Докл. Акад. Наук СССР
Том 254 (1980), № 6

Soviet Math. Dokl.
Vol. 22 (1980), No. 2

PROPERTIES OF RANDOM MATRICES CONNECTED WITH UNCONDITIONAL CONVERGENCE ALMOST EVERYWHERE

UDC 517.5

B. S. KAŠIN

1°. The following notation will be used. D_N is the collection of all $N \times N$ matrices $B = \{\epsilon_{ij}\}$ with elements $\epsilon_{ij} = \pm 1$; by assigning to each matrix $B \in D_N$ the measure $\mu_N(B) = 2^{-N^2}$, we introduce the natural measure μ_N on D_N. $S(N)$ is the group of permutations of the tuple of numbers $(1, \ldots, N)$. For every finite set Ω let $|\Omega|$ be the number of elements in Ω. Finally, for the vector $x \in R^N$ with coordinates $(x)_j$, $1 \leqslant j \leqslant N$,

$$\|x\|_{l_2^N} = \Big(\sum_{j=1}^{N} (x)_j^2\Big)^{1/2}; \quad B_2^N = \{x \in R^N : \|x\|_{l_2^N} \leqslant 1\}.$$

2°. In studying the unconditional convergence almost everywhere of series in some system $\Phi = \{\varphi_n(x)\}_1^\infty$, $x \in (0, 1)$, of functions there usually arises the problem of estimating the quantity

$$(1) \quad \alpha_N(\Phi) = \sup_{\{\sigma(j)\}=\sigma\in S(N)} \ \sup_{\{a_j\}\in B_2^N} \Big(\int_0^1 \Big(\sup_{1\leqslant M=M(x)\leqslant N} \sum_{j=1}^{M} a_{\sigma(j)}\varphi_{\sigma(j)}(x)\Big)^2 dx\Big)^{1/2}.$$

It follows from the Men′šov-Rademacher theorem (see [1]) that $\alpha_N(\Phi) \leqslant c \ln N$ for any orthonormal system Φ. It is known [2] that $\alpha_N(\chi) \asymp \ln^{1/2} N$ for the system χ of Haar functions. The orders $\alpha_N(\Phi)$ have not yet been determined for the trigonometric system and the Walsh system, but it has been shown that for these systems $\alpha_N(\Phi)$ grows more rapidly than $\ln^\gamma N$, $\gamma > 0$ (concerning the value of γ see [3]–[6]).

The finite-dimensional analogue of the problem of estimating $\alpha_N(\Phi)$ is the problem of estimating the following quantity for a given matrix $B = \{\beta_{ij}\}$, $1 \leqslant i, j \leqslant N$:

$$(2) \quad \alpha_N(B) = N^{-1/2} \sup_{\{\sigma(j)\}=\sigma\in S(N)} \ \sup_{\{a_j\}\in B_2^N} \ \sup_{\{M_i\}_1^N, 1\leqslant M_i\leqslant N} \Big(\sum_{i=1}^{N}\Big(\sum_{j=1}^{M_i} a_{\sigma(j)} b_{i,\sigma(j)}\Big)^2\Big)^{1/2}.$$

Indeed, if the system $\Phi = \{\varphi_j(x)\}_1^N$ is given by the matrix B, i.e., $\varphi_j(x) = \beta_{ij}$ for $x \in ((i-1)/N, i/N)$, $1 \leqslant i, j \leqslant N$, then $\alpha_N(\Phi) = \alpha_N(B)$.

The function $\alpha_N(B)$ satisfies the inequalities

$$N^{-1/2}\|B\| \leqslant \alpha_N(B) \leqslant cN^{-1/2}\cdot \ln N \|B\|, \qquad \|B\| \equiv \sup_{\{a_j\}\in B_2^N} \Big(\sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_j\beta_{ij}\Big)^2\Big)^{1/2}.$$

It was shown in [7] that $\|B\| \asymp N^{1/2}$ for the "majority" of matrices $B \in D_N$. In the present note we give an estimate of the mean value on D_N of the quantity $\alpha_N(B)$. It is easy to see that $\alpha_N(B) \geqslant 1$ for $B \in D_N$; it turns out (see the theorem) that the reverse inequality $\alpha_N(B) \leqslant C_0$ is also true for the majority of matrices $B \in D_N$.

We remark that while the Walsh matrices U_{2^k}, $1 \leqslant k < \infty$, already provide an example of a sequence of matrices with $U_{2^k} \in D_{2^k}$, $\|U_{2^k}\| \cdot 2^{-k/2} = 1$, $1 \leqslant k < \infty$, it was apparently

1980 *Mathematics Subject Classification*. Primary 15A52, 42C10; Secondary 42A28.

not known whether there existed a sequence of matrices $B_{n_k} \in D_{n_k}$ with $n_k \longrightarrow \infty$ and $\alpha_{n_k}(B_{n_k}) \leqslant C_0$, $1 \leqslant k < \infty$.

3°. THEOREM 1. *There exists a constant* C_0 *such that*

$$\mu_N E_1(C_0, N) \equiv \mu_N\{B \in D_N\colon\ \alpha_N(B) \geqslant C_0\} \leqslant 10^{-N}, \quad N = 1, 2, \ldots .$$

In the proof we use a well-known inequality: for $y \geqslant 0$ and any set of numbers $\{\beta_j\}_1^R$

$$(3) \qquad 2^{-R}\left|\left\{\{\epsilon_j\}_{j=1}^R\colon\ \epsilon_j = \pm 1 \text{ and } \left|\sum_{j=1}^R \beta_j\epsilon_j\right| \geqslant y\left(\sum_{j=1}^R \beta_j^2\right)^{1/2}\right\}\right| \leqslant 2e^{-y^2/2}.$$

LEMMA. *For* $N = 1, 2, \ldots$ *there is a set* $\Omega_N = \{e\} \subset B_2^N$ *with* $|\Omega_N| \leqslant C_1^N$ *such that:*

1) *for any* z, $\|z\|_{l_2^N} = 1$, *there is an* $e \in \Omega_N$ *with* $\|z - e\|_{l_2^N} \leqslant 1/2$;

2) *for* $e = \{e_j\} \in \Omega_N$ *and* $\sigma = \{\sigma(j)\} \in S(N)$ *the vector* $e_\sigma = \{e_{\sigma(j)}\}$ *is in* Ω_N *(and, consequently, the number of distinct permutations of* e_σ *does not exceed* C_1^N*); and*

3) $e_j \neq 0$, $1 \leqslant j \leqslant N$, *for* $e = \{e_j\} \in \Omega_N$.

We remark that the properties 2) and 3) of the set Ω_N are needed only for a certain convenience of the estimates. Without property 2) the lemma is well known, and a method for constructing a set Ω_N with all the needed properties is given in [8], Lemma 1. It is easy to see that property 1) of Ω_N ensures that

$$(4) \qquad \sup_{e \in \Omega_N} (z, e) \geqslant 1/2\, \|z\|_{l_2^N}, \qquad z \in R^N;$$

for any linear operator $T\colon l_2^N \longrightarrow l_2^N$

$$(4') \qquad \|T\| \leqslant 2\sup_{e \in \Omega_N} \|T(e)\|_{l_2^N}.$$

Let us show that for every $K \geqslant 1$ there is a constant C_K such that, for any vectors $a = \{a_j\} \in \Omega_N$ and $c = \{c_i\} \in B_2^N$

$$(5) \qquad \mu_N(a, b, C_K) \equiv \mu_N\left\{B = \{\epsilon_{ij}\}\colon \sup_{\sigma \in S(N),\, \{M_i\}} \sum_{i=1}^N \sum_{j=1}^{M_i} a_{\sigma(j)} c_i \epsilon_{i,\sigma(j)} \geqslant C_K N^{1/2}\right\} \leqslant K^{-N}.$$

We fix the vectors $a \in \Omega_N$ and $c \in B_2^N$ and construct for a the dyadic expansions (well known in the theory of orthogonal series; see [9], Chapter XI, §10, for details) into sums of vectors:

$$(6) \qquad a = \sum_{\nu=0}^{2^s - 1} r_\nu^s(a), \qquad s = 0, 1, \ldots, s_0,$$

where $r_0^0 = a$, $r_0^1 = (a_1, \ldots, a_{l-1}, a_l', 0, \ldots, 0)$, $r_1^1 = (0, \ldots, 0, a_l'', a_{l+1}, \ldots, a_N)$, and, in general, if the r_ν^{s-1}, $s \in [1, s_0]$, have already been defined, then the tuple $\{r_\nu^s\}$ is constructed in such a way that the following relations hold:

1) $r_\nu^{s-1} = r_{2\nu}^s + r_{2\nu+1}^s$, $0 \leqslant \nu \leqslant 2^{s-1} - 1$;

2) $\|r_\nu^s\|_{l_2^N}^2 \leqslant 2^{-s}$, $\|r_\nu^s\|_{l_\infty^N} \leqslant \|a\|_{l_\infty^N}$, $0 \leqslant \nu \leqslant 2^s - 1$;

3) if Δ_ν^s is the set of nonzero coordinates of the vector r_ν^s, then $\Delta_\nu^s = \{j\colon a_\nu^s \leqslant j \leqslant b_\nu^s\}$ and $[a_{2\nu}^s, b_{2\nu+1}^s] = [a_\nu^{s-1}, b_\nu^{s-1}]$; $b_{2\nu}^s \leqslant a_{2\nu+1}^s$. The number s_0 is so chosen that $\sup_\nu |\Delta_\nu^s| \leqslant 2$ for $s = s_0$. It is also convenient to define $r_{2^s}^s = 0$, $1 \leqslant s \leqslant s_0$.

By the properties of the expansions, any sum of the form $S = \Sigma_1^M a_j y_j$, $1 \leqslant M \leqslant N$, can be represented in the form

$$S = \sum_{s=1}^{s_0} \sum_{j=1}^{N} (r_{\nu(M)}^s)_j y_j + \delta, \qquad |\delta| \leqslant 2\max_{1\leqslant j\leqslant N} |a_j y_j|.$$

Consequently,

$$(7) \qquad \sum_{i=1}^{N} \sum_{j=1}^{M_i} a_{\sigma(j)} c_i \epsilon_{i,\sigma(j)} = \sum_{s=1}^{s_0} P_s(\{M_i\}, \sigma, B) + \delta',$$

where

$$a_\sigma = \{a_{\sigma(j)}\}_{j=1}^N, \qquad |\delta'| \leqslant 2 \sum_{i=1}^{N} |c_i| \leqslant 2N^{1/2},$$

$$(8) \qquad P_s(\{M_i\}, \sigma, B) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} c_i (r_{\nu(M_i)}^s(a_\sigma))_j \epsilon_{i,\sigma(j)} = \sum_{i,j=1}^{N} c_i (r_{\nu(M_i)}^s(a_\sigma))_{\sigma^{-1}_{(j)}} \epsilon_{ij}.$$

It is clear that for a given s the number R_s^N of distinct sums of the form (8), regarded as functions of matrices defined on D_N, does not exceed $N! \cdot N^N \leqslant N^{2N}$. Moreover, by using the property 2) of Ω_N and the properties of the expansions (6), the number R_s^N can be estimated as follows:

$$R_s^N \leqslant C^N \cdot 2^{sN} I_s^N N^{2^{s+1}},$$

where I_s^N is the number of distinct partitions of the tuple $(1, \ldots, N)$ into 2^s parts. Since $I_s^N \leqslant 2^{sN}$, the final result is

$$(9) \qquad R_s^N \leqslant \min(\exp(2N\ln N),\ \exp(C''N + 2sN + 2^s \ln N)).$$

Since $\|r_\nu^s\|_{l_2^N}^2 \leqslant 2^{-s}$, $0 \leqslant \nu \leqslant 2^s$ [see property 2) of the expansion (6)], each sum (8) has the form $\Sigma_{i,j=1}^N \beta_{i,j} \epsilon_{i,j}$, where $\Sigma_{i,j=1}^N \beta_{ij}^2 \leqslant 2^{-s} \Sigma_1^N c_i^2 \leqslant 2^{-s}$. It follows from the last inequality and (3) that for each $\alpha > 0$

$$(10) \qquad \mu_N\Big\{B \in D_N\colon \sup_{\{M_i\},\sigma} P_s(\{M_i\}, \sigma, B) \geqslant \alpha N^{1/2}\Big\} \leqslant R_s^N \exp(-N\alpha^2 \cdot 2^s).$$

Using (7), (9) and (10), we get that

$$\mu_N(a, b, C_K) \leqslant \sum_{s=1}^{s_0} \mu_N\Big\{B \in D_N\colon \sup_{\{M_i\},\sigma} P_s(\{M_i\}, \sigma, B) \geqslant \frac{N^{1/2}}{5s^2}(C_K - 2)\Big\}$$

$$\leqslant \sum_{s=1}^{\min(s_0, N^{1/8})} \exp\Big\{C''N + 2sN + 2^s \ln N - \frac{2^s N}{25 \cdot s^4}(C_K - 2)\Big\}$$

$$+ \sum_{s > \min(s_0, N^{1/8})} \exp\Big\{2N\ln N - \frac{2^s N}{25 s^4}(C_K - 2)\Big\} \leqslant K^{-N},$$

if the constant C_K is sufficiently large. The estimate (5) is proved.

By the definition of the set $E_1(C_0, N)$ (see Theorem 1), for $B \in E_1(C_0, N)$ we have

$$\sup_{z \in B_2^N, \{M_i\}, \sigma} \Big(\sum_{i=1}^{N} \Big(\sum_{j=1}^{M_i} z_{\sigma(j)} \epsilon_{i\sigma(j)}\Big)^2\Big)^{1/2} \geqslant C_0 N^{1/2}.$$

Then, by the properties (4) and (4′) of Ω_N, we have for $B \in E_1(C_0)$ that

$$\sup_{e, e' \in \Omega_N, \{M_i\}, \sigma} \sum_{i=1}^{N} \sum_{j=1}^{M_i} e_{\sigma(j)} e_i' \epsilon_{i\sigma(j)} \geqslant \tfrac{1}{4} C_0 N^{1/2},$$

and if C_0 is so large that $C_0/4 > C_{K_0}$, where C_{K_0} is determined in (5) from the number $K_0 = 10 \cdot C_1^2$ (the constant C_1 is defined in the lemma), then

$$\mu_N(E_1(C_0, N)) \leqslant |\Omega_N|^2 \cdot (10 \cdot C_1^2)^{-N} \leqslant 10^{-N},$$

which is what was required to prove.

Theorem 1 can be carried over also to the group O^n of orthogonal $n \times n$ matrices. More precisely, we have

THEOREM 2. *$N^{1/2} \cdot \int_{O^N} \alpha_N(U) d\widetilde{\mu}_N \leqslant C^0$, $N = 1, 2, \ldots$, where C^0 does not depend on N, $\widetilde{\mu}_N$ is Haar measure on O^N, and $U \in O^N$.*

The author thanks Professor L. Carleson for attention to the problem considered.

Steklov Institute of Mathematics
Academy of Sciences of the USSR

Received 13/JUNE/80

BIBLIOGRAPHY

1. Stefan Kaczmarz and Hugo Steinhaus, *Theorie der Orthogonalreihen,* PWN, Warsaw, 1935; reprint, Chelsea, New York, 1951.

2. P. L. Ul'janov, Mat. Sb. **63** (**105**) (1964), 356. (Russian)

3. Sabûrô Nakata, Tôhoku Math. J. (2) **26** (1974), 407.

4. S. V. Bočkarev, Dokl. Akad. Nauk SSSR **239** (1978), 509; English transl. in Soviet Math. Dokl. **19** (1978).

5. ———, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 1025; English transl. in Math. USSR Izv. **15** (1980).

6. Sabûrô Nakata, Anal. Math. **5** (1979), 201.

7. G. Bennett, V. Goodman and C. M. Newman, Pacific J. Math. **59** (1975), 359.

8. B. S. Kashin [Kašin], Anal. Math. **4** (1978), 27.

9. N. K. Bari, *Trigonometric series,* Fizmatgiz, Moscow, 1961; English transl., *A treatise on trigonometric series.* Vols. I, II, Macmillan, New York, and Pergamon Press, Oxford, 1964.

Translated by H. H. McFADEN