the number m in the case of octahedra. More precisely, the following theorem is valid.

THEOREM I.6. a) Let $m^\lambda < n < \theta\cdot m$, $\lambda > 0$, and $\theta < 1$. Then

$$0 < c_\theta \cdot n^{-1/2} < d_n(B_1^m, l_\infty^m) < C_\lambda \cdot n^{-1/2};$$

b) For arbitrary m and n such that $m \geqslant n$

$$d_n(B_1^m, l_\infty^m) \leqslant (C/n^{1/2})(1 + \ln(m/n))^{1/2}.$$

Ismagilov's following estimate [2] of the diameter $d_n(B_1^m, l_\infty^m)$ was the best estimate till the author's work:

$$d_n(B_1^m, l_\infty^m) \leqslant (C/n^{1/2})\cdot(m/n)^{1/2}.$$

In Chap. II of the dissertation estimates of the Weyl multiplier are given and certain other problems connected with the a.e. convergence of orthogonal series are considered.

Definition 2. Let $\{\varphi_n(x)\}_{n=1}^\infty$ be an orthonormal system of functions that are defined on the segment [0, 1]. A sequence $\{\lambda_n\}$ $(\lambda_1 \geqslant 0, \lambda_{n+1} \geqslant \lambda_n)$ is called a Weyl multiplier for the a.e. convergence of series with respect to this system if it follows from the condition

$$\sum_{n=1}^\infty c_n^2 \lambda_n < \infty$$

that the series

$$\sum_{n=1}^\infty c_n \varphi_n(x) \tag{2}$$

converges a.e. on the segment [0, 1].

A Weyl multiplier $\{\lambda_n\}$ is called an exact Weyl multiplier if for an arbitrary sequence $\beta_n = o(\lambda_n)$ there exists a series of form (2) that diverges on a set of positive measure, although

$$\sum_{n=1}^\infty c_n^2 \beta_n < \infty.$$

Definition 3. An orthonormal system $\{\varphi_n(x)\}_{n=1}^\infty$ is called a convergence system if the sequence $\lambda_n \equiv 1$, n = 1, 2, . . ., is a Weyl multiplier for a.e. convergence of series with respect to this system.

Weyl has proved that the sequence $\lambda_n = \sqrt{n}$ is a Weyl multiplier for a.e. convergence of series with respect to an arbitrary orthonormal system. Then many authors (see [18] for details) successively strengthened this result. Finally, Men'shov and, independently, Rademacher proved the following theorem in 1921.

THEOREM A (Men'shov, Rademacher). The sequence $\lambda_n = \log^2 n$ is a Weyl multiplier for the a.e. convergence of series with respect to an arbitrary orthonormal system.

At the same time, Men'shov obtained a result showing that this theorem is sharp.

THEOREM B (D. Men'shov). There exists an orthonormal system $\Phi_0 = \{\varphi_n(x)\}$ for which the sequence $\lambda_n = \log^2 n$ is an exact Weyl multiplier.

The problem of obtaining estimates of the Weyl multiplier for some concrete orthonormal system Φ that is sharper than that following from Theorem A is often very complicated. For example, in the case where Φ is the trigonometric system, the final result was obtained only in 1966 by Carleson [19], who proved that the trigonometric system is a convergence system.

Since the functions $\{\varphi_n(x)\}$ in the system Φ_0, constructed by Men'shov, are essentially unbounded, it is possible that for an arbitrary equibounded orthonormal system we can take a Weyl multiplier that grows shower than $\log^2 n$. However, in 1938 Men'shov [20] proved that for each $K > 1$ there exists an orthonormal system $\Phi = \{\varphi_n(x)\}$ such that

1) $|\varphi_n(x)| \leqslant K$, $n = 1, 2, \ldots$, $x \in [0, 1]$;

2) the sequence $\{\log^2 n\}$ is an exact Weyl multiplier for it.

The case $K = 1$, not figuring in Men'shov's theorem, is also interesting. In this case all the functions $\varphi_n(x)$ of the system Φ are such that $|\varphi_n(x)| \equiv 1$, which together with orthogonality imposes a very hard restriction on the system Φ.

As early as 1927 Kolmogorov and Men'shov had proved the following theorem.

THEOREM C. There exists an orthonormal system $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in [0,1[$, such that

1) $|\varphi_n(x)| = 1$, $x \in [0,1]$, $n = 1, 2, \ldots$;

2) no sequence $\{\beta_n\}$ with $\beta_n = o(\log n)$ is a Weyl multiplier for a.e. convergence of series with respect to this system.

In 1969 Tandori [22] obtained the following result in the same direction.

THEOREM D. For an arbitrary positive number ε there exist orthonormal systems $\{\varphi_n(x)\}$ with $|\varphi_n(x)| = 1$ for $x \in [0,1]$, $n = 1, 2, \ldots$, and a series of the form (2) such that

1) $\sum_{n=1}^{\infty} c_n^2 \log n (\log\log n)^{1-\varepsilon} < \infty$;

2) series (2) converges a.e. after a certain change of order of terms.

In Sec. 1 of Chap. II a certain result, strengthening Theorems C and D, is proved.

THEOREM II.1. There exists an orthonormal system $\{\varphi_n(x)\}$ with $|\varphi_n(x)| = 1$ for $x = [0, 1]$, $n = 1, 2, \ldots$, such that the sequence $\lambda_n = \log^2 n$ is an exact Weyl multiplier for the a.e. convergence of series with respect to this system.

Using an intermediate result, obtained in the proof of Theorem II.1, the author has proved the following theorem at the end of Sec. 1.

THEOREM II.2. There exists a continuous function

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

on $[-\pi, \pi]$ such that

1) the modulus of continuity

$$\omega(\delta, f) = O(1/\log(1/\delta));$$

2) $|a_1| > |a_2| > \ldots$;

3) the Fourier series of the function $f(x)$ diverges for $x = 0$.

This result is a strengthening of a theorem of Salem [23] that established an analogous statement without the condition 1). Let us observe that by virtue of the Dini–Lipschitz theorem (see [24, p. 280]) the smoothness of the function $f(x)$, ensured by the condition 1), cannot be augmented.

Weyl multipliers have sense when considered only for infinite orthonormal systems. In Sec. 2 of Chap. II a finite-dimensional analog of the notion of a Weyl multiplier is studied.

Let $\Phi = \Phi(n) = \{\varphi_i(x)\}_{i=1}^{n}$, $x \in [0,1]$, be a certain orthonormal set of functions. Let us define the operator of majorant of partial sums $S_\Phi^*: l_2^n \to L^2(0,1)$ in the following manner: If $y = \{y_i\}_{i=1}^{n} \in l_2^n$, then

$$S_\Phi^*(y) = f(x) = \sup_{1 \leqslant r \leqslant n} \left| \sum_{i=1}^{r} y_i \varphi_i(x) \right|. \tag{3}$$

Let $s(\Phi)$ be the norm of the operator S_Φ^*, i.e.,

$$s(\Phi) = \sup_{\|y\|_{l_2^n} \leqslant 1} \| S_\Phi^*(y) \|_{L^2}.$$

In the study of a.e. convergence of orthogonal series there often arises the problem of estimation of the number $s(\Phi)$ for a given set Φ. For example, the fundamental lemma in the Men'shov–Rademacher theorem (see Theorem A, formulated earlier) states that the inequality

$$s(\Phi) \leqslant C \ln n$$

holds for arbitrary set $\Phi = \Phi(n)$. At the same time, Men'shov gave, during the proof of Theorem B, for each $n \geqslant 1$ an example of an orthonormal set $\Phi_0 = \Phi_0(n)$ such that

$$s(\Phi_0) \geqslant c \ln n. \tag{4}$$

In Sec. 2 of Chap. II it is elucidated as to what is the "mean value" of the norm $s(\Phi)$. In order to give a precise meaning to the notion "mean value," we consider a set of orthonormal systems $Q^n = \{\Phi\}$ of the following form: If $\Phi \in Q^n$, then $\Phi = \{\varphi_i(x)\}_{i=1}^{n}$ and, moreover, each function $\varphi_i(x)$ is constant on the intervals $((j-1)/n, j/n)$, $1 \leqslant j \leqslant n$.

There exists a natural one-to-one correspondence between the systems $\Phi \in Q^n$ and the elements of the group O^n of orthogonal matrices of order n; namely: a matrix $A = \{a_{ij}\} \in O^n$ of the following form is put in correspondence with a system $\Phi = \{\varphi_i\}$:

$$a_{ij} = n^{-1/2}\varphi_i((j - 1/2)/n), \quad 1 \leqslant i, j \leqslant n.$$

With the help of this correspondence the Haar measure μ_n, defined on the group O^n, is carried over to the set Q^n.

The following theorem is proved.

THEOREM II.3. There exist absolute constants C and $\gamma > 0$ such that for arbitrary $n \geqslant 1$ and $t \geqslant 0$

$$\mu_n\{\Phi \in Q^n: s(\Phi) \geqslant t\} \leqslant (C \cdot e^{-\gamma(t)})^n.$$

The following corollaries follow from Theorem II.3.

COROLLARY II.1. There exists a constant B such that

$$\mu_n\{\Phi \in Q^n: s(\Phi) \geqslant B\} < e^{-n}$$

for n = 1, 2, ...

Corollary II.1 shows that the example of the system $\Phi_0(n)$, indicated by Men'shov (see (4)), is a rare exception from the general rule (let us observe that the system $\Phi_0(n)$ with property (4) can be chosen to lie in Q^n).

COROLLARY II.2. Let S^n be the set of all permutations of the set of the numbers 1, 2, . . ., n, $\sigma \in S^n$, and Φ_σ be the set of functions Φ ($\Phi \in Q^n$) arranged in the order σ. Then

$$\mu_n\{\Phi \in Q^n: \max_{\sigma \in S^n} s(\Phi_\sigma) > C \ln^{1/2} n\} < n^{-n}.$$

The following corollary follows from Theorem I.4 and Corollary II.1.

COROLLARY II.3. There exists a set of systems $E_n \subset Q^n$ with $\mu_n(E_n) < 2^{-n}$ such that the following inequalities are valid for each system $\Phi = \{\varphi_i\} \in Q^n$, that does not belong to E_n:

1) $s(\Phi) < B$;

2) for an arbitrary set $\{a_i\}_{i=1}^n$ such that $\sum_{i=1}^n a_i^2 = 1$,

$$\max_{1 \leqslant r \leqslant n} m\left\{x \in [0, 1]: \left|\sum_{i=1}^r a_i\varphi_i(x)\right| > c_1\right\} > c_2,$$

where B, c_1, and c_2 are positive absolute constants.

Corollary II.3 shows that "majority" of the systems Φ from Q^n have the properties that were discovered only for "lacunary" systems (e.g., systems of independent functions).

In Sec. 3 at the end of Chap. II the connection between a.e. summability and a.e. convergence of orthogonal series is considered.

Ul'yanov has posed the following problem: Does there exist a complete orthonormal system $\{\varphi_n(x)\}$, in $L^2(0, 1)$ that is equibounded (i.e., $|\varphi_n(x)| < M$, $x \in [0, 1]$, $n = 1, 2, \ldots$) and is such that series (2) is a.e. convergent if and only if it is summable by the method of arithmetic means. (See [25] for more details about the reasons for the formulation of this problem.) The following theorem, proved in Sec. 3 of Chap. II, gives an affirmative solution of this problem.

THEOREM II.4. There exists a permutation $\{u_{k_n}(x)\}_{n=1}^\infty$ of the Walsh orthonormal system $\{u_k(x)\}$, such that the series (2) with respect to the system $\varphi_n(x) = u_{k_n}(x)$, $n = 1, 2, \ldots$, is a.e. convergent if and only if it is a.e. summable by the method of arithmetic means.

The last, third, chapter of the dissertation consists of two sections. Section 2 of Chap. III is devoted to the problems of representation of measurable functions by a.e. convergent series. An extensive theory of representation of functions by series has been created to the present time. Side by side with the problems of representation of functions the problems of uniqueness of the representation are always considered.

The results of the author relate to that part of the theory of representation of functions by series which was started in 1916 by D. Men'shov by proving the following theorem.

THEOREM E (D. Men'shov). There exists a trigonometric series

$$\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

for which $a_{n_0} \neq 0$ for a certain n_0 and which converges a.e. on the segment $[0, 2\pi]$ to the function $f(x) \equiv 0$.

Let us recall that the problem of existence of such series was posed by Luzin.

Definition 4. Let a system of measurable functions $\{f_n(x)\}_{n=1}^{\infty}$, $x \in [0, 1]$. A series $\sum_{n=1}^{\infty} a_n f_n(x)$ is called a zero series in the sense of a.e. convergence if it converges a.e. on [0, 1] to $f(x) \equiv 0$ and $a_{n_0} \neq 0$ for a certain n_0.

Theorem E asserts, in other words, that there exists a zero series in the sense of a.e. convergence with respect to the trigonometric system.

As in Definition 4 we define zero series in the sense of convergence in measure; moreover, it is clear that each zero series in the sense of a.e. convergence is a zero series in the sense of convergence in measure.

The problem of existence of a zero series is naturally carried over from the trigonometric system to an arbitrary complete orthonormal system. This problem in such a general formulation was first considered, probably, by Marcinkiewicz (see [26, p. 312]), who obtained the following result in 1937.

THEOREM F (Marcinkiewicz). For an arbitrary complete orthonormal system $\{\varphi_n(x)\}_{n=1}^{\infty}$ there exists a series $\sum_{n=1}^{\infty} a_n\varphi_n(x)$, $a_{n_0} \neq 0$, $1 \leqslant n_0 < \infty$, for which a certain sequence of partial sums $S_{m_\nu}(x)$, $\nu = 1, 2, \ldots$, converges a.e. on the segment [0, 1] to $f(x) \equiv 0$.

In 1956 Talalyan [27] strengthened this result.

THEOREM G (A. A. Talalyan). With respect to an arbitrary complete orthonormal system there exists a zero series in the sense of convergence in measure.

The example of Men'shov's zero series was carried over to the Haar and the Walsh systems (see [28, 29]). Later on many authors found conditions on a complete orthonormal system $\{\varphi_n(x)\}$ under which zero series in the sense of a.e. convergence with respect to this system exist. Till now the existence of zero series in the sense of a.e. convergence for a wide class of complete orthonormal systems has been proved (see [30-32] for details). However, the problem of the existence of a zero series with respect to an arbitrary complete orthonormal system has remained open.

The following theorem, proved in Sec. 2, is the main result of Chap. III.

THEOREM III.3. There exists a complete (in $L^2(0, 1)$) orthonormal system $\{\psi_n(x)\}_{n=1}^{\infty}$ such that the series

$$\sum_{n=1}^{\infty} a_n\psi_n(x) \tag{5}$$

converges a.e. if and only if

$$\sum_{n=1}^{\infty} a_n^2 < \infty. \tag{6}$$

The following corollary follows immediately from Theorem II.3.

COROLLARY III.1. There exists a complete orthornormal system $\{\psi_n(x)\}$ such that there exists no zero series in the sense of a.e. convergence with respect to it. Consequently, the representation of an arbitrary function by an a.e. convergent series with respect to this system is unique.

Let us observe that the problem of existence of systems, mentioned in Theorem III.3, was first posed by Ul'yanov [33].

Definition 5. Let a class $\mathcal{F}$ of functions that are defined on the segment [0, 1] and a system $\{f_n(x)\}_{n=1}^{\infty}$ be given. The system $\{f_n(x)\}_{n=1}^{\infty}$ is called a representation system in the sense of a.e. convergence for the class $\mathcal{F}$ if for each function $g(x) \in \mathcal{F}$ there exists a series $\sum_{n=1}^{\infty} a_n f_n(x)$ that converges to g(x) a.e. on [0, 1].

Talalyan [27, 34] has proved that each complete orthonormal system $\{\varphi_n(x)\}$ is a representation system in the sense of convergence in measure for the class of all measurable (not necessarily a.e. finite) functions.

In 1965 Talalyan and Arutyunyan [35] discovered that no series with respect to the Haar (and Walsh) system can converge to infinity on a set of positive measure. Hence it follows readily that the Haar system is not a representation system for the class of all measurable functions. However, Bari (see [34]) proved still earlier that each a.e. finite measurable function can be represented by an a.e. convergent series with respect to the Haar system. An analogous statement is valid for the Walsh system also. It was not known for a sufficiently long time (see, e.g., [34]) whether each complete orthonormal system is a representation system in the sense of a.e. convergence for the class of all a.e. finite measurable functions. It follows from Theorem II.3 that this question has a negative answer. Moreover, the following theorem holds.

THEOREM III.4. There exist a continuous function $f(x)$, $x \in [0, 1]$, and a complete orthonormal system $\{\psi_n(x)\}_{n=1}^{\infty}$ such that there does not exist any series of the form (5) that converges a.e. to f(x).

In the first section of Chap. III the properties of orthogonal convergence systems are studied. The results of Sec. 1 are essentially used in the proof of Theorem III.3.

If $\Phi = \{\varphi_n(x)\}$ is an orthonormal convergence system, then the operator of majorant of partial sums, $S_\Phi^*: l^2 \to L^0(0, 1)$, acting in the space $L^0(0, 1)$ of all a.e. finite measurable functions is defined by an equation similar to Eq. (3).

Namely, if $a = \{a_n\}_{n=1}^{\infty} \in l^2$, then

$$S_\Phi^*(a) = f(x) = \sup_{1 \leqslant r < \infty} \left| \sum_{n=1}^{r} a_n \varphi_n(x) \right|.$$

Olevskii [36] has shown that there exist a convergence system $\Phi_1 = \{\varphi_n(x)\}$ and an element $a \in l^2$ such that

$$S_{\Phi_1}^*(a) \notin \bigcup_{p>0} L^p(0, 1).$$

Nikishin ([37]; see also [38]) has proved the following theorem.

THEOREM H (E. M. Nikishin). Let $\Phi = \{\varphi_n(x)\}$ be an orthonormal convergence system. Then for each $\varepsilon > 0$ there exists a set $E_\varepsilon \subset [0, 1]$, $mE_\varepsilon > 1 - \varepsilon$, and a constant C_ε such that

$$m\{x \in E_\varepsilon: S_\Phi^*(a) \geqslant y\} \leqslant C_\varepsilon (\|a\|_{l^2}/y)^2 \tag{7}$$

for arbitrary $a \in l^2$.

It follows immediately from this theorem that for each convergence system Φ the operator S_Φ^* is a bounded operator from l^2 into the space $L^p(E_\varepsilon)$ ($E_\varepsilon \subset [0, 1]$, $mE_\varepsilon > 1 - \varepsilon$), for arbitrary $p < 2$.

The problem of the possibility of replacing the "weak-type" inequality (7) by a stronger "strong-type" inequality is important for applications (see [37, p. 134]).

It is proved in Sec. 1 that Theorem H canno- be strengthened.

THEOREM III.1. There exists an orthonormal convergence system $\Phi_2 = \{\varphi_n(x)\}_{n=1}^{\infty}$ such that for arbitrary set E of position measure $mE > 0$

$$S_\Phi^*(a) \notin L^2(E)$$

for a certain element $a = \{a_n\} \in l^2$ that depends on E.

The following theorem is proved at the end of Sec. 1 of Chap. III.

THEOREM III.2. Let L be an arbitrary subspace of $L^2(0, 1)$. Then there exists an orthonormal basis $\{\varphi_n(x)\}$ in L such that $\{\varphi_n(x)\}$ is a convergence system.

Theorem III.2 enables us to avoid special constructions for proving certain theorems about the a.e. convergence of orthogonal series.

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