# STABILITY OF UNCONDITIONAL CONVERGENCE ALMOST EVERYWHERE

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We will investigate the properties of series of functions which are unconditionally convergent almost everywhere on [0, 1]. We will establish the following theorem: If the series $\Sigma_{k=1}^{\infty} f_k(x)$ converges unconditionally almost everywhere, then there exists a sequence $\{\beta_k\}_1^{\infty}, \beta_k \uparrow \infty$ such that if $\lambda_k \leq \beta_k$, $k = 1, 2, \ldots$, the series $\Sigma_{k=1}^{\infty} \lambda_k f_k(x)$ converges unconditionally almost everywhere.

We begin by recalling some definitions.

Let B be a Banach space. A series

$$\sum_{k=1}^{\infty} x_k \qquad (x_k \in B) \tag{1}$$

is said to be unconditionally convergent if it converges (in the norm of B) after an arbitrary permutation of its terms. Suppose that we are given a series

$$\sum_{k=1}^{\infty} f_k(x), \tag{2}$$

where the $f_k(x)$ ($k = 1, 2, \ldots$) are measurable functions defined on a set $E \subset (-\infty, \infty)$ and are finite almost everywhere. The series (2) is said to be unconditionally convergent almost everywhere on E (a.e. on E) if after each permutation of its terms the resulting series

$$\sum_{k=1}^{\infty} f_{n_k}(x) \tag{3}$$

converges almost everywhere on E. Here the set (of measure zero) of points of divergence of the series (3) depends on the permutation $\{n_k\}$.

Orlicz (see [1]) showed that the series (1) is unconditionally convergent if and only if each of the following partial sums is convergent (in B):

$$\sum_{k=1}^{\infty} x_{p_k} \qquad (p_1 < p_2 < \ldots).$$

It follows that convergence of the series

$$\sum_{k=1}^{\infty} \varepsilon_k x_k \tag{4}$$

for arbitrary choices of $\varepsilon_k = \pm 1$ is equivalent to unconditional convergence of the series (1). The analogous assertion is true for unconditional convergence in measure for series of the form (2) (see Orlicz [3]). By using convergence of all series of the form (4) we can show that if the series (1) is unconditionally convergent, then the series

$$\sum_{k=1}^{\infty} \lambda_k x_k$$

is also unconditionally convergent for arbitrary $\{\lambda_k\}$ with $|\lambda_k| \leq c = \text{const}$ ($k = 1, 2, \ldots$).

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Similar questions can be asked concerning series of the form (2). It is clear that if the series (2) is unconditionally convergent almost everywhere on E, the series

$$\sum_{k=1}^{\infty} \varepsilon_k f_k(x) \tag{5}$$

converges almost everywhere on E for an arbitrary choice of the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$, where $\varepsilon_k = \pm 1$. For a long time it was not known whether convergence almost everywhere on E of all series of the form (5) was a sufficient condition for unconditional convergence almost everywhere of the series (2). A negative answer to this question was given by Ul'yanov (see [2], Theorem 5). This fact indicates the principal difference between unconditional convergence of series in a Banach space and unconditional convergence almost everywhere. Ul'yanov has communicated the following problem to us. If the series (2) is unconditionally convergent almost everywhere on a set E, does every series

$$\sum_{k=1}^{\infty} \lambda_k f_k(x) \tag{6}$$

with $|\lambda_k| \leq c = \text{const} < \infty$ $(k = 1, 2, \ldots)$ have the same property?

In this article we give a positive answer to the above question.

THEOREM 1. If the series (2) converges unconditionally almost everywhere on a set E, then for each sequence $\{\lambda_k\}_{k=1}^{\infty} \in l_\infty$ the series (6) also converges unconditionally almost everywhere on E.

It is clear that unconditional convergence a.e. on E of the series (2) is equivalent to unconditional convergence a.e. on E of all series of the form (5). Therefore Theorem 1 is a corollary of the following result.

THEOREM 2. Suppose that the series (5) converges a.e. on E for each choice of $\varepsilon_k = \pm 1$ $(k = 1, 2, \ldots)$. Then the series (6) converges a.e. on E for each $\{\lambda_k\}_{k=1}^{\infty} \in l_\infty$.

For the proof of Theorem 2 we first establish the following result.

LEMMA. Suppose that we are given functions $f_i(x) \in L^2(E)$ $(1 \leq i \leq m)$ and numbers $\lambda_i$ with $|\lambda_i| \leq 1$ for $1 \leq i \leq m$. Then there exists a set of $\varepsilon_i = \pm 1$ $(1 \leq i \leq m)$ such that

$$\Big\| \sum_{i=1}^{k} \lambda_i f_i - \sum_{i=1}^{k} \varepsilon_i f_i \Big\|_{L^2(E)}^2 \leqslant 4 \sum_{i=1}^{k} \| f_i \|_{L^2(E)}^2 \quad \text{for} \quad 1 \leqslant k \leqslant m. \tag{7}$$

The lemma will be proved by induction. For k = 1 the result is clear. Let $1 \leq k < m$ and let $\varepsilon_i = \pm 1$ with $1 \leq i \leq k$ be chosen such that (7) is true. If we set

$$g = \sum_{i=1}^{k} \lambda_i f_i - \sum_{i=1}^{k} \varepsilon_i f_i,$$

we obtain

$$\sum_{i=1}^{k+1} \lambda_i f_i - \sum_{i=1}^{k} \varepsilon_i f_i - f_{k+1} = g + (\lambda_{k+1} - 1) f_{k+1} \ (\text{here} \quad \varepsilon_{k+1} = 1),$$
$$\sum_{i=1}^{k+1} \lambda_i f_i - \sum_{i=1}^{k} \varepsilon_i f_i + f_{k+1} = g + (\lambda_{k+1} + 1) f_{k+1} \ (\text{here} \quad \varepsilon_{k+1} = -1),$$

where, by assumption, we have

$$\| g \|_{L^2}^2 \leqslant 4 \sum_{i=1}^{k} \| f_i \|_{L^2}^2. \tag{8}$$

It suffices to show [see (8)] that either

$$\| g + (\lambda_{k+1} - 1) f_{k+1} \|_{L_2}^2 \leqslant \| g \|_{L}^2 + 4 \| f_{k+1} \|_{L^2}^2 \leqslant 4 \sum_{i=1}^{k+1} \| f_i \|_{L^2}^2$$

or

$$\| g + (\lambda_{k+1} + 1) f_{k+1} \|_{L^2}^2 \leqslant \| g \|_{L^2}^2 + 4 \| f_{k+1} \|_{L^2}^2 \leqslant 4 \sum_{i=1}^{k+1} \| f_i \|_{L^2}^2,$$

but

$$\| g + (\lambda_{k+1} - 1) f_{k+1} \|_{L^2}^2 \leqslant \int_E (g + (\lambda_{k+1} - 1) f_{k+1})^2 \, dx \doteq$$
$$= \int_E g^2 \, dx + (\lambda_{k+1} - 1)^2 \int_E f_{k+1}^2 \, dx + 2 (\lambda_{k+1} - 1) \int_E fg \, dx \leqslant$$
$$\leqslant \| g \|_{L^2}^2 + 4 \| f_{k+1} \|_{L^2}^2 + 2 (\lambda_{k+1} - 1) \int_E fg \, dx. \tag{9}$$

Analogously

$$\|g+(\lambda_{k+1}+1)f_{k+1}\|^2_{L^2}\leqslant\|g\|^2_{L^2}+4\|f_{k+1}\|^2_{L^2}+2(\lambda_{k+1}+1)\int_E fg\,dx. \tag{9'}$$

Since $|\lambda_{k+1}|<1$, either $2(\lambda_{k+1}-1)\int_E fg\,dx\leqslant 0$ or $2(\lambda_{k+1}+1)\int_E fg\,dx\leqslant 0$. But then from (9) and (9') it follows that either

$$\|g+(\lambda_{k+1}-1)f_{k+1}\|^2_{L^2}\leqslant\|g\|^2_{L^2}+4\|f_{k+1}\|^2_{L^2}$$

or

$$\|g+(\lambda_{k+1}+1)f_{k+1}\|^2_{L^2}\leqslant\|g\|^2_{L^2}+4\|f_{k+1}\|^2_{L^2}.$$

The lemma is proved.

Proof of Theorem 2. Without loss of generality we can assume that $E=[0,1]$. Suppose that Theorem 2 is false. Then there exists a series of the form (2) for which all of the series (5) converge a.e. on $[0,1]$ but for some $\{\lambda_k\}\in l_\infty$ with $|\lambda_k|\leq 1$ $(k=1,2,\ldots)$ the series (6) diverges at each point of some set $A\subset[0,1]$ with $\mu A>0$. By a theorem of Orlicz (see [3]) the convergence of all of the series (5) implies that $\sum_{k=1}^{\infty}f_k^2(x)<\infty$ for almost all $x\in[0,1]$. Thus there exists a set $A_1\subset[0,1]$ such that $\mu A_1>1-\mu A$ and $\sum_{k=1}^{\infty}\int_{A_1}f_k^2(x)<\infty$.

We set $A_2=A\cdot A_1$ and $g_k(x)=f_k(x)\cdot\chi_{A_1}(x)$, where $\chi_{A_1}(x)$ is the characteristic function of the set $A_1$. Then we have

$$\left.\begin{array}{l}\text{a) the series } \sum_{k=1}^{\infty}\varepsilon_k g_k(x) \text{ converges a.e. on } [0,1] \text{ for arbitrary } \varepsilon_k=\pm1,\\[2mm] \text{b) the series } \sum_{k=1}^{\infty}\lambda_k g_k(x) \text{ diverges at each point of } A_2, \text{ where } \mu A_2>0,\\[2mm] \text{c) } \sum_{k=1}^{\infty}\|g_k\|^2_{L^2(0,1)}<\infty.\end{array}\right\} \tag{10}$$

We will show that the three statements in (10) cannot be true simultaneously. We will find a contradiction. First we note that part b) of (10) implies that there exists a number $\varepsilon_0>0$, a set $A_3\subset A_2$ with $\mu A_3>0$, a sequence of natural numbers $1=M_1<\ldots<M_\nu<\ldots$, and a sequence of measurable functions $N_\nu(x)$ on $[0,1]$ with $M_\nu\leq N_\nu(x)<M_{\nu+1}$ such that

$$\left|\sum_{k=M_\nu}^{N_\nu(x)}\lambda_k g_k(x)\right|\geqslant\varepsilon_0 \quad\text{for all}\quad \begin{cases}x\in A_3,\\ \nu=1,2,\ldots\end{cases} \tag{11}$$

(see [4], pp. 817-819).

Let

$$\delta_\nu=\sum_{k=M_\nu}^{M_{\nu+1}-1}\|g_k\|^2_{L^2(0,1)}\qquad(\nu=1,2,\ldots).$$

From part c) of (10) it follows that $\delta_\nu\to0$ as $\nu\to\infty$. We fix $\nu$. We introduce a sequence of functions $U_k(x)$ with $M_\nu\leqslant k<M_{\nu+1}$, by setting for each $x\in[0,1]$

$$U_k(x)=\begin{cases}g_k(x) & \text{for}\quad k\leqslant N_\nu(x),\\ 0 & \text{for}\quad k>N_\nu(x).\end{cases} \tag{12}$$

The functions $U_k(x)$ are measurable on $[0,1]$ since the functions $N_\nu(x)$ are measurable. From the definition of $U_k(x)$ it follows that [see (11)]

$$\left|\sum_{k=M_\nu}^{M_{\nu+1}-1}\lambda_k U_k(x)\right|\geqslant\varepsilon_0 \quad\text{for}\quad x\in A_3. \tag{13}$$

By virtue of the Lemma there exists a sequence $\{\varepsilon_k\}_{M_\nu}^{M_{\nu+1}-1}$ with $\varepsilon_k=\pm1$ such that [see (12)]

$$\left\|\sum_{k=M_\nu}^{M_{\nu+1}-1}\lambda_k U_k(x)-\sum_{k=M_\nu}^{M_{\nu+1}-1}\varepsilon_k U_k(x)\right\|^2_{L^2(0,1)}\leqslant4\sum_{k=M_\nu}^{M_{\nu+1}-1}\|U_k\|^2_{L^2}\leqslant4\sum_{k=M_\nu}^{M_{\nu+1}-1}\|g_k(x)\|^2_{L^2(0,1)}=4\delta_\nu.$$

Hence by the Chebyshev inequality we have

$$\mu\mathcal{I}_\nu \equiv \mu\left\{x\colon x\in[0,1],\ \left|\sum_{k=M_\nu}^{M_{\nu+1}-1}\lambda_k U_k(x) - \sum_{k=M_\nu}^{M_{\nu+1}-1}\varepsilon_k U_k(x)\right| > \frac{\varepsilon_0}{2}\right\} < 4\delta_\nu\left(\frac{2}{\varepsilon_0}\right)^2.$$

But $\delta_\nu \to 0$ and therefore $\mu\mathcal{I}_\nu < \frac{1}{2}\mu A_3$ for $\nu > \nu_0$.

Thus it follows that

$$\left|\sum_{k=M_\nu}^{M_{\nu+1}-1}\lambda_k U_k(x) - \sum_{k=M_\nu}^{M_{\nu+1}-1}\varepsilon_k U_k(x)\right| \leqslant \frac{\varepsilon_0}{2} \tag{14}$$

for $x\in A_3 - \mathcal{I}_\nu$ and $\nu > \nu_0$.

From (13) and (14) we have

$$\left|\sum_{k=M_\nu}^{M_{\nu+1}-1}\varepsilon_k U_k(x)\right| \geqslant \frac{\varepsilon_0}{2} \quad \text{for} \quad x\in A_3-\mathcal{I}_\nu \text{ and } \nu>\nu_0$$

or [see (12)]

$$\left|\sum_{k=M_\nu}^{N_\nu(x)}\varepsilon_k g_k(x)\right| \geqslant \frac{\varepsilon_0}{2} \quad \text{for} \quad x\in A_3-\mathcal{I}_\nu \text{ and } \nu>\nu_0. \tag{15}$$

Let $D = \overline{\lim\limits_{\nu\to\infty}}(A_3 - \mathcal{I}_\nu)$. Since $\mu(A_3-\mathcal{I}_\nu) \geqslant \frac{1}{2}\mu A_3$ for $\nu \geq \nu_0$, we have $\mu D \geqslant \frac{1}{2}\mu A_3$.

For each $\nu = 1, 2, \ldots$ we have constructed a sequence $\{\varepsilon_k\}_{k=M_\nu}^{M_{\nu+1}-1}$ and thus we have constructed a series $\sum_{k=1}^\infty \varepsilon_k g_k(x)$, which [see (15)] diverges at each point of D, which contradicts part a) of (10).

The theorem is proved.

COROLLARY. If the hypothesis of Theorem 2 is satisfied, there exists a sequence $\{\beta_k\}_{k=1}^\infty$ with $\beta_k \uparrow\infty$ as $k \to \infty$, for which the series $\sum_{k=1}^\infty \lambda_k f_k(x)$ converges a.e. on E if $|\lambda_k| \leq \beta_k (k = 1, 2, \ldots)$.

Remark. By virtue of the lemma we can show that if the series (2) converges in measure on E for each choice of signs, then the series (6) converges in measure on E for each $\{\lambda_k\}\in l_\infty$.

It is well known that convergence in measure of a sequence of measurable functions defined on (0, 1) can be described by a metric. For instance, we can let the distance be given by

$$\rho(f, g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx. \tag{16}$$

It is easy to see that the series (2) converges almost everywhere on (0, 1) if and only if for each $\varepsilon > 0$ there exists a number $\tilde{N}_0 = \tilde{N}_0(\varepsilon)$ such that

$$\rho\left(0, \sum_{k=M}^{N(x)} f_k(x)\right) < \varepsilon \quad \text{for each number} \quad M > \tilde{N}_0 \tag{17}$$

and each bounded, integer-valued, measurable function N(x) on [0, 1] for which $N(x) \geq M$ for $x \in [0, 1]$.

THEOREM 3. Suppose that the series (2) is unconditionally convergent almost everywhere on (0, 1). Then there exists a sequence $\{\beta_k\}_{k=1}^\infty$ with $\beta_k \uparrow \infty$ as $k \to \infty$ such that the series (6) is unconditionally convergent a.e. on (0, 1) if

$$|\lambda_k| \leqslant |\beta_k| \quad \text{for} \quad k \geqslant 1.$$

Proof. By virtue of Theorem 1 it suffices to find a sequence $\{\beta_k\}_{k=1}^\infty$ with $\beta_k \uparrow\infty$ as $k \to \infty$ such that the series

$$\sum_{k=1}^\infty \beta_k f_k(x) \tag{18}$$

converges unconditionally almost everywhere on (0, 1). We construct such a sequence $\{\beta_k\}_{k=1}^\infty$. Unconditional convergence a.e. of the series (2) implies [see (17)] that for each $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that for each permutation $\{n_k\}_{k=1}^\infty$ the distance

$$\rho\left(0, \sum_{k=M_1(x)}^{M_2(x)} f_{n_k}(x)\right) < \varepsilon \tag{19}$$

for arbitrary integer-valued, bounded, measurable functions $M_1(x)$ and $M_2(x)$ satisfying the conditions

$$M_1(x) \leqslant M_2(x) \text{ and } n_k \geqslant N_0$$

for

$$M_1(x) \leqslant k \leqslant M_2(x), \; x \in [0, 1].$$

Let

$$N_\nu = N_0(3^{-\nu}), \quad 1 \leqslant \nu < \infty.$$

We can assume that $N_1 < N_2 < \ldots$. We set $\beta_k = \nu$ for $N_\nu \leq k < N_{\nu+1}$.

We will show that the series (18) converges unconditionally a.e. on (0, 1). Suppose the contrary; that is, suppose there exists a permutation such that the series

$$\sum_{k=1}^{\infty} \beta_{m_k} f_{m_k}(x)$$

does not converge a.e. on (0, 1). Then [see (17)] there exist a number $\varepsilon > 0$, natural numbers $M_\nu$ ($\nu = 1, 2, \ldots$), and measurable integer-valued functions $N_\nu(x)$ ($\nu = 1, 2, \ldots$) for which

$$\rho\left(0, \sum_{k=M_\nu}^{N_\nu(x)} \beta_{m_k} f_{m_k}(x)\right) > \varepsilon_0,$$

$$M_\nu \leqslant N_\nu(x) < M_{\nu+1} \quad \text{for} \quad x \in [0, 1], \; \nu = 1, 2, \ldots \tag{20}$$

We pick a natural number such that $2^{-s} < \varepsilon_0$. There exists $\nu_0$ such that $m_k > N_s$ for all $k \geq M_{\nu_0}$. It is then clear that

$$F(x) \equiv \sum_{k=M_{\nu_0}}^{N_{\nu_0}(x)} \beta_{m_k} f_{m_k}(x) = \sum_{j=s}^{\infty} \sum_{\substack{k=M_{\nu_0} \\ N_j \leqslant m_k < N_{j+1}}}^{N_{\nu_0}(x)} \beta_{m_k} f_{m_k}(x) \equiv \sum_{j=s}^{\infty} A_j(x). \tag{21}$$

In Eq. (21) the inner sum for $A_j(x)$ can be written in the form

$$A_j(x) = \sum_{k=R_j(x)}^{R_j'(x)} \beta_{p_k} f_{p_k}(x),$$

where $\{p_k\}_{k=1}^{\infty}$ is some permutation of the natural numbers, and the bounded, integer-valued, measurable functions $R_j(x)$ and $R_j'(x)$ satisfy the inequalities $R_j(x) \leq R_j'(x)$ and $N_j \leq p_k < N_{j+1}$ for $R_j(x) \leq k \leq R_j'(x)$ if $x \in [0, 1]$ and $j \geq s$.

By virtue of the definition of the sequence $\{\beta_k\}$ the functions

$$A_j(x) = j \sum_{k=R_j(x)}^{R_j'(x)} f_{p_k}(x) \qquad (j \geqslant s). \tag{22}$$

Furthermore, the function $\rho(f, g)$ has the properties [see (16)]

$$\rho(0, f+g) \leqslant \rho(0, f) + \rho(0, g)$$

and

$$\rho(0, jf) \leqslant j\rho(0, f) \qquad (j = 1, 2, \ldots),$$

and therefore [see (19), (21), and (22)]

$$\rho(0, F) \leqslant \sum_{j=s}^{\infty} \rho(0, A_j) \leqslant \sum_{j=s}^{\infty} j\rho\left(0, \sum_{k=R_j(x)}^{R_j'(x)} f_{p_k}(x)\right) \leqslant \sum_{j=s}^{\infty} j \cdot \frac{1}{3^j} < \frac{1}{2^s} < \varepsilon_0,$$

which contradicts (20) for $\nu = \nu_0$. The proof is complete.

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