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ON SYSTEMS OF VECTORS IN A HILBERT SPACE

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ABSTRACT. Let $\omega = \{x_j\}_1^n$ be a system of unit vectors in a Hilbert space H such that any triple $(x_1, x_2, x_3) \subset \omega$ of vectors contains a pair of orthogonal vectors. The properties of such systems are studied in this article. It is proved that for any such system and any numbers $\alpha_1, \ldots, \alpha_n$

$$\left\| \sum_{j=1}^n \alpha_j x_j \right\|_H \leqslant 2^{1/3} n^{1/6} \left(\sum \alpha_j^2 \right)^{1/2}$$

and that there exists a system $\{x_j^0\}_1^n$ satisfying the indicated condition for which

$$\left\| \sum_{j=1}^n x_j^0 \right\|_H \geqslant c_1 n^{2/3} \ln^{-1/2} n, \quad c_1 > 0, n = 1, 2, \ldots.$$

Bibliography: 4 titles.

We consider the properties of systems $\omega = \{x_j\}$ of unit vectors in a real Hilbert space H that satisfy a condition generalizing the orthogonality property, namely: any triple $(x_{j_1}, x_{j_2}, x_{j_3}) \subset \omega$ with $j_1 \neq j_2 \neq j_3 \neq j_1$ contains a pair of orthogonal vectors.

Let Ω_n denote the class of systems $\omega = \{x_j\}_1^n$ having the indicated property, and let

$$\Delta_n = \sup_{\{x_j\}_{j=1}^n \in \Omega_n} \left\| \sum_{j=1}^n x_j \right\|_H .$$

L. Lovász posed the problem of determining the order of Δ_n. It was shown in [1] that for $n = 3, 4, \ldots$([1])

$$c_1' n^{4/3 - \ln 3/2 \ln 2} \leqslant \Delta_n \leqslant c_2' n^{2/3}, \qquad c_1' > 0.$$

In this article we use another method to prove a lower estimate for Δ_n which differs from the upper estimate by only a logarithmic factor.

THEOREM 1. $c_1 n^{2/3} \ln^{-1/2} n \leqslant \Delta_n$, $n = 3, 4, \ldots,$ $c_1 > 0$.

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([1]) $4/3 - \ln 3/2 \ln 2 \in (54/100, 55/100)$.

Moreover, for completeness of the exposition we prove in a simple way the following generalization of the upper estimate for Δ_n:

THEOREM 2. *If* $\omega = \{x_j\} \in \Omega_n$, *then for any numbers* $\alpha_1, \ldots, \alpha_n$

$$\left\| \sum_{j=1}^{n} \alpha_j x_j \right\|_H \leqslant 2^{1/3} n^{1/6} \left(\sum_{j=1}^{n} \alpha_j^2 \right)^{1/2}. \tag{1}$$

PROOF OF THEOREM 1. The system $\{x_j\} \in \Omega_n$ we need will have the property that its Gram matrix $G = \{(x_j, x_l)\}$ is a Toeplitz matrix, i.e., $(x_j, x_l) = a_{|j-l|}$. In this case the problem of estimating $\|\Sigma_1^n x_j\|_H$ reduces to estimating an appropriate trigonometric polynomial.

LEMMA (see [2], VII. Abschnitt, Aufgabe 68). *Let* $P(t)$ *be a real nonnegative polynomial of the form*

$$1 + 2\sum_{k=1}^{n-1} a_k \cos kt. \tag{2}$$

Then there exists a system $\{x_j\}_1^n \subset H$ *such that*

$$(x_j, x_j) = 1, \qquad (x_j, x_l) = a_{|j-l|}, \qquad 1 \leqslant j, l \leqslant n.$$

It is not hard to see that the system $\{x_j\}$ constructed in the lemma belongs to Ω_n if

$$a_{r_1} a_{r_2} a_{r_1+r_2} = 0, \quad r_1 \geqslant 1, r_2 \geqslant 1, \qquad r_1 = r_2 \leqslant n-1. \tag{3}$$

Moreover,

$$\left\| \sum_{j=1}^{n} x_j \right\|_H^2 = \sum_{j,l=1}^{n} (x_j, x_l) = n + 2\sum_{k=1}^{n-1} (n-k) a_k. \tag{4}$$

To prove the theorem it thus suffices to determine a polynomial satisfying the conditions of the lemma, the relations (3), and the inequality

$$n + 2\sum_{k=1}^{n-1} (n-k) a_k \geqslant c_1^2 n^{4/3} \ln^{-1} n. \tag{5}$$

It is known ([3], p. 220) that for each $n \geqslant 3$ there exists a set Γ of $s \geqslant c_3 n^{1/3}$ ($c_s > 0$) positive integers $k_1, \ldots, k_s$ less than $n/2$ such that the equality

$$k_1 + k_2 + k_3 = k_1' + k_2' + k_3' \qquad (k_1, k_2, k_3, k_1', k_2', k_3' \in \Gamma)$$

holds only if the unordered collections (k_1, k_2, k_3) and (k_1', k_2', k_3') coincide.

It follows from well-known probabilistic arguments (see, for example, [4], Chapter VI, Theorem 4) that the signs $+$ and $-$ can be chosen in such a way that for $q(t) = \Sigma_{k\in\Gamma} \varepsilon_k e^{ikt}$, $\varepsilon_k = \pm 1$, we have

$$\max_{t\in[-\pi,\pi]} |q(t)| < C_4 (s \ln n)^{1/2}. \tag{6}$$

Let $\Gamma_+ = \{k \in \Gamma\colon \varepsilon_k = +1\}$ and $\Gamma_- = \{k \in \Gamma\colon \varepsilon_k = -1\}$, and let $|\Gamma_+|$ and $|\Gamma_-|$ be the numbers of elements in Γ_+ and Γ_-, respectively. We represent $q(t)$ in the form

$$q(t) = q_+(t) - q_-(t) \equiv \sum_{k\in\Gamma_+} \exp(ikt) - \sum_{k\in\Gamma_-} \exp(ikt). \tag{7}$$

We let

$$P(t) = 1 + 2\left(C_4^2 s \ln n\right)^{-1}\left(q_+(t)\overline{q_-(t)} + \overline{q_+(t)}q_-(t)\right)$$

and show that it is the desired polynomial.

Clearly, $P(t)$ has the form

$$1 + 4\left(C_4^2 s \ln n\right)^{-1} \sum_{k_+ \in \Gamma_+,\, k_- \in \Gamma_-} \cos((k_+ - k_-)t). \tag{8}$$

We verify that $P(t)$ is nonnegative. By (6), for $t \in [-\pi, \pi]$

$$\begin{aligned}\operatorname{Re}\left(q_+(t)\overline{q_-(t)}\right) &= \operatorname{Re}\left(q_+(t)\overline{q_+(t)}\right) - \operatorname{Re}\left[q_+(t)\left(\overline{q_+(t)} - \overline{q_-(t)}\right)\right] \\ &\geqslant |q_+(t)|^2 - |q_+(t)|\,|q_+(t) - q_-(t)| \\ &\geqslant |q_+(t)|^2 - |q_+(t)| C_4 (s \ln n)^{1/2} \geqslant -\tfrac{1}{4} C_4^2 s \ln n,\end{aligned}$$

whence

$$q_+(t)\overline{q_-(t)} + \overline{q_+(t)}q_-(t) = 2\operatorname{Re}\left(q_+(t)\overline{q_-(t)}\right) \geqslant -\tfrac{1}{2} C_4^2 s \ln n,$$

and, consequently, $P(t) \geqslant 0$.

Let us verify the conditions (3). Suppose that $a_{r_1}, a_{r_2}, a_{r_1+r_2} \neq 0$ for some r_1 and r_2 with $r_1, r_2 \geqslant 1$ and $r_1 + r_2 \leqslant n-1$. This means (see (8)) that there are numbers k_1, $k_2, k_3, k_1', k_2', k_3' \in \Gamma$ such that

$$r_1 = k_1 - k_1', \qquad r_2 = k_2 - k_2', \qquad r_1 + r_2 = k_3 - k_3'$$

and in each pair (k_1, k_1'), (k_2, k_2'), (k_3, k_3') one element belongs to Γ_+ and the other belongs to Γ_-. Hence $k_1 + k_2 + k_3' = k_1' + k_2' + k_3$. But, on the other hand, the collection $(k_1, k_2, k_3, k_1', k_2', k_3')$ contains exactly three elements of Γ_+; therefore, the unordered collections (k_1, k_2, k_3') and (k_1', k_2', k_3) cannot coincide, and, by the properties of Γ, the last equality is impossible. The condition (3) is proved.

We verify inequality (5) for $P(t)$. It can be assumed that n is large enough that $C_4^2 \ln n (c_3 n^{1/3})^{-1} < 1/4$. Then, by (6) and the estimate $s \geqslant c_3 n^{1/3}$,

$$\begin{aligned}\big|\,|\Gamma_+| - |\Gamma_-|\,\big| &= |q_+(0) - q_-(0)| \leqslant C_4 s (\ln n/s)^{1/2} \\ &\leqslant C_4 s \ln^{1/2} n \left(c_3 n^{1/3}\right)^{-1/2} \leqslant s/2,\end{aligned} \tag{9}$$

which implies that $|\Gamma_+| \geqslant s/4$ and $|\Gamma_-| \geqslant s/4$.

Taking the form of the polynomial $P(t)$ (see (8)) into account, we get

$$\begin{aligned} n + 2\sum_{k=1}^{n-1} (n-k) a_k &\geqslant n \sum_{k=1}^{[n/2]} a_k \geqslant n |\Gamma_+|\,|\Gamma_-| 4\left(C_4^2 s \ln n\right)^{-1} \\ &\geqslant 4n\left(\frac{s}{4}\right)^2 \left(C_4^2 s \ln n\right)^{-1} \geqslant C_1^2 n^{4/3} \ln^{-1} n.\end{aligned}$$

Theorem 1 is proved.

It is possible that Δ_n has order $n^{2/3}$. We could show this by means of the arguments given above if instead of the polynomial $q(t)$ we could construct a polynomial $Q(t)$ of the form $\sum_{k \in \Gamma} b_k e^{ikt}$ with real b_k such that

$$\sum_{k \in \Gamma} |b_k| \geqslant c_5 s^{1/2} \max_{t \in [-\pi, \pi]} |Q(t)|.$$

PROOF OF THEOREM 2. Let l_2^n be the space of sequences $\{\alpha_j\}_1^n$ with norm $\|\{\alpha_j\}\|_{l_2^n} = (\sum \alpha_j^2)^{1/2}$, and let $\|Q\|$ be the norm of an $n \times n$ matrix Q as an operator from l_2^n to itself. We remark that the conditions $\omega \in \Omega_n$ implies that the vectors in ω which are not orthogonal to x_j are pairwise orthogonal. Using the Bessel inequality, we find that

$$\sum_{\substack{k=1\\k\neq j}}^{n} (x_j, x_k)^2 \leqslant 1, \qquad j = 1,2,\ldots,n. \tag{10}$$

Suppose next that $T\colon l_2^n \to H$ is a linear operator acting in the following way: $T(\{\alpha_j\}_1^n) = \sum_1^n \alpha_j x_j$. We need to show that $\|T\| \leqslant 2^{1/3} n^{1/6}$.

Since we always have that $\|T\|^2 = \|G\|$, $G = \{(x_j, x_k)\}$, it suffices to estimate $\|G\|$, i.e., the maximal eigenvalue of G. Clearly, $\|G\|^3 = \|G^3\|$. Let us now estimate the diagonal elements $(G^3)_{jj}$, $1 \leqslant j \leqslant n$, of the matrix G^3. Using the fact that $\omega \in \Omega_n$ and (10), we have

$$\begin{aligned}(G^3)_{jj} &= \sum_{k=1}^{n} a_{jk}\left(\sum_{l=1}^{n} a_{jl}a_{kl}\right) = \sum_{k,l=1}^{n} a_{jk}a_{jl}a_{kl}\\ &= a_{jj}^3 + \sum_{\substack{k=1\\k\neq j}}^{n} a_{jj}a_{jk}a_{jk} + \sum_{\substack{l=1\\l\neq j}}^{n} a_{jj}a_{jl}a_{jl} + \sum_{\substack{k=1\\k\neq j}}^{n} a_{jk}a_{kk}a_{jk} \leqslant 4.\end{aligned} \tag{11}$$

Since G^3 is positive definite, (11) gives us that any element $(G^3)_{jk}$ satisfies the inequality

$$(G^3)_{jk} \leqslant 1/2\big((G^3)_{jj} + (G^3)_{kk}\big) \leqslant 4.$$

Therefore, $\|G^3\| \leqslant 4n$, which, by the indicated equality $\|T\|^6 = \|G^3\|$, implies the desired statement.

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