7. L. I. Kamynin and B. N. Khimchenko, "On the maximum principle for second-order parabolic equations," Dokl. Akad. Nauk SSSR, 200, No. 2, 282-285 (1971).
8. L. I. Kamynin and B. N. Khimchenko, "On the maximum principle for a second-order elliptic-parabolic equation," Sib. Mat. Zh., 13, No. 4, 773-789 (1972).
9. L. I. Kamynin and B. N. Khimchenko, "On the applications of the maximum principle to second-order parabolic equations," Dokl. Akad. Nauk SSSR, 204, No. 3, 529-532 (1972).
10. L. I. Kamynin and B. N. Khimchenko, "Analogs of the Giraud theorem for a second-order parabolic equation," Sib. Mat. Zh., 14, No. 1, 86-110 (1973).
11. L. I. Kamynin and B. N. Khimchenko, "On theorems of Giraud type for a second-order elliptic operator which becomes weakly degenerate near the boundary," Dokl. Akad. Nauk SSSR, 224, No. 4, 752-755 (1975).
12. O. A. Oleinik and E. V. Radkevich, "Second-order equations with nonnegative characteristic form," Mat. Analiz, No. 1, Izd-vo VINITI, Moscow (1971).
13. A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall (1964).

COEFFICIENTS IN THE EXPANSION OF A CERTAIN CLASS OF FUNCTIONS WITH RESPECT TO COMPLETE SYSTEMS

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For each integer N > 1 we define on the segment [0, 1] a collection of continuous functions $\{\chi_j(x)\}_{j=1}^N$ in the following manner:

$$\chi_j^N(x)=\begin{cases}0, & x\leqslant (j-1)/N,\\ 1, & x\geqslant j/N,\\ \text{linear on} & [(j-1)/N,\ j/N],\end{cases}$$

j = 1, 2, . . . , N.

In [1] Bochkarev proved the following:

THEOREM A. Let $\{\psi_k(x)\}_{k=1}^\infty$ be an orthonormal, complete system of functions (ONCS) in $L_2[0, 1]$ which is uniformly bounded, i.e., $|\psi_k(x)| \le M$ for $x \in [0, 1]$ and k = 1, 2, ... Then if $\chi_j^N(x) \overset{L_2}{=} \sum_{k=1}^\infty a_{k,j}\psi_k(x)$ and $S_j^N=\sum_{k=1}^\infty |a_{kj}|$, $j=1,\ldots,N$, then

$$\frac{1}{N}\left(\sum_{j=1}^N S_j^N\right)\geqslant c\ln N \tag{1}$$

for all $N = 2^r$, $1 < r < \infty$, where $c > 0$ is some absolute constant.

In the case where $\{\psi_k(x)\}$ is the trigonometric system, the validity of inequality (1) and its accuracy can be verified directly.

The proof of Bochkarev is based on certain inequalities for the Haar system.

The purpose of the present work is to obtain inequalities which are not as strong as (1) but under weaker requirements on the system $\{\psi_k(x)\}$. We moreover prove an inequality from which the estimate (1) is obtained as a corollary.

Inequality (1) and also the inequalities in Theorem 1 of this paper remain valid (as will be evident from the proof) if the system $\{\chi_j^N(x)\}$ is replaced by a system of functions which is similar to it in a certain sense

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(cf. Remark 1). Inequalities for the coefficients in the expansion of the functions defined above with respect to complete systems make it possible to obtain rather precise results in a number of problems of function theory. In particular, the following results can be derived in a rather simple way from an inequality analogous to inequality (1) for the Fourier coefficients of the functions $\{\chi_j^{N,m}(x)\}$.

For any ONCS $\{\psi_k(x)\}_{k=1}^{\infty}$, $|\psi_k(x)| \leq M$ for $k = 1, 2, \ldots, x \in [0, 1]$, there exists a function of bounded variation with modulus of continuity $\omega(\delta, f) = o(\log^{-2}\delta^{-1})$ such that the series of its Fourier coefficients with respect to the system $\{\psi_k(x)\}$ does not converge absolutely.

This result was first obtained by Bochkarev in [2], where the history of the problem is given as well as its complete solution in the class of ONC systems.

Properties of the Hilbert matrix play a basic role in this work. For each N we define a collection of piecewise constant (with constancy interval $A_i = ((i-1)/N, i/N)$) functions as follows:

$$f_j^N(x) = \begin{cases} 1/(i-j) & \text{for} \quad x \in A_i \text{ and } \quad i \neq j, \\ 0 & \text{for} \quad x \in A_i \text{ and } \quad i = j,\ 1 \leqslant i, j \leqslant N. \end{cases} \tag{2}$$

In the sequel we shall drop the index indicating dependence on N.

The following property of the Hilbert matrix is well known (cf. [3, Theorem 314]): for $1 < p < \infty$, $1/q = 1 - 1/p$

$$\left|\sum_{j=1}^{N}\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{a_j b_i}{i-j}\right| \leqslant C_p^* \left(\sum_{j=1}^{N} |a_j|^p\right)^{1/p} \left(\sum_{i=1}^{N} |b_i|^q\right)^{1/q}.$$

Therefore, for any $p > 1$ and any collection of numbers $\{a_j\}_{j=1}^{N}$ we have

$$\left\|\sum_{j=1}^{N} a_j f_j^N(x)\right\|_{L_p} = \frac{1}{N^{1/p}} \left(\sum_{i=1}^{N} \left|\sum_{j=1}^{N} a_j f_j^N\left(\frac{i-1/2}{N}\right)\right|^p\right)^{1/p} = \frac{1}{N^{1/p}} \sup_{\sum_{i=1}^{N} |b_i|^q = 1} \sum_{i=1}^{N} b_i \left(\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{a_j}{i-j}\right) \leqslant C_p \left(\frac{\sum_{j=1}^{N} |a_j|^p}{N}\right)^{1/p}. \tag{3}$$

In order to obtain the corollary of the theorem proved below, we will need well-known properties of the system of Rademacher functions. Namely, suppose there is given a polynomial in the Rademacher system $P(t) = \sum_{k=1}^{N} c_k r_k(t)$. It is known (cf. [4]) that there exist absolute constants $C_1 > 0$ and $\alpha > 0$ such that the measure

$$\mu\left\{t : \left|\sum_{k=1}^{N} c_k r_k(t)\right| \geqslant C_1 \sqrt{\sum_{k=1}^{N} c_k^2}\right\} > \alpha. \tag{4}$$

Further, for any $y > 0$ the following inequality holds (cf. [5, p. 73]):

$$\mu\left\{t : \left|\sum_{k=1}^{N} c_k r_k(t)\right| \geqslant y \left(\sum_{k=1}^{N} c_k^2\right)^{1/2}\right\} \leqslant 2e^{-y^2/4}. \tag{5}$$

It follows immediately from (5) that there exists a constant C_2 greater than zero such that

$$\mu\left\{t : \sup_{1\leqslant r<r'\leqslant N} \left\{\left|\sum_{k=r}^{r'} r_k(t)\right| \Big/ (r'-r)^{1/2} \log^{1/2} N\right\} < C_2\right\} \geqslant 1 - \alpha/10, \tag{6}$$

where the constant α in formula (6) is the same as in (4).

We shall also need the following simple fact.

Suppose there is given a collection of measurable functions $\{a_k(t)\}_{k=1}^{R}$ such that

1) $a_k(t) \geqslant 0$ for $t \in [0, 1]$, 2) $\mu\{t : t \in [0, 1],\ a_k(t) \geqslant \beta_k\} \geqslant \alpha$,

where the β_k are nonnegative numbers and $\alpha > 0$. Then

$$\mu\left\{t : \sum_{k=1}^{R} a_k(t) \geqslant \left(\sum_{k=1}^{R} \beta_k\right) \frac{\alpha}{2}\right\} \geqslant \alpha/2. \tag{7}$$

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*The C_p denote constants depending only on the parameters indicated and are different in different formulas.

Indeed, let

$$\bar{a}_k(t) = \begin{cases} 0, & \text{if} \quad a_k(t) < \beta_k, \\ \beta_k, & \text{if} \quad a_k(t) \geq \beta_k. \end{cases}$$

We consider the set $E = \left\{t : t \in [0, 1] \text{ and } \sum_{k=1}^{R} \bar{a}_k(t) \geq \left(\sum_{k=1}^{R} \beta_k\right) \frac{\alpha}{2}\right\}$.

Since $\bar{a}_k(t) \leq a_k(t)$, it suffices to prove that $\mu E \geq \alpha/2$. Since $\mu\{t : \bar{a}_k(t) \geq \beta_k\} \geq \alpha$ for $k = 1, 2, \ldots, R_1$, we have

$$\alpha\left(\sum_{k=1}^{R} \beta_k\right) \leqslant \int_0^1 \sum_{k=1}^{R} \bar{a}_k(t)\,dt = \int_E \sum_{k=1}^{R} a_k(t)\,dt + \int_{[0,1]\setminus E} \sum_{k=1}^{R} \bar{a}_k(t)\,dt \leqslant \mu E\left(\sum_{k=1}^{R} \beta_k\right) + \frac{\alpha}{2}\left(\sum_{k=1}^{R} \beta_k\right).$$

From the last inequality it follows that $\mu E \geq \alpha - \alpha/2 \geq \alpha/2$. The estimate (7) has been proved.

We have the following:

THEOREM 1. Let $1 < p < \infty$, $q^{-1} + p^{-1} = 1$, and suppose that $\{\psi_k(x)\}$ is some system of functions satisfying the conditions: $\psi_k(x) \in L_p(0, 1)$ and $\|\psi_k(x)\|_{L_p} = 1$ for all $k \geq 1$. Suppose further that for some M and $N > 1$ the functions $\chi_j^N(x) = \sum_{k=1}^{M} a_{k,j}\psi_k(x) + \Delta_j^N(x)$, where $\|\Delta_j^N(x)\|_{L_1} \leq 1/N$ for $j = 1, 2, \ldots, N$. Then

$$\sum_{k=1}^{M}\left(\frac{1}{N} \cdot \sum_{j=1}^{N} |a_{k,j}|^q\right)^{1/q} \geqslant C_q \ln N.$$

Proof. For small N the theorem is true, because

$$\sum_{k=1}^{M}\left(\frac{1}{N} \cdot \sum_{j=1}^{N} |a_{k,j}|^q\right)^{1/q} \geqslant \sum_{k=1}^{M} \sum_{j=1}^{N} \frac{|a_{k,j}|}{N},$$

where the last sum is less than C_q by the hypotheses of the theorem. For $\ln N > 20$ we have

$$\int_0^1 \sum_{j=1}^{N} \chi_j^N(x) f_j^N(x)\,dx = \sum_{j=1}^{N} \int_0^1 \chi_j^N(x) f_j^N(x)\,dx \geqslant \sum_{j=1}^{N-1} \frac{1}{N} \cdot \sum_{i=j+1}^{N} \frac{1}{i-j} = \sum_{j=1}^{N-1} \frac{1}{N}\left(\sum_{r=1}^{N-j} \frac{1}{r}\right) \geqslant \frac{\ln N}{10}.$$

But

$$\sum_{j=1}^{N} \chi_j^N(x) f_j(x) = \sum_{j=1}^{N} f_j(x)\left[\sum_{k=1}^{M} a_{k,j}\psi_k(x) + \Delta_j(x)\right],$$

and $|f_j^N(x)| \leq 1$, $j = 1, 2, \ldots, N$; therefore, $\sum_{j=1}^{N} \int_0^1 f_j^N(x)\Delta_j^N(x)\,dx \leqslant \sum_{j=1}^{N} \|\Delta_j^N(x)\|_{L_1} \leqslant 1$. Thus

$$\frac{\ln N}{20} \leqslant \frac{\ln N}{10} - 1 \leqslant \sum_{j=1}^{N} \int_0^1 f_j^N(x)\left(\sum_{k=1}^{M} a_{k,j}\psi_k(x)\right)dx \leqslant \sum_{k=1}^{M} \int_0^1 \psi_k(x)$$
$$\times \left(\sum_{j=1}^{N} a_{k,j} f_j(x)\right)dx \leqslant \sum_{k=1}^{M}\left(\int_0^1 |\psi_k(x)|^p\,dx\right)^{1/p}\left(\int_0^1 \left|\sum_{j=1}^{N} a_{kj} f_j\right|^q dx\right)^{1/q}.$$

By (3) and the normalization of the system $\{\psi_k(x)\}$, the last sum does not exceed $C_q\left(\sum_{k=1}^{M}\left(\frac{1}{N}\sum_{j=1}^{N} |a_{kj}|^q\right)^{1/q}\right)$ which proves the theorem.

From Theorem 1 the following corollary immediately follows:

COROLLARY 1. If $\{\psi_k(x)\}_{k=1}^{\infty}$ is a normalized basis in the space $L_p(0, 1)$ for some $1 < p < \infty$, then $\sum_{k=1}^{\infty}\left(\frac{1}{N}\sum_{j=1}^{N} |a_{kj}|^q\right)^{1/q} \geqslant C_q \ln N$ for all $N > 1$, where the a_{kj} are the coefficients in the expansion of χ_j^N with respect to the basis $\{\psi_k(x)\}$.

Remark 1. For $N > 1$, $1 < m < N/2$, we define on [0, 1] the sequence of functions $\{\chi_j^{N,m}(x)\}_{j=1}^{N}$ as follows: let $\alpha_j^{N,m} = N^{-1} \min(N-1, j+m)$; then

$$\chi_j^{N,m}(x)=\begin{cases}1 & \text{if} \quad x\in[j/N,\alpha_j^{N,m}],\\ 0 & \text{if} \quad x\leqslant (j-1)/N \quad \text{or} \quad x\geqslant \alpha_j^{N,m}+1/N,\\ \text{linear for} & x\in[(j-1)/N,j/N] \text{ и } x\in[\alpha_j,\alpha_j+1/N].\end{cases}$$

The following inequality is proved in the same way as Theorem 1: let $\{\psi_k(x)\}$ be a basis in the space L_p, $\|\psi_k\|_{L_p}=1$, $k=1,2,\ldots,1<p<\infty$,

$$\chi_j^{m,N}(x)=\sum_{k=1}^{\infty} a_{kj}^{m,N}\psi_k(x), \quad j=1,2,\ldots,N,$$

then

$$\sum_{k=1}^{\infty}\left(\frac{1}{N}\sum_{j=1}^{N}|a_{kj}^{m,N}|^q\right)^{1/q}\geqslant C_q\ln m. \tag{7'}$$

We will show that Corollary 1 can be applied.

COROLLARY 2. Let $\{\psi_k(x)\}$ be a basis in the space $L_p(0,1)$ ($p\geq 2$) with $\|\psi_k(x)\|_{L_p}=1$, $k=1,2,\ldots$. Then there exists a function $f(x)\in \mathrm{Lip}\,1/2$ such that $f(x)\overset{L_p}{=}\sum_{k=1}^{\infty}a_k\psi_k(x)$ and $\sum_{k=1}^{\infty}|a_k|=\infty$.

Proof. We set

$$g_t^N(x)=\frac{1}{\sqrt{N}}\left(\sum_{j=1}^{N}r_j(t)\chi_j^N(x)\right),$$

where the $r_j(t)$ are the Rademacher functions.

Let

$$g_t^N(x)\overset{L_p}{=}\sum_{k=1}^{\infty}a_k^t\psi_k(x).$$

It is clear that

$$a_k^t=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}r_j(t)a_{k,j}.$$

It follows from the definition of the functions $\chi_j^N(x)$ that for $|x-y|\leq 2/N$ we have

$$|g_t^N(x)-g_t^N(y)|\leqslant 2|x-y|\sqrt{N}\leqslant 2\sqrt{2}|x-y|^{1/2}.$$

Further, by the piecewise linearity of the functions $g_t^N(x)$

$$\sup_{\substack{|x-y|>2/N\\ x,y\in[0,1]}}\left|\frac{g_t^N(x)-g_t^N(y)}{|x-y|^{1/2}}\right|\leqslant 3\sup_{1\leqslant r<r'\leqslant N}\left|\frac{g_t^N\left(\frac{r}{N}\right)-g_t^N\left(\frac{r'}{N}\right)}{\left(\frac{r'}{N}-\frac{r}{N}\right)^{1/2}}\right|.$$

But

$$\left|g_t^N\left(\frac{r'}{N}\right)-g_t^N\left(\frac{r}{N}\right)\right|=\left|\sum_{j=r}^{r'}r_j(t)\right|\cdot\frac{1}{\sqrt{N}},$$

and it remains only to use the estimate (6) to see that

$$\mu\left\{t:\sup_{x,y\in[0,1]}\left|\frac{g_t^N(x)-g_t^N(y)}{|x-y|^{1/2}}\right|\leqslant C_3\log^{1/2}N\right\}\geqslant 1-\alpha/10. \tag{8}$$

Further, let

$$\beta_k=\left(\frac{1}{N}\sum_{j=1}^{N}a_{kj}^2\right)^{1/2}.$$

It then follows immediately from (4) that

$$\mu\{t:|a_k^t|\geqslant C_1\beta_k\}>\alpha \quad \text{for all} \quad k\geqslant 1.$$

From the last inequality and estimate (7) we find that

$$\mu\left\{t: \sum_{k=1}^{\infty} |a_k^t| \geqslant C_1 \frac{\alpha}{2}\left(\sum_{k=1}^{\infty} \beta_k\right)\right\} \geqslant \alpha/2.$$

Since $p \geq 2$, it follows that $q = 1/(1 - 1/p) \leq 2$, and therefore

$$\sum_{k=1}^{\infty} \beta_k \geqslant \sum_{k=1}^{\infty}\left(\frac{1}{N}\sum_{j=1}^{N} |a_{k,j}|^q\right)^{1/q}.$$

By Corollary 1 the last sum is not less than $C_q \ln N$, and hence

$$\mu\left\{t: \sum_{k=1}^{\infty} |a_k^t| \geqslant C_q \frac{\alpha}{2} \ln N\right\} \geqslant \alpha/2. \tag{8'}$$

From (8) and (8') we find that there exists a point t_0 and a function $F_N(x) = g_{t_0}^N(x) / \ln^{1/2} N$ such that

1) $F_N(x) \overset{L_p}{=} \sum_{k=1}^{\infty} a_k \psi_k(x)$ and $\sum_{k=1}^{\infty} |a_k| \geqslant A_q \log^{1/2} N$;

2) $|F_N(x) - F_N(y)| \leq B|x - y|^{1/2}$ for $x, y \in [0, 1]$ ($A_q > 0$, and B is an absolute constant). From the fact that the number N may be chosen arbitrarily large in relations 1 and 2 the assertion of Corollary 2 follows immediately.

Remark 2. In the assertion of Corollary 2 an additional condition can be imposed on the function $f(x)$: $f(0) = f(1) = 0$.

The assertion of Corollary 2 was established by S. N. Bernstein in the case where $\{\psi_k(x)\}$ is the trigonometric system and by B. I. Golubov in the case where $\{\psi_k(x)\}$ is the Haar system; it was then extended by Mityagin [6] to arbitrary complete orthonormal systems in $L_2(0, 1)$.

Remark 3. By arguments similar to those used in the proof of Corollary 2 [using inequality (7') and the estimate (7)] it is possible to show that if $\psi_k(x)$ is a normalized basis of the space $L_p(0, 1)$, $1 < p < 2$, then there exists a function $f(x) \in \mathrm{Lip}\, 1/q$ ($q^{-1} + p^{-1} = 1$) such that

$$f(x) \overset{L_p}{=} \sum_{k=1}^{\infty} a_k \psi_k(x), \quad f(0) = f(1) = 0, \quad \sum_{k=1}^{\infty} |a_k| = \infty.$$

The index $1/q$ cannot be improved as is shown by the example

$$\psi_k(x) = \frac{1}{\|\chi_k\|_{L_p}} \cdot \chi_k(x), \quad k = 1, 2, \ldots,$$

where $\{\chi_k(x)\}_{k=1}^{\infty}$ is the Haar system.

It is easy to see that the inequality of Bochkarev [cf. (1)] is the limiting case of Theorem 1 (when $p \to \infty$). However, the proof in the limiting case is more complicated [since we cannot apply the estimate (3)], and the important feature is that it is necessary to impose an additional condition on the system $\{\psi_k(x)\}$, for example, to require the orthogonality of the system $\{\psi_k(x)\}$. Here we shall prove a result somewhat stronger than the inequality (1) (cf. Corollary 3). It is easily deduced from the assertion proved below.

Suppose that $\{\psi_k(x)\}_{k=1}^{\infty}$ is an ONCS and $|\psi_k(x)| \leq M$ for $k = 1, 2, \ldots$. Suppose, as before, that $\chi_j^N(x) = \sum_{k=1}^{\infty} a_{kj} \psi_k(x)$, $S_j \equiv S_j^N = \sum_{k=1}^{\infty} |a_{kj}|$; for each $\alpha > 0$ we set

$$S_\alpha^j = \sum_{\substack{k \\ \alpha/2 < |a_{k,j}| \leqslant \alpha}} |a_{k,j}|; \quad j = 1, 2, \ldots, N.$$

It is clear that $S_\alpha^j = 0$ if $\alpha \geq 2M$. We have the following result.

There exists an absolute constant $C_0 > 0$ such that for any $m = 2, 4, 8, \ldots, 2^{[\log_2 N - 3]}$ and $\alpha_m = mM/N$ we have

$$\frac{1}{N}\left(\sum_{r=0}^{\infty}\sum_{j=1}^{N} \frac{S_{\alpha_m \cdot 2^r}^j}{2^r} + \sum_{r=0}^{2[\log N]+1}\sum_{j=1}^{N} \sqrt{\frac{S_{\alpha_m \cdot 2^{-r}}^j}{2^r}}\right) + \left(\frac{1}{N^2}\sum_{j=1}^{N}\sqrt{S_j}\right) \geqslant C_0. \tag{9}$$

Proof. (We remark that the arguments presented here have common points with the proof of a weak-type inequality for the Hilbert transform, cf. [7].)

Let m be fixed. We consider the system of piecewise constant functions $\{f_j^m(x)\}_{j=1}^N$ with constancy interval $1/N$ which is defined as follows:

$$f_j^m\left(\frac{r-1/2}{N}\right)=\begin{cases}-1/m & \text{if} \quad m\leqslant r-j<2m,\ 2m<r<N-2m,\\ 1/m & \text{if} \quad m\leqslant j-r<2m,\ 2m<r<N-2m,\\ 0 & \text{for other} \quad j \text{ and } r,\end{cases}$$
$$r,\ j=1,\ 2,\ \ldots,\ N.$$

It follows from the definition of the system $\{f_j^m(x)\}_{j=1}^N$ that

a) the support of the function $f_j^m(x)$ lies on the segment

$$[(j-2m)/N,\ (j+2m)/N];$$

$$\text{b)} \quad \sum_{j=1}^{N} f_j^m(x)\equiv 0; \tag{10}$$

$$\text{c)} \quad \|f_j^m(x)\|_{L_1}\leqslant 2/N; \quad \|f_j^m(x)\|_{L_2}\leqslant 2\sqrt{1/mN}.$$

We shall further require the inequality

$$|a_{k,j}-a_{k,j+1}|\leqslant M/N, \tag{11}$$

which follows from the inequality

$$|a_{k,j}-a_{k,j+1}|=\left|\int_0^1 \psi_k(x)(\chi_j(x)-\chi_{j+1}(x))\,dx\right|\leqslant M\int_{\frac{j-1}{N}}^{\frac{j}{N}} 1\,dx\leqslant\frac{M}{N}.$$

It is easy to see that for any of the admissible values of m

$$\frac{1}{20}\leqslant\int_0^1\sum_{j=1}^{N} f_j^m\chi_j(x)\,dx=\int_0^1\sum_{j=1}^{N} f_j^m(x)\left(\sum_{k=1}^{\infty}a_{k,j}\psi_k(x)\right)dx=\sum_{k=1}^{\infty}\sum_{j=1}^{N}\int_0^1 a_{k,j}f_j^m(x)\psi_k(x)\,dx=R_1^m+R_2^m,$$

where

$$R_1^m=\sum_{k=1}^{\infty}\int_0^1\psi_k(x)\left(\sum_{\substack{j\\ |a_{k,j}|>\alpha_m}} a_{k,j}f_j^m(x)\right)dx,$$

$$R_2^m=\sum_{j=1}^{N}\left(\sum_{\substack{k\\ |a_{k,j}|\leqslant\alpha_m}} a_{k,j}\int_0^1\psi_k(x)f_j^m(x)\,dx\right).$$

We shall estimate the sum R_1^m. Let $F_k^m(x)=\sum_{\substack{j\in[1,N]\\ |a_{k,j}|>\alpha_m}} a_{kj}f_j^m(x)$. Then, by virtue of the fact that $|\psi_k(x)|\leq M$, we have

$$R_1^m\leqslant M\sum_{k=1}^{\infty}\int_0^1|F_k^m(x)|\,dx. \tag{12}$$

We now estimate each term of the sum (12) separately:

$$\int_0^1|F_k^m(x)|\,dx=\int_{E_1^k}|F_k^m(x)|\,dx+\int_{E_2^k}|F_k^m(x)|\,dx,$$

where

$$\begin{cases}E_1^k=\{x:(r-1)/N<x<r/N, \quad \max\limits_{j\in[r-2m,r+2m]}|a_{kj}|>8\alpha\ \},\\ E_2^k=[0,1]/E_1^k.\end{cases} \tag{13}$$

Further, since $x_0=(r-1/2)/N$ and $|j-r|>2m$ we have

$$f_j^m(x_0)=0, \tag{14}$$

and hence

$$\int_{E_2^k}|F_k^m(x)|\,dx\leqslant\int_0^1\sum_{\alpha_m\leqslant|a_{kj}|\leqslant 8\alpha_m}|a_{kj}|\cdot|f_j^m(x)|\,dx\leqslant\sum_{\alpha_m\leqslant|a_{kj}|\leqslant 8\alpha_m}|a_{kj}|\int_0^1|f_j^m(x)|\,dx\leqslant\frac{2}{N}\sum_{\substack{k\\ \alpha_m\leqslant|a_{kj}|\leqslant 8\alpha_m}}|a_{kj}|. \tag{15}$$

If the point $x_0 = (r - 1/2)/N \in E_1^k$, then it follows from (13) and (11) that

$$\min_{j \in [r-2m,\, r+2m]} |a_{kj}| \geqslant 2mM/N \geqslant 2\alpha_m. \tag{16}$$

Using (14) and (16), we obtain

$$F_k^m\left(\frac{r-1/2}{N}\right) = \sum_{j=1}^{N} a_{kj} f_j^m(x_0) = \sum_{j=r-2m}^{r+2m} a_{kj} f_j^m(x_0) = \sum_{j=r-2m}^{r+2m} (a_{kj} - a_{k,r-2m}) f_j^m(x_0) + a_{k,r-2m} \sum_{j=r-2m}^{r+2m} f_j^m(x_0).$$

Because of (14) and (10) of b) above, the last sum is equal to zero. Further, by (11) for $|j - (r-2m)| \leq 4m$ we have the inequality $|a_{kj} - a_{k,r-2m}| \leq 4\alpha m$. Moreover, $|f_j^m(x_0)| \leq 1/m$, $j = 1, 2, \ldots, N$. Therefore, for $x_0 = (r-1/2)/N \in E_1^k$ the estimate $|F_k^m(x_0)| \leqslant 4\alpha_m \cdot \frac{1}{m}\left(\sum_{j \in [r-2m, r+2m]} 1\right)$ holds and by (16)

$$|F_k^m(x_0)| \leqslant \frac{4\alpha_m}{m}\left(\sum_{\substack{j \in [r-2m, r+2m] \\ |a_{kj}| \geqslant 2\alpha_m}} 1\right). \tag{17}$$

It follows immediately from inequality (17) that

$$\int_{E_1} |F_k^m(x)|\, dx \leqslant \frac{16\alpha_m}{N}\left(\sum_{|a_{k,j}| \geqslant 2\alpha_m} 1\right).$$

Combining the last estimate with (15), we find that $\int_0^1 |F_k^m(x)|\, dx \leqslant \frac{20\alpha_m}{N}\left(\sum_{|a_{kj}| \geqslant \alpha_m} 1\right)$ and hence [cf. (12)]

$$R_1^m \leqslant \frac{20M}{N}\left(\sum_{k=1}^{\infty} \sum_{\substack{j=1 \\ |a_{kj}| \geqslant \alpha_m}}^{N} \alpha_m\right) \leqslant \frac{40M}{N}\left(\sum_{r=0}^{\infty} \sum_{j=1}^{N} \frac{S^j_{\alpha_m \cdot 2^r}}{2^r}\right). \tag{18}$$

We now estimate the sum R_2^m; it can be estimated in the rough way. We decompose the sum R_2^m into two parts

$$R_2^m \leqslant \sum_{j=1}^{N} \sum_{r=0}^{2[\log N]+1} \sum_{\substack{k \\ \alpha_m \cdot 2^{-r-1} < |a_{kj}| \leqslant \alpha_m \cdot 2^{-r}}} |a_{kj}| \left|\int_0^1 \psi_k(x) f_j^m(x)\, dx\right| + \sum_{j=1}^{N} \sum_{\substack{k \\ |a_{kj}| \leqslant \alpha_m/N^2}} |a_{kj}| \left|\int_0^1 \psi_k f_j^m(x)\, dx\right|.$$

In the first term for fixed j and r we apply the Cauchy inequality and use the orthogonality of the functions $\psi_k(x)$:

$$A_{jr} \equiv \sum_{\substack{k \\ \alpha_m \cdot 2^{-r-1} \leqslant |a_{kj}| \leqslant \alpha_m \cdot 2^{-r}}} |a_{kj}| \cdot \left|\int_0^1 \psi_k(x) f_j^m(x)\, dx\right|$$

$$\leqslant \left(\sum_{\substack{k \\ \alpha_m \cdot 2^{-r-1} < |a_{kj}| \leqslant \alpha_m \cdot 2^{-r}}} a_{kj}^2\right)^{1/2} \left(\sum_{k=1}^{\infty}\left(\int_0^1 \psi_k(x) f_j^m(x)\, dx\right)^2\right)^{1/2} \leqslant \left(\alpha_m \cdot 2^{-r} S^j_{\alpha_m \cdot 2^{-r}}\right)^{1/2} \cdot \|f_j^m(x)\|_{L_2};$$

recalling the definition of the number α_m and (10) of c) above, we obtain

$$A_{jr} \leqslant C\left(S^j_{\alpha_m \cdot 2^{-r}}\right)^{1/2} / N\sqrt{2^r}. \tag{19}$$

Similarly,

$$\sum_{j=1}^{N} \sum_{\substack{k \\ |a_{kj}| \leqslant \alpha_m/N^2}} |a_{kj}| \left|\int_0^1 \psi_k(x) f_j^m(x)\, dx\right| \leqslant \sum_{j=1}^{N}\left(\sum_{\substack{k \\ |a_{kj}| < mM/N^3}} a_{kj}^2\right)^{1/2} \|f_j^m(x)\|_{L_2}$$

$$\leqslant \sum_{j=1}^{N} C\left(S_j \frac{mM}{N^3} \cdot \frac{1}{mN}\right)^{1/2} \leqslant \frac{C}{N^2}\left(\sum_{j=1}^{N} \sqrt{S_j}\right).$$

Summing inequality (19) on j and r, we find

$$R_2^m \leqslant \frac{C}{N}\left(\sum_{j=1}^{N}\sum_{r=0}^{2[\log N]+1}\sqrt{\frac{S^j_{\alpha_m\cdot 2^{-r}}}{2^r}}\right)+\frac{C}{N^2}\left(\sum_{j=1}^{N}\sqrt{S_j}\right).$$

Combining the estimates obtained from R_2^m and R_1^m [cf. (18)], and recalling that $R_1^m + R_2^m \geq 1/20$, we obtain the desired estimate (9). The assertion has been proved.

If inequality (9) is summed for $m = 2, 4, \ldots, 2^{[\log N]-3}$ then we find that for $\alpha = [M/N]\cdot 2^{[\log N]} \geq M/2$

$$\frac{1}{N}\left(\sum_{j=1}^{N}\sum_{r=0}^{3[\log N]} S^j_{2^{-r}\alpha}+\sqrt{S^j_{2^{-r}\alpha}}\right)+\left(\sum_{j=1}^{N}\sqrt{S_j}\right)\frac{3\log N}{N^2}\geqslant C\log N. \tag{20}$$

It may be assumed that $C < 1$.

Inequality (1) follows easily from inequality (20). Indeed, we will show that the inequality

$$\frac{1}{N}\sum_{j=1}^{N}S_j\leqslant\frac{C^2}{27}\log N \tag{21}$$

contradicts inequality (20). We assume that the estimate (21) holds. Then also

$$\frac{1}{N}\left(\sum_{j=1}^{N}\sum_{r=0}^{3[\log N]} S^j_{2^{-r}\alpha}\right)\leqslant\frac{C^2}{27}\log N. \tag{22}$$

It is obvious that

$$\frac{1}{N}\left(\sum_{j=1}^{N}\sum_{r=0}^{3[\log N]}\frac{C}{8}\right)\leqslant\frac{C}{8}(3\log N+1),$$

and since $\sqrt{S} \leq S\cdot(8/C) + C/8$, it follows that

$$\sum_{j=1}^{N}\sum_{r=0}^{3[\log N]}\sqrt{S^j_{2^{-r}\alpha}}\leqslant\frac{8}{27}C\log N+\frac{3}{8}C\log N+\frac{3}{8}C. \tag{23}$$

If inequalities (22) and (23) are added and it is noted that (21) implies that $\frac{\log N}{N^2}\left(\sum_{j=1}^{N}\sqrt{S_j}\right)=O(1)$, then we obtain an inequality contradicting the estimate (20). Hence

$$\frac{1}{N}\sum_{j=1}^{N}S_j\geqslant\frac{C^2}{27}\log N.$$

The inequality (9) is deduced in a similar way and is stronger than inequality (1).

COROLLARY 3. Suppose that for some $p \geq 1$ and $N \geq 2$

$$\frac{1}{N}\left(\sum_{j=1}^{N}S_j\right)\leqslant(\log N)^p. \tag{24}$$

Then there exist positive constants A and B depending only on p and not on N such that for $1 < Y < \tilde{Y} = Y\cdot(\log N)^B < N$ we have

$$\frac{1}{N}\left(\sum_{\substack{k,j\\ 1/\tilde{Y}\leqslant|a_{kj}|\leqslant 1/Y}}|a_{kj}|\right)\geqslant A\log\log N.$$

Corollary 3 shows that under the condition (24) the coefficients of the functions $\chi_j^N(x)$ with respect to any orthonormal bounded system behave in the average over j much like those with respect to the trigonometric system.

LITERATURE CITED

1. S. V. Bochkarev, "On the absolute convergence of Fourier series with respect to bounded systems," Mat. Zametki, 15, No. 3, 363-370 (1974).

2. S. V. Bochkarev, "On the absolute convergence of Fourier series with respect to bounded, complete, orthonormal systems of functions," Mat. Sb., 93, No. 2, 203-217 (1974).
3. G. H. Hardy, D. Littlewood, and G. Polya, Inequalities, Cambridge Univ. Press (1952).
4. A. Zygmund, Trigonometric Series, Cambridge Univ. Press (1968).
5. V. V. Petrov, Sums of Independent Random Variables [in Russian], Nauka, Moscow (1972).
6. B. S. Mityagin, "On the absolute convergence of series of Fourier coefficients," Dokl. Akad. Nauk SSSR, 157, No. 5, 1047-1050 (1964).
7. E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press (1971).

SPHERICAL QUADRATURE FORMULAS EXACT TO ORDERS 25-29

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§ 1

References [1-3] set forth a method for obtaining Gauss-type spherical quadrature formulas that are invariant under G_8^*, the octahedral group with inversion, and that are exact for all polynomials of degree n or less. Maximal precision of these quadratures is obtained by special choice both of nodes and of weights. By use of results of Sobolev [4], by passage to even symmetric polynomials which generate algebras of polynomials invariant relative to G_8^*, and by use of certain changes of variables (which involve first distinguishing polynomials invariant relative to G_8^* which vanish on axes and on planes of symmetry of the group G_8^*, and then passing to elementary symmetric functions in the values of symmetric polynomials of each type), significant reduction has been achieved in the algebraic order of the nonlinear system of equations which defines the weights and nodes of quadrature; cases are indicated in [3] where this system of equations can be broken into subsystems.

However, despite these simplifications, the matter of obtaining Gauss-type quadratures for large values of n involves a number of difficult problems, owing to the following circumstances:

a) As n grows, both the algebraic order of the system of nonlinear equations and the number of its unknowns grow very rapidly.

b) For some values of n the system of equations does not break up into subsystems, for example, for n = 21, 31, 33 [2, 3].

c) Even when the system does break up into subsystems, one of these subsystems will be a so-called moment system in two variables, and computational methods for solution of such systems of high order are as yet not thoroughly worked out.

We shall deem the problem of finding parameters for a Gauss-type quadrature comparatively simple to solve if the solution algorithm reduces after some changes of variables to application (possibly several times) of the following algorithms:

a) solution of a system of linear equations; and

b) finding the roots of an algebraic equation in one unknown. Note that finding a solution of the moment systems in one variable

$$\sum_{i=1}^{p} B_i u_i^k = c_k, \quad k = 0, 1, \ldots, 2p-1, \tag{1.1}$$

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