ON A PROPERTY OF FUNCTIONAL SERIES

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The question of the convergence of functional series everywhere in the segment [0, 1] is considered. Let $F = \{f\}$ be the set of such functions in [0, 1] for each of which there is a transposition of the series $\sum_{k=1}^{\infty} f_k(x)$, which converges to it everywhere in [0, 1]. An example of a series is constructed such that the set F consists just of an identical zero, but $\sum_{k=1}^{\infty} |f_k(x_0)| = \infty$ $(x_0 \in [0,1])$ for any point of the segment [0, 1].

Let be given the number series

$$\sum_{n=1}^{\infty} a_n. \tag{1}$$

Let A denote the set of all those numbers a for each of which there is a transposition $\tau \equiv \{n_k\} = (n_1, n_2, \ldots, n_k, \ldots)$ of the natural number series such that

$$a = \sum_{k=1}^{\infty} a_{n_k} = \sum_{\tau} a_k.$$

It is well known that if the set A consists of one point, then the series (1) is absolutely convergent. This results directly from the Riemann theorem. This fact also holds for series of the form (1) with elements a_n from N-space [Steinitz-Levy theorem (see [1, 2])].

An analogous question is investigated herein for the case when $a_n = a_n(x)$ are continuous functions in the segment [0, 1]. More accurately, let the functions $f_n(x) \in C(0, 1)$ and $F = \{f(x)\}$ be a set of all those functions $f(x)$ defined in [0, 1] for each of which there is a transposition $\tau = \{n_k\}$ such that the series

$$\sum_{k=1}^{\infty} f_{n_k}(x) = \sum_{\tau} f_n(x) \tag{2}$$

converges at each point $x \in [0, 1]$ to $f(x)$.

Could it be asserted that if the set $F = \{f(x)\}$ consists of just one function then the series (2) will converge absolutely to some points of the segment [0, 1]? The answer to this question is given by the following

THEOREM. There exists a series

$$\sum_{n=1}^{\infty} f_n(x) \quad (x \in [0, 1]; \; f_n(x) \in C(0, 1); \quad n = 1, 2, \ldots) \tag{3}$$

such that the set F consists of one function $f(x) \equiv 0$ and at the same time

$$\sum_{n=1}^{\infty} |f_n(x)| = \infty \qquad \text{for all} \qquad x \in [0, 1]. \tag{4}$$

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Proof. Let A be the set of all transpositions $\tau = \{n_k\}$ of the natural number series such that

$$n_k \leqslant 27k^3 \quad \text{for} \quad k = 1, 2, \ldots . \tag{5}$$

Let us map the set A in the segment [0, 1] according to the rule

$$x = \varphi(\tau) = 0,\ \underbrace{00\ldots0}_{n_1}\ \underbrace{11\ldots1}_{n_2}\ \underbrace{00\ldots0}_{n_3}\ \underbrace{11\ldots1}_{n_4}\ldots,$$

where the decimal fraction written out yields the number $x \in [0, 1]$. Let $B = \varphi(A)$ and $\overline{B}$ be the closure of the set B. It is easy to see that if $x \in \overline{B}$, then

$$x = 0,\ \underbrace{00\ldots0}_{p_1}\ \underbrace{11\ldots1}_{p_2}\ldots, \tag{6}$$

where $p_i \leq 27i^3$ $(i = 1, 2, \ldots)$ and $p_j \neq p_k$ for $j \neq k$. Let us note that the sequence $\{p_i\}$ is not generally a transposition of the natural number series since some natural numbers may be missing in the set of numbers p_i $(i = 1, 2, \ldots)$.

Let us define the sequence of functions $\{\varphi_n(x)\}_{n=1}^{\infty}$ in the set $\overline{B}$. Let $x \in \overline{B}$; then x is represented as the decimal fraction (6). Let us put

$$\varphi_{2j-1}(x) = \frac{1}{\sqrt[3]{p_j}}, \quad \varphi_{2j}(x) = -\frac{1}{\sqrt[3]{p_j}} \qquad (j = 1, 2, \ldots). \tag{7}$$

It is clear that the function $\varphi_n(x)$ $(n = 1, 2, \ldots)$ retains its sign in the set $\overline{B}$. Moreover, the functions $\varphi_n(x)$ are continuous in the set $\overline{B}$ for any $n = 1, 2, \ldots$. This results from the fact that if j is a fixed integer, and $x_0 \in \overline{B}$, then there is a $\delta > 0$ such that the functions $\varphi_{2j-1}(x)$ and $\varphi_{2j}(x)$ are constant in the set $\overline{B} \cap (x_0 - \delta, x_0 + \delta)$.

Let us note first that for any point $x \in \overline{B}$ we will have [see (5)-(7)]

$$\varphi_{2j-1}(x) = \frac{1}{\sqrt[3]{p_j}} \geqslant \frac{1}{3\sqrt[3]{j^3}} = \frac{1}{3j}, \qquad (j = 1, 2, \ldots),$$

and hence

$$\sum_{n=1}^{\infty} |\varphi_n(x)| = \infty \quad \text{for} \quad x \in \overline{B}. \tag{8}$$

Furthermore, $\sum_{n=1}^{2k} \varphi_n(x) = 0$ for $x \in \overline{B}$ and $k = 1, 2, \ldots$, but $\lim_{n \to \infty} \varphi_n(x) = 0$ for $x \in \overline{B}$ [see (7)]. Hence, the series

$$\sum_{n=1}^{\infty} \varphi_n(x) \tag{9}$$

converges in the set $\overline{B}$ to the function $\varphi(x) \equiv 0$. Let us note that besides the functions $\varphi_n(x)$, the function $\varphi_n(x)$ which is unique for each φ_n is also a member of the series (9).

We now prove that if the series (9) converges after some transposition $\tau' = \{m_i\}$ to some function $\Phi(x)$ everywhere in $\overline{B}$, then $\Phi(x) = \varphi(x) \equiv 0$ for $x \in \overline{B}$, i.e.,

$$\Phi(x) = \sum_{i=1}^{\infty} \varphi_{m_i}(x) = \sum_{\tau'} \varphi_n(x) = 0. \tag{10}$$

Let us take an arbitrary natural number N and let us call the function $\varphi_j(x)$ contracted into the section from 1 to N in the transposition τ', if among the functions $\{\varphi_{m_i}(x)\}_{i=1}^{N}$ there is both the function $\varphi_j(x)$ and the function $\varphi_j(x)$. Let $K(\tau', N)$ denote the number of functions not reduced in the transposition τ' in the section from 1 to N.

Let us consider the series

$$\sum_{i=1}^{\infty} \varphi_{m_i}(x_0) = \sum_{\tau'} \varphi_n(x_0) \tag{11}$$

at an arbitrary point $x_0 \in \overline{B}$. Let us call the number $a_j = \varphi_j(x_0)$ reduced in the section from 1 to N in the cross τ' if among the numbers $\{a_{m_i}\}_{i=1}^N$ there are both the numbers a_j and $-a_j$. Let $K(x_0, \tau', N)$ denote the number of terms in the series (11) not reduced in the section from 1 to N.

Let us note that when the point $x_0 \in \overline{B}$ [see (6)], then $p_i \neq p_j$ for $i \neq j$, and hence, if $\varphi_n(x_0) = \varphi_m(x_0)$, then $n = m$. Consequently, for any $x_0 \in \overline{B}$, τ', and N the function $K(x_0, \tau', N) = K(\tau', N)$, and the functions $\varphi_{m_i}(x)$ not being reduced on the section from 1 to N in the series (10), are at the same places as the terms $\{a_{m_i}\}$ of the series (11) not being reduced in the same section.

For a given transposition $\tau' = \{m_i\}$ [see (10)] two cases are possible:

$$\text{I. } \overline{\lim_{N\to\infty}} K(\tau', N) < \infty. \quad \text{II. } \overline{\lim_{N\to\infty}} K(\tau', N) = \infty.$$

Members of the series (11) tend to zero, and moreover, for each member $\varphi_{m_i}(x_0)$ of the series (11) there is a member $\varphi_{m_i}(x_0) = -\varphi_{m_i}(x_0)$ of the same series. Hence, in case I, i.e., when $K(\tau', N) \leq M = \text{const}$ for all $N = 1, 2, \ldots$, in the partial sum

$$\sigma_N(x_0) = \sum_{i=1}^{N} \varphi_{m_i}(x_0)$$

the terms not being reduced ($\leq M$ in quantity) have an arbitrarily large number as $N \to \infty$. This means that $\sigma_N(x_0) \to 0$ as $N \to \infty$. Therefore, the series (11) converges to zero at each point $x_0 \in \overline{B}$, i.e., $\Phi(x) \equiv \varphi(x)$ and (10) is valid in case I.

Now, let us analyze case II. We prove that it is impossible. For this case we construct the divergent series

$$\sum_{i=1}^{\infty} s_i, \tag{12}$$

which is a transposition of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n}}\right) \tag{13}$$

and for which $s_i = \varphi_{m_i}(x_1)$ $(i = 1, 2, \ldots)$ at some point $x_1 \in B$. This fact will contradict the convergence of the series (10) to the function $\Phi(x)$ in the set $\overline{B}$. Let us select two partial series from the series (13)

$$T_1 \equiv \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt[3]{2k}} - \frac{1}{\sqrt[3]{2k}}\right), \quad T_2 \equiv \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right).$$

The remaining terms in (13) form some partial series which we denote by

$$T_3 \equiv \sum_{i=1}^{\infty} \left(\frac{1}{\sqrt[3]{r_i}} - \frac{1}{\sqrt[3]{r_i}}\right). \tag{14}$$

It is clear that $r_i \leq 3i$ $(i \geq 1)$. We construct the series (12) by induction.

First Step in the Induction. Let us select a natural number p_1 such that

$$\sum_{k=1}^{p_1} \frac{1}{\sqrt[3]{2k}} > 2\sqrt{p_1}. \tag{15}$$

After this, let us find the natural number N_1 such that

$$K(\tau', N_1) > 2p_1 \tag{16}$$

and the partial sum of the series (10)

$$\sigma_{N_1}(x) = \sum_{i=1}^{N_1} \varphi_{m_i}(x) \tag{17}$$

contains all functions $\{\varphi_n(x)\}_{n=1}^{2p_1}$ (which are naturally reduced). Let $\alpha^+(\tau', N)$ [or $\alpha^-(\tau', N)$] denote the number of positive (or negative) terms from 1 to N not reduced in the series (10). By virtue of (16), either $\alpha_1 = \alpha^+(\tau', N_1) > p_1$, or $\beta_1 = \alpha^-(\tau', N_1) > p_1$. Without limiting the generality (this will be seen from the reasoning below), we can consider that

$$\alpha_1 > p_1. \tag{18}$$

Two cases are possible: a) $\beta_1 < p_1$, b) $\beta_1 \geq p_1$. In case a) we set terms of the series T_1 successively in the first p_1 places of the positive functions not being reduced in (17):

$$\frac{1}{\sqrt[3]{2\cdot 1}},\quad \frac{1}{\sqrt[3]{2\cdot 2}},\quad \ldots,\quad \frac{1}{\sqrt[3]{2\cdot p_1}},$$

i.e., if $\varphi_{m_{i_q}}(x)$ $(i_1 < i_2 < \ldots < i_{\alpha_1})$ are positive functions not being reduced in the partial sum (17), then, [see (18)]

$$s_{m_{i_q}} = \frac{1}{\sqrt[3]{2\cdot q}} \quad \text{for} \quad 1 \leqslant q \leqslant p_1.$$

Analogously, we successively set members of the series T_2 in the β_1 places of the negative functions not reduced in (17):

$$-\frac{1}{2\cdot 1+1},\ -\frac{1}{2\cdot 2+1},\ \ldots,\ -\frac{1}{2\cdot \beta_1+1}.$$

Furthermore, we set terms from the series T_2 successively at the places of the functions $\varphi_{m_{i_q}}(x)$ with $p_1 < q < 2p_1 - \beta_1$:

$$\frac{1}{2(\beta_1+1)+1},\ \ldots,\ \frac{1}{2p_1+1}.$$

Therefore, we have determined s_i for some values of i $(1 \leq i \leq N_1)$ within the quantity $p_1 + \beta_1 + (p_1 - \beta_1) = 2p_1$. We set members of the series T_3 at the places of the functions $\varphi_{m_{i_q}}(x)$ with $2p_1 - \beta_1 < q \leq \alpha_1$ and at the places of all the functions reduced in (16) so that:

1) If there are functions $\varphi_{2j-1}(x)$ and $-\varphi_{2j-1}(x) = \varphi_{2j}(x)$ at any two places in (17), then we put $(1/\sqrt[3]{r_j})$ and $-(1/\sqrt[3]{r_j})$, respectively, at these places;

2) At the place of the functions $\varphi_{m_{i_q}}(x) \equiv \varphi_{2\nu-1}(x)$ with $2p_1 - \beta_1 < q < \alpha_1$, we put $(1/\sqrt[3]{r_\nu})$;

3) If the first member of the series T_3 unused in 1) and 2) has the number $i_0 \leq p_1$, then at the place of the function $\varphi_{m_{\alpha_1}}(x) = \varphi_{2\nu-1}(x)$, we put $(1/\sqrt[3]{r_{i_0}})$. If $i_0 > p_1$, then at the place of $\varphi_{m_{\alpha_1}}(x) = \varphi_{2\nu-1}(x)$, we put $(1/\sqrt[3]{r_\nu})$ [as in 2)].

We have therefore constructed s_i for all $1 \leq i \leq N_1$. The numbers s_i $(1 \leq i \leq N_1)$ have been constructed so that if the number $s_i = (1/\sqrt[3]{l})$ [or $-(1/\sqrt[3]{l})$] has been compared to the functions $\varphi_{m_i}(x) = \varphi_{2h-1}(x)$ or $\varphi_{m_i}(x) = \varphi_{2h}(x)$, then [see (14) and the choice of N_1]

$$l \leqslant 27\, h^3. \tag{19}$$

Let us note that [see (15)]

$$A_1 \equiv \sum_{i=1}^{N_1} s_i \geqslant \sum_{i=1}^{p_1} \frac{1}{\sqrt[3]{2i}} - \sum_{i=1}^{p_1} \frac{1}{2i+1} \equiv R_1 > \sqrt{p_1}. \tag{20}$$

In case b), i.e., when $\beta_1 \geq p_1$, we construct s_i as follows. We put the number $(1/\sqrt[3]{2\cdot q})$ successively for $1 \leq q \leq p_1$ [or $-1/(2q+1)$ for $1 \leq q \leq p_1$] at the first p_1 places of the positive (or negative) functions not reduced in (17).

We put members of the series T_3 at the remaining places in (17) exactly as in sections 1), 2), and 3) case a).

Let $s_i^{(1)}$ denote the numbers thus constructed for $1 \le i \le N_1$. If we alter the construction just so that we put successively the number $1/(2q+1)$ with $1 \le q \le p_1$ (or $-(1/\sqrt[3]{2q})$ with $1 \le q \le p_1$) in the first p_1 places of the positive (or negative) functions not reduced in (17), then we obtain the number $s_i^{(2)}$ with $1 \le i \le N_1$. But then, [see (20)]

$$\left|\sum_{i=1}^{N_1} s_i^{(2)} - \sum_{i=1}^{N_1} s_i^{(1)}\right| = 2R_1,$$

and hence, we have for $\alpha = 1$ (or for $\alpha = 2$)

$$\left|\sum_{i=1}^{N_1} s_i^{(\alpha)}\right| \geqslant R_1. \tag{21}$$

Let us put $s_i = s_i^{(\alpha)}$ for $1 \le i \le N_1$ and that α for which (21) is valid. We have constructed s_i $(1 \le i \le N_1)$ as in both case a) and case b) so that (19) is valid and $|A_1| > \sqrt{p_1}$. The first step in the induction is thereby terminated.

Second Step in the Induction. For integer $n \ge 2$ let us have constructed p_{n-1}, N_{n-1}, and s_i for $1 \le i \le N_{n-1}$. Let us select p_n such that

$$\sum_{k=p_{n-1}+1}^{p_n} \frac{1}{\sqrt[3]{2k}} \geqslant 2\sqrt{p_n},$$

and let us find the natural number N_n such that $K(\tau', N_n) > 2p_n$, and the partial sum

$$\sigma_{N_n}(x) = \sum_{i=1}^{N_n} \varphi_{m_i}(x) \tag{22}$$

contains all functions $\{\varphi_i(x)\}_{i=1}^{2p_n}$ and $\{\varphi_{m_i}(x)\}_{i=1}^{N_{n-1}}$ which have been reduced in (22). We will construct the s_i with $N_{n-1} < i \le N_n$ almost exactly as in the first step of the induction, with the sole exception that if the number s_i $(1 \le i \le N_{n-1})$ has been compared to the function $f_m(x)$ in the partial sum $\sigma_{N_{n-1}}(x)$, then we compare the number $-s_i$ to the function $-f_m(x)$ in the second step.

The sequence $\{s_i\}_{i=1}^{\infty}$ thus constructed satisfies the following conditions:

1°) the sum is $\left|\sum_{i=1}^{N_n} s_i\right| > \sqrt{p_n}$;

2°) if the number $s_i = (1/\sqrt[3]{l})$ has been compared to the function $\varphi_{m_i}(x) \equiv \varphi_{2n-1}(x)$ $(1 \le i < \infty)$, then $l \le 27\, n^3$;

3°) if the number s_i has been compared to the function $f_m(x)$, then the number $s_j = -s_i$ is compared to the function $-f_m(x)$;

4°) the sequence $\{s_i\}_{i=1}^{\infty}$ is some transposition of the sequence $\left\{\frac{1}{\sqrt[3]{n}}, -\frac{1}{\sqrt[3]{m}}\right\}_{n,m=1}^{\infty}$.

Let us show that there exists a point $x_1 \in B$, for which

$$\varphi_{m_i}(x_1) = s_i \qquad (1 \leqslant i < \infty).$$

Indeed, by construction a number $s_{j_k} = (1/\sqrt[3]{n_k})$, i.e.,

$$\varphi_{2k-1}(x) \Rightarrow \frac{1}{\sqrt[3]{n_k}} \quad (1 \leqslant k < \infty),$$

and

$$-\varphi_{2k-1}(x_0) \equiv \varphi_{2k}(x_0) \Rightarrow -\frac{1}{\sqrt[3]{n_k}}, \tag{23}$$

where the sign $\Rightarrow$ denotes comparison, has been compared to each positive function $\varphi_{m_{j_\alpha}}(x) = \varphi_{2k-1}(x)$ from the series (10). Let us take the point

$$x_1 = 0,\ \underbrace{00\ldots0}_{n_1}\ \underbrace{11\ldots1}_{n_2}\ \underbrace{00\ldots0}_{n_3}\ \underbrace{11\ldots1}_{n_4}\ldots\ .$$

By virtue of 2°) and 4°) the point $x_1 \in B$, because $n_k \leq 27\,k^3$ and $\{n_k\}_{k=1}^{\infty}$ are the transposition of a natural series. But then [see (7) and 3°)]

$$\varphi_{m_{j_\alpha}}(x_1) \equiv \varphi_{2k-1}(x_1) = \frac{1}{\sqrt[3]{n_k}} = s_{j_k}.$$

Hence [see 1°, (23), and (24)]

$$\left|\sum_{j=1}^{N_n} \varphi_{m_j}(x)\right| = \left|\sum_{j=1}^{N_n} s_j\right| > \sqrt{p_n} \qquad (n = 1, 2, \ldots),$$

i.e., [see (22)], $\overline{\lim_{n\to\infty}}\,|\sigma_{N_n}(x_1)| = \infty$. It hence follows that the series (10) diverges at the point x_1 and the impossibility of case II is thereby proved. The functions $\overline{B} \subset [0, 1]$ with $0 \leqslant m \equiv \inf_{x\in\overline{B}} x < \sup_{x\in\overline{B}} x = M \leqslant 1$. Let $[m, M] - \overline{B} = \bigcup_{i=1}^{\infty} (a_i, b_i)$, where (a_i, b_i) are adjacent intervals in $\overline{B}$. Let us define the functions $f_k(x) \in C(0, 1)$ follows:

$$f_{2j-1}(x) = \begin{cases} \varphi_{2j-1}(x) & \text{for} \quad x \in \overline{B}, \\ \varphi_{2j-1}(m) & \text{for} \quad x \in [0, m], \\ \varphi_{2j-1}(M) & \text{for} \quad x \in [M, 1], \\ \varphi_{2j-1}(a_i) & \text{for} \quad x \in \left[a_i, a_i + \frac{i-1}{i}(b_i - a_i)\right] \\ & \qquad (1 \leqslant i < \infty), \\ \text{linear in} & \quad \left[a_i + \frac{i-1}{i}(b_i - a_i), b_i\right] \\ & \qquad (1 \leqslant i < \infty), \end{cases}$$

and let

$$f_{2j}(x) \equiv -f_{2j-1}(x) \quad (j = 1, 2, \ldots).$$

The functions $f_n(x)$ are those desired. Indeed, the series (4) diverges at each point $x \in [0, 1]$ since $f_n(x) = \varphi_n(a)$ for $n \geq N(x)$ for some point $a = a(x) \in \overline{B}$. On the other hand, evidently the series (3) converges to zero at each point $x \in [0, 1]$, and the set consists only of one function $f(x) \equiv 0$. The theorem is proved.

LITERATURE CITED

1. E. Steinitz, "Bedingt konvergente reihen und konvexe systeme," J. Reine u. Angew. Math., 143, 128–175 (1913).
2. P. Levy, "Sur les series semi-convergents," Nouv. Ann. d. Math., 5, 506–511 (1905).