RANDOM SETS OF UNIFORM CONVERGENCE

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In [1], P. L. Ul'yanov posed the question of characterization of sets $\sigma \subset \mathbb{Z}$ for which the functions $\{e^{2\pi inx}\}_{n\in\sigma}$ form a basis for the subspace of C(0, 1) which they generate. In other words one must determine for which sets σ the quantity

$$U(\sigma) = \sup_{s,\{a_n\}} \frac{\left\| \sum_{n\in\sigma\cap[-s,s]} a_n e^{2\pi inx} \right\|_\infty}{\left\| \sum_{n\in\sigma} a_n e^{2\pi inx} \right\|_\infty}$$

is finite, where the supremum is taken with respect to all $s \in \mathbb{N}$ and all finite sequences $\{a_n\}_{n\in\sigma} \neq 0$.

We shall call a set σ for which $U(\sigma)$ is finite a set of uniform convergence (U. C. set).

The article [1] contains two basic questions: one of them is to determine whether $\sigma = \{k^2\}_{k\in N}$ is a set of uniform convergence, and the second, of more general character, is to determine how dense a U. C. set can be.

The first problem is completely solved by K. I. Oskolkov [2] (cf. also [3]), who showed that for any polynomial P(x) with integer coefficients $\sigma = \{P(n)\}_{n\in N}$ is not a U. C. set.

The problem of possible density of sets of uniform convergence remains open. It can be reduced to the question of finding for given N a subset $\sigma \subset \{-N, \ldots, N\}$ with maximal number of elements for which $U(\sigma)$ is bounded by a constant independent of N. This problem can also be considered for other orthonormal systems, in particular for a Walsh system $\{W_i\}_{i=1}^{\infty}$.

The simplest examples of sets of uniform convergence are Sidon sets, for which it is known that their density is quite small. More precisely, for Sidon sets $\sigma \in \mathbb{Z}$, $|\sigma \cap [-N, N]| \le C \log N$, where $C < \infty$ is a constant, N = 1, 2, ...; sets of larger density of order $(\log N)^2$ were constructed in [4], of order $(\log N)^k$, $k \in \mathbb{N}$ in [5]. U. C. sets of greater density are not yet known.

The goal of the present note is to consider random subsets $\sigma \subset \{-N, \ldots, N\}$ of the trigonometric system $\{e^{2\pi inx}\}_{n=-N}^{N}$ or $\sigma \subset \{1, \ldots, N\}$ for the Walsh system $\{W_i\}_{i=1}^{2^r=N}$. In both cases estimates of the Lebesgue constant (i.e., the norm of the operators of partial sums) guarantee that $U(\sigma) \le C \log N$ for all σ and N and some constant C.

It is shown in [6] (cf. also [7, p. 283]) that for any uniformly bounded orthonormal system $\{\varphi_n\}_{n=1}^{N}$ a random set $\sigma \subset \{1, \ldots, N\}$ with number of elements $|\sigma| \le (1/6) \log N$ is a Sidon set with Sidon constant independent of N. Consequently, for a random set σ with $|\sigma| \le (1/6) \log N$ we have $U(\sigma) \le C$ with constant C independent of N.

The basic result of this paper is that for a random subset $\sigma \in \{1, \ldots, N\}$ with number of elements $>> \log N$, $U(\sigma) \to \infty$ as $N \to \infty$ and in addition if the number of elements of the random set σ satisfies $|\sigma| \ge N^{\varepsilon}$ for some $\varepsilon > 0$, then with high probability $U(\sigma)$ has maximal order, i.e., log N.

First we consider the case of a Walsh system. For natural numbers q and N such that $1 \le q \le N$, we denote by S_N^q the family of all sets $\sigma \subset \{1, \ldots, N\}$ with $|\sigma| = q$ and by ν the normalized counting measure on S_N^q. For $\sigma \subset S_N^q$ let

$$U(\sigma) = \sup \left\{ \frac{\left\| \sum_{j=1}^{s} a_j W_j \right\|_\infty}{\left\| \sum_{j=1}^{N} a_j W_j \right\|_\infty}; \quad 1 \le s \le N, \quad \operatorname{supp}(\{a_j\}) \subset \sigma, \right\}.$$

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THEOREM 1. There exists an absolute constant $c > 0$ such that if $N = 2^r$ for some natural number r and $1 \le q \le N/2$, then

$$\nu\left\{\sigma \in S_N^q : U(\sigma) \le c \log\left(2 + \frac{q}{\log N}\right)\right\} < \frac{1}{N^2}.$$

Proof. When q is not very large compared with log N one can find a constant $c > 0$ such that

$$c \log\left(2 + \frac{q}{\log N}\right) < 1,$$

and hence the measure considered in the theorem is zero since $U(\sigma) \ge 1$ for any set σ. Hence we assume below that N is sufficiently large and $q \ge 20 \log N$.

Let $\delta = q/N$ and we note that $0 < \delta \le 1/2$. Further, let $\{\xi_i\}^N_{i=1}$ be a collection of independent random variables defined on a probability space (Ω, Σ, μ) and assuming values 0 or 1 with mean δ. For $\omega \in \Omega$ let

$$\sigma(\omega) = \{i;\, 1 \le i \le N,\, \xi_i(\omega) = 1\}.$$

We show below that for some constant $c > 0$

$$\mu\left\{\omega \in \Omega;\, U(\sigma(\omega)) < c \log\left(2 + \frac{q}{\log N}\right)\right\} \le \frac{5}{N^3}. \tag{*}$$

Thus we prove Theorem 1 since

$$\mu\{\omega \in \Omega;\, |\sigma(\omega)| = q = \delta N\} \ge \frac{B}{\sqrt{N}}$$

for some constant $B > 0$ that is independent of N and $\left(\frac{5}{N^3}\right)\Big/\left(\frac{B}{\sqrt{N}}\right) < \frac{1}{N^2}$ for N sufficiently large.

To prove (*) we need some auxiliary lemmas.

LEMMA 1. Let $b = (b_1, b_2, \ldots, b_{2^r})$ be a sequence such that for $s = 1, 2, \ldots, r - 1$ and some $\beta_0 > 0$ the set

$$\Delta_s = \left\{k;\, \frac{1}{2^{s+1}} < |b_k| \le \frac{1}{2^s}\right\}$$

has cardinality $|\Delta_s| \ge \beta_0 2^s$. Then for any sequence $a = (a_1, a_2, \ldots, a_{2^r})$ with $\|a\|_1 \le \lambda$, for some $1 \le \lambda \le (r - 2)\beta_0/8$ one has

$$\|a - b\|_2 \ge 2^{-4\lambda/\beta_0} \frac{\sqrt{\beta_0}}{8}.$$

Proof. We fix $a = (a_1, a_2, \ldots, a_{2^r})$ with $\|a\|_1 \le \lambda$ and for each s let $\lambda_s = \sum_{k\in\Delta_s} |a_k|$. Since

$$\lambda \ge \sum_{s=1}^{(8\lambda/\beta_0)+1} \lambda_s \ge \frac{8\lambda}{\beta_0} \min_{1\le s\le(8\lambda/\beta_0)+1} \lambda_s,$$

we can find a natural number $1 \le s_0 \le 8\lambda/\beta_0 + 1$ such that $\lambda_{s_0} \le \beta_0/8$. Let

$$\Delta'_{s_0} = \left\{k \in \Delta_{s_0};\, |a_k| \ge \frac{1}{2^{s_0+2}}\right\}$$

and we note that

$$\frac{\beta_0}{8} \ge \sum_{k\in\Delta'_{s_0}} |a_k| \ge \frac{|\Delta'_{s_0}|}{2^{s_0+2}},$$

i.e.,

$$|\Delta'_{s_0}| \le \frac{\beta_0 2^{s_0+2}}{8} = \frac{\beta_0 2^{s_0}}{2}.$$

Consequently

$$\|a-b\|_2^2 \geq \sum_{k\in\Delta_{s_0}} |a_k-b_k|^2 \geq \sum_{k\in\Delta_{s_0}\setminus\Delta'_{s_0}} \left(|b_k|-|a_k|\right)^2 \geq \frac{|\Delta_{s_0}\setminus\Delta'_{s_0}|}{2^{2s_0+4}} \geq \frac{\beta_0 2^{s_0}}{2^{2s_0+5}} = \frac{\beta_0}{2^{s_0+5}} = \frac{\beta_0}{32}\cdot\frac{1}{2^{s_0}},$$

and we get

$$\|a-b\|_2 \geq \sqrt{\frac{\beta_0}{32}}\cdot\frac{1}{2^{s_0/2}} \geq \frac{\sqrt{\beta_0}}{8}\frac{1}{2^{4\lambda/\beta_0}}.$$

LEMMA 2. Let us assume that $\{\varphi_i\}^m_{i=1}$ is a collection of elements of a Hilbert space H such that for some $0 < \varepsilon < 1$ and any vector $c = (c_1, c_2, \ldots, c_m)$ one has

$$(1-\varepsilon)\|c\|_2 \leq \left\|\sum_{i=1}^m c_i\varphi_i\right\| \leq (1+\varepsilon)\|c\|_2.$$

Then

$$(1-\varepsilon)^4\|c\|_2^2 \leq \sum_{k=1}^m \left|\left(\sum_{i=1}^m c_i\varphi_i, \varphi_k\right)\right|^2 \leq (1+\varepsilon)^4\|c\|_2^2$$

for the same $\varepsilon > 0$ and each $c \in l_2^m$.

Proof. Our assumption implies that

$$(1-\varepsilon)^2\|c\|^2 \leq \sum_{i,j=1}^m c_i c_j(\varphi_i, \varphi_j) \leq (1+\varepsilon)^2\|c\|_2^2$$

for any $c \in l_2^m$, i.e., the matrix $G = \{(\varphi_i, \varphi_j)\}^m_{i,j=1}$ is positive definite and its eigenvalues $(\lambda_1, \ldots, \lambda_m)$ satisfy the inequalities

$$(1-\varepsilon)^2 \leq \lambda_i \leq (1+\varepsilon)^2; \quad 1 \leq i \leq m.$$

Hence the eigenvalues of the matrix $GG^* = G^2$ lie between $(1-\varepsilon)^4$ and $(1+\varepsilon)^4$. In particular,

$$(1-\varepsilon)^4\|c\|_2^2 \leq (c, GG^*c) \leq (1+\varepsilon)^4\|c\|_2^2$$

for any $c \in l_2^m$. This finishes the proof since

$$\sum_{k=1}^m \left|\left(\sum_{i=1}^m c_i\varphi_i, \varphi_k\right)\right|^2 = \sum_{k=1}^m \sum_{i,j=1}^m c_i c_j(\varphi_i, \varphi_k)(\varphi_j, \varphi_k) = (c, GG^*c).$$

LEMMA 3. Under the hypotheses of Lemma 2, for the element $f = \Sigma_{i=1}^m c_i\varphi_i$ one has

$$\left\|f - \sum_{i=1}^m (f, \varphi_i)\varphi_i\right\| \leq 3\sqrt{\varepsilon}\left(\sum_{i=1}^m |(f, \varphi_i)|^2\right)^{1/2},$$

if ε is sufficiently small.

Proof. By Lemma 2

$$0 \leq \left\|f - \sum_{i=1}^m (f, \varphi_i)\varphi_i\right\|^2 = \|f\|^2 + \left\|\sum_{i=1}^m (f, \varphi_i)\varphi_i\right\|^2 - 2\sum_{i=1}^m |(f, \varphi_i)|^2$$

$$\leq \|f\|^2 - (1-\varepsilon)^4\|c\|_2^2 + (2\varepsilon+\varepsilon^2)\sum_{i=1}^m |(f, \varphi_i)|^2 \leq \left[(1+\varepsilon)^2 - (1-\varepsilon)^4 + (2\varepsilon+\varepsilon^2)(1+\varepsilon)^4\right]\|c\|_2^2.$$

Hence, if ε is sufficiently small, we get

$$0 \leq \left\|f - \sum_{i=1}^m (f, \varphi_i)\varphi_i\right\|^2 \leq 8.5\,\varepsilon\|c\|_2^2 \leq 9\,\varepsilon\sum_{i=1}^m |(f, \varphi_i)|^2.$$

LEMMA 4. There exists an $\varepsilon_0 > 0$ such that as soon as a collection $\{\varphi_i\}_{i=1}^m$ of elements of the Hilbert space H is given satisfying the condition

$$(1-\varepsilon)\|c\|_2 \le \Big\|\sum_{i=1}^m c_i\varphi_i\Big\|_H \le (1+\varepsilon)\|c\|_2,$$

for some $0 < \varepsilon < \varepsilon_0$ and any $c = (c_1, \ldots, c_m) \in l_2^m$, one has that for any $z \in H$ and any vector $c = (c_1, \ldots, c_m)$

$$\Big\|z - \sum_{i=1}^m c_i\varphi_i\Big\| \ge (1-\varepsilon)\Big[\sum_{i=1}^m ((z,\varphi_i) - c_i)^2\Big]^{1/2} - 3\sqrt{\varepsilon}\Big(\sum_{i=1}^m |(z,\varphi_i)|^2\Big)^{1/2}.$$

Proof. Let R be the orthogonal projection from H to $[\varphi_i]_{i=1}^m$. Then

$$\Big\|z - \sum_{i=1}^m c_i\varphi_i\Big\| \ge \Big\|Rz - \sum_{i=1}^m c_i\varphi_i\Big\|$$

and $(Rz, \varphi_i) = (z, \varphi_i)$; $1 \le i \le m$, i.e., it is enough to establish the inequality for Rz instead of z. In other words we can assume that $z \in L$. Then by our hypotheses and Lemma 3 we find that

$$\begin{aligned}\Big\|z - \sum_{i=1}^m c_i\varphi_i\Big\| &\ge \Big\|\sum_{i=1}^m [(z,\varphi_i) - c_i]\varphi_i\Big\| - \Big\|z - \sum_{i=1}^m (z,\varphi_i)\varphi_i\Big\| \\ &\ge (1-\varepsilon)\Big[\sum_{i=1}^m ((z,\varphi_i) - c_i)^2\Big]^{1/2} - 3\sqrt{\varepsilon}\Big(\sum_{i=1}^m |(f,\varphi_i)|^2\Big)^{1/2}.\end{aligned}$$

To estimate $U(\sigma)$ we need

LEMMA 5. Let $\{W_j\}_{j=1}^{N=2^r}$ be the first N Walsh functions defined on [0, 1], and let $\sigma \subset \{1, 2, \ldots, N\}$. Then

$$U(\sigma) = \max_{\substack{1 \le p \le N \\ z \in Z(\sigma)}} |(v_p, z)|,$$

where $v_p = (\overbrace{1, 1, \ldots, 1}^{p \text{ times}}, 0, \ldots, 0)$ and

$$Z(\sigma) = \Big\{z = (z_1, \ldots, z_N) \in \mathbb{R}^N,\ \mathrm{supp}(z) \subset \sigma,\ \Big\|\sum_{i=1}^N z_i W_i\Big\|_\infty \le 1\Big\}.$$

Proof. We fix $1 \le p \le N$ and $z \in Z(\sigma)$. Then for any $0 < \tau < 1/N$

$$|(v_p, z)| = \Big|\sum_{j=1}^p z_j\Big| = \Big|\sum_{j=1}^p z_j W_j(\tau)\Big| \le \Big\|\sum_{j=1}^p z_j W_j\Big\|_\infty \le U(\sigma).$$

To prove the opposite inequality we fix a function $f = \Sigma^N_{k=1}\hat{f}(k)W_k$ such that $\{\hat{f}(k)\} \subset \sigma$, $\|f\|_\infty = 1$ and for some $1 \le p \le N$ and $0 \le \tau_0 \le 1$ one has

$$U(\sigma) = \sum_{k=1}^p \hat{f}(k)W_k(\tau_0).$$

Let

$$g(x) = \sum_{k=1}^N \hat{f}(k)W_k(\tau_0)W_k(x), \qquad x \in [0, 1],$$

and we note that $g(x) = \Sigma^N_{k=1}\hat{f}(k)W_k(x \oplus_d \tau_0)$, where $x \oplus_d \tau_0$ means addition of x and τ_0 modulo 2 (cf. [7, p. 135]). Hence

$$g(x) = f(x \oplus_d x_0)$$

and so $\|g\|_\infty \le 1$. Consequently, $z^0 = \{\hat{f}(k)W_k(\tau_0)\}_{k=1}^N \in Z(\sigma)$ and thus

$$U(\sigma) = (v_p, z^0) \le \max_{\substack{1 \le p \le N \\ z \in Z(\sigma)}} |(v_p, z)|.$$

LEMMA 6. For $N = 2^r$ we consider the discrete Walsh system $W_i = (w_{i,j})^N_{j=1}$, $1 \le i \le n$, as a collection of vectors normalized in l_∞^N, i.e., such that $|w_{i,j}| = 1$ for all $1 \le i, j \le N$. In addition let $\{W^{(j)}\}_{j=1}^N$ be the columns of the Walsh matrix. Then for any $\sigma \subset \{1, 2, \ldots, N\}$

$$U(\sigma) = \inf\Big\{\lambda;\ \forall\, 1 \le p \le N,\ R_\sigma v_p = \sum_{j=1}^N \lambda_j R_\sigma W^{(j)} \quad \text{c} \quad \sum_{j=1}^N |\lambda_j| \le \lambda\Big\},$$

where $v_p = (\overbrace{1,\ldots,1}^{p \text{ times}},0,\ldots,0)$ and R_σ is the orthogonal projection operator to $[e_i]_{i\in\sigma}$ ($\{e_i\}^N_{i=1}$ denotes the canonical basis in $\mathbb{R}^N$).

Proof. First we show that for any $1 \le p \le N$ and $\sigma \subset \{1, 2, \ldots, N\}$

$$R_\sigma v_p \in \operatorname{conv}\{\pm U(\sigma) R_\sigma W^{(j)};\ 1 \le j \le N\}.$$

Indeed if this conclusion failed for some $1 \le p \le N$, then the standard argument based on the property of separability of convex sets would imply the existence of a vector $b = (b_1, \ldots, b_N)$ such that $(b, R_\sigma v_p) > 1$, but $|(b, U(\sigma)R_\sigma W^{(j)})| \le 1$ for all $1 \le j \le N$. Then we set $z = U(\sigma)R_\sigma b$ and note that for the p considered $|(z, v_p)| > U(\sigma)$. On the other hand, $|(z, W^{(j)})| \le 1$ for all $1 \le j \le N$, i.e., $z \in Z(\sigma)$, which by Lemma 5 implies $|(z, v_p)| \le U(\sigma)$, and we arrive at a contradiction.

As an immediate corollary we have

$$R_\sigma v_p = \sum_{j=1}^N \mu_j U(\sigma) R_\sigma W^{(j)},$$

for any $1 \le p \le N$ with $\Sigma^N_{j=1}|\mu_j| \le 1$. Consequently,

$$\inf\Big\{\lambda;\ \forall\, 1 \le p \le N,\ R_\sigma v_p = \sum_{j=1}^N \lambda_j R_\sigma W^{(j)} \quad \text{c} \quad \sum_{j=1}^N |\lambda_j| \le \lambda\Big\} \le U(\sigma).$$

On the other hand, if for some $1 \le p \le N$ we have that

$$R_\sigma v_p = \sum_{j=1}^N \lambda_j R_\sigma W^{(j)}$$

with $\Sigma^N_{j=1}|\lambda_j| \le \lambda$, then for $z \in Z(\sigma)$

$$|(v_p, z)| = |(R_\sigma v_p, z)| = \sum_{j=1}^N \lambda_j (z, W^{(j)}) \le \lambda \max_{1\le j\le N} |(z, W^{(j)})| \le \lambda,$$

i.e., $U(\sigma) \le \lambda$.

For completeness of the exposition we also cite the following familiar probabilistic result.

LEMMA 7. Let $0 < \delta \le 1/2$ and let $\{\xi_k\}^N_{k=1}$ be a collection of independent random variables defined on a probability space (Ω, Σ, μ) and assuming values 0 or 1 with mean δ. Then for any $|a_k| \le 1$, $1 \le k \le N$, and $0 \le \gamma \le \delta N$ we have

$$\mu\Big\{\omega \in \Omega;\quad \Big|\sum_{k=1}^N a_k(\xi_k(\omega) - \delta)\Big| \ge \gamma\Big\} \le 2e^{-\gamma^2/(4\delta N)}.$$

Proof. We fix $1 \le k \le N$ and let $X_k(\omega) = \xi_k(\omega) - \delta$ and we note that for $0 \le t \le 1$

$$\int_\Omega e^{tX_k(\omega)}\, d\mu(\omega) = \int_\Omega \Big[1 + tX_k(\omega) + \sum_{j=2}^\infty \frac{t^j}{j!} X_k^j(\omega)\Big]\, d\mu(\omega)$$

$$\le 1 + t^2 \int_\Omega X_k^2(\omega)\, d\mu(\omega) \sum_{j=2}^\infty \frac{1}{j!} = 1 + t^2\delta(1-\delta)(e-2) \le e^{t^2\delta(1-\delta)}.$$

In particular

$$\int e^{ta_k X_k(\omega)}\, d\mu(\omega) \le e^{t^2\delta(1-\delta)},$$

provided that $0 \leq t \leq 1$. Consequently, by Theorem 15 of [8, p. 52] we get that

$$\mu\Big\{\omega \in \Omega;\ \sum_{k=1}^{N} a_k(\xi_k(\omega) - \delta) \geq \gamma\Big\} \leq e^{-\gamma^2/(4\delta(1-\delta)N)} \leq e^{-\gamma^2/(4\delta N)}$$

provided $0 \leq \gamma \leq 2\delta(1 - \delta)N$, and in particular if $0 \leq \gamma \leq \delta N$. Lemma 7 is proved.

Before proving Theorem 1 we note that for $1 \leq j \leq N$

$$|(v_p, W^{(j)})| \leq \frac{N}{j} \tag{i}$$

for each $1 \leq p \leq N$, and if

$$p = [N/3] \quad \text{and} \quad b_j = \frac{(v_p, W^{(j)})}{N}, \quad 1 \leq j \leq N, \tag{ii}$$

then the cardinality of the set

$$\Delta_s = \Big\{k;\ \frac{1}{2^{s+1}} < |b_k| \leq \frac{1}{2^s}\Big\}$$

satisfies $|\Delta_s| \geq 2^s/4$ for all $0 < s < \log_2 N$.

For $\sigma \subset \{1, 2, \ldots, N\}$ and $p = [N/3]$ it follows from Lemma 6 that

$$R_\sigma v_p = \sum_{j=1}^{N} \nu_j^p(\sigma) R_\sigma W^{(j)},$$

where $\Sigma^N_{j=1}|\nu^p(\sigma)| \leq U(\sigma)$.

For fixed δ and N such that $\delta N \geq 20 \log N$, let

$$m = \Big[\Big(\frac{\delta N}{20 \log N}\Big)^{3/8}\Big],$$

and we note here that

$$\Big\|R_\sigma v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_\sigma W^{(j)}\Big\|_2^2 = \Big(\sum_{h=m+1}^{N} \nu_h(\sigma) R_\sigma W^{(h)}, R_\sigma v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_\sigma W^{(j)}\Big)$$

$$\leq U(\sigma) \max_{m<h\leq N} |(v_p, R_\sigma W^{(h)})| + U(\sigma)^2 \max_{\substack{1\leq j\leq m\\ m<h\leq N}} |(W^{(h)}, R_\sigma W^{(j)})|.$$

To estimate the right side of the last inequality from above we first apply Lemma 7 for fixed h, $m < h \leq N$, and $0 \leq \gamma \leq \delta N$, and we get

$$\mu\Big\{\omega \in \Omega;\ \Big|\sum_{i=1}^{p} w_{i,h}(\xi_i(\omega) - \delta)\Big| \geq \gamma\Big\} \leq 2e^{-\gamma^2/(4\delta N)},$$

where $\{\xi_i\}^N_{i=1}$, as usual, are (0, 1)-valued independent random variables with mean δ for some $0 < \delta < 1$, defined on a probability space (Ω, Σ, μ). Consequently,

$$\mu\Big\{\omega \in \Omega,\ \max_{m<h\leq N} \Big|\sum_{i=1}^{p} w_{i,h}(\xi_i(\omega) - \delta)\Big| \geq \gamma\Big\} \leq 2Ne^{-\gamma^2/(4\delta N)},$$

and so with probability $\geq 1-2Ne^{-\gamma^2/(4\delta N)}$ we have

$$\max_{m<h\leq N} |(R_\sigma v_p, W^{(h)}) - \delta(v_p, W^{(h)})| \leq \gamma.$$

By remark (i) above

$$\max_{m<h\le N} |(R_\sigma v_p, W^{(h)}| \le \frac{\delta N}{m} + \gamma$$

with probability $\ge 1-2Ne^{-\gamma^2/(4\delta N)}$.

Let $\gamma = \sqrt{20\delta N \log N}$ and we note that $0 < \gamma \le \delta N$. Then for $\varepsilon = (\delta N/20 \log N)^{-1/16}$ one has

$$m = \left[\varepsilon^2 \sqrt{\frac{\delta N}{20 \log N}}\right],$$

using which, we easily derive that

$$\max_{m<h\le N} |(R_\sigma v_p, W^{(h)}| \le \left(1 + \frac{1}{\varepsilon^2}\right)\sqrt{20\delta N \log N} \le \frac{2}{\varepsilon^2}\sqrt{20\delta N \log N}$$

with probability $\ge 1 - 2/N^4$.

An analogous calculation using Lemma 7 and the orthogonality of the Walsh matrix shows that

$$\mu\left\{\omega \in \Omega;\ \left|\sum_{i=1}^{N} w_{i,h} w_{i,j} \xi_i(\omega)\right| \ge \gamma\right\} \le 2e^{-\gamma^2/(4\delta N)}$$

for any $0 < \gamma \le \delta N$, $m < h \le N$, and $1 \le j \le m$. Hence with probability $\ge 1 - 2/N^3$ we have

$$\max_{\substack{1\le j\le m\\ m<h\le N}} |(W^{(h)}, R_\sigma W^{(j)})| \le \sqrt{20\delta N \log N}.$$

In sum we get that

$$\begin{aligned}\left\|R_\sigma v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_\sigma W^{(j)}\right\|_2^2 &\le \frac{2}{\varepsilon^2}\, U(\sigma)\sqrt{20\delta N \log N} + U(\sigma)^2\sqrt{20\delta N \log N}\\ &\le \left(\frac{2}{\varepsilon^2} + U(\sigma)\right) U(\sigma)\sqrt{20\delta N \log N}\end{aligned}$$

with probability $\ge 1 - 4/N^3$.

To estimate the left side of the last inequality from below we set

$$\varphi_j = \frac{R_\sigma W^{(j)}}{\sqrt{\delta N}}; \qquad 1 \le j \le m,$$

and we note that arguing as above we get

$$\mu\left\{\omega \in \Omega;\ \left|\sum_{i=1}^{N} w_{i,j} w_{i,l}(\xi_i(\omega) - \delta)\right| \ge \gamma\right\} \le 2e^{-\gamma^2/(4\delta N)}$$

for any $1 \le j, l \le m$. This implies that with probability $\ge 1 - 2/N^3$

$$\left|(R_\sigma W^{(j)}, R_\sigma W^{(l)}) - \delta(W^{(j)}, W^{(l)})\right| \le \sqrt{20\delta N \log N}, \qquad 1 \le j, l \le m.$$

Hence, with the same probability $\ge 1 - 2/N^3$

$$\left|(\varphi_j, \varphi_l) - \left(\frac{W^{(j)}}{\sqrt{N}}, \frac{W^{(l)}}{\sqrt{N}}\right)\right| \le \sqrt{\frac{20 \log N}{\delta N}}, \qquad 1 \le j, l \le m.$$

Hence for each vector $c = (c_j)^m_{j=1}$

$$\left| \left\| \sum_{j=1}^{m} c_j \varphi_j \right\|^2 - \|c\|_2^2 \right| \leq \sum_{j,l=1}^{m} |c_j||c_l| \left| (\varphi_j, \varphi_l) - \left(\frac{W^{(j)}}{\sqrt{N}}, \frac{W^{(l)}}{\sqrt{N}} \right) \right|$$

$$\leq \left(\sum_{j=1}^{m} |c_j| \right)^2 \sqrt{\frac{20 \log N}{\delta N}} \leq m\|c\|_2^2 \sqrt{\frac{20 \log N}{\delta N}} \leq \varepsilon^2 \|c\|_2^2,$$

i.e.,

$$(1-\varepsilon)\|c\|_2 \leq \left\| \sum_{j=1}^{m} c_j \varphi_j \right\| \leq (1+\varepsilon)\|c\|_2.$$

Consequently, using Lemma 4 applied when $z = R_\sigma v_p/\sqrt{\delta N}$ and $c_j = \nu_j^p(\sigma)$, $1 \leq j \leq m$, we get that

$$\left\| \frac{R_\sigma v_p}{\sqrt{\delta N}} - \sum_{j=1}^{m} \nu_j^p(\sigma) \frac{R_\sigma W^{(j)}}{\sqrt{\delta N}} \right\|_2 \geq (1-\varepsilon) \left(\sum_{j=1}^{m} \left| \left(\frac{R_\sigma v_p}{\sqrt{\delta N}}, \frac{W^{(j)}}{\sqrt{\delta N}} \right) - \nu_j^p(\sigma) \right|^2 \right)^{1/2}$$

$$- 3\sqrt{\varepsilon} \left(\sum_{j=1}^{m} \left| \left(\frac{R_\sigma v_p}{\sqrt{\delta N}}, \frac{R_\sigma W^{(j)}}{\sqrt{\delta N}} \right) \right|^2 \right)^{1/2},$$

i.e.,

$$\left\| R_\sigma v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_\sigma W^{(j)} \right\|_2 \geq (1-\varepsilon)\sqrt{\delta N} \left(\sum_{j=1}^{m} \left| \left(\frac{R_\sigma v_p}{\sqrt{\delta N}}, \frac{W^{(j)}}{\sqrt{\delta N}} \right) - \nu_j^p(\sigma) \right|^2 \right)^{1/2}$$

$$- \frac{3\sqrt{\varepsilon}}{\sqrt{\delta N}} \left(\sum_{j=1}^{m} \left| (R_\sigma v_p, R_\sigma W^{(j)}) \right|^2 \right)^{1/2}.$$

Previously we made the assumption that $q(\log N)^{-1} = \delta N(\log N)^{-1}$ is large enough (or equivalently that ε is small enough) that we can use Lemma 4. Repeating calculations made earlier we conclude that

$$\left| (v_p, R_\sigma W^{(j)}) - \delta(v_p, W^{(j)}) \right| \leq \sqrt{20\delta N \log N}, \quad 1 \leq j \leq m,$$

with probability $\geq 1 - 2/N^4$. This implies that

$$\begin{aligned}
&\left\| R_\sigma v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_\sigma W^{(j)} \right\| \\
&\geq (1-\varepsilon)\sqrt{\delta N} \left(\sum_{j=1}^{m} \left| \left(\frac{v_p}{\sqrt{N}}, \frac{W^{(j)}}{\sqrt{N}} \right) - \nu_j^p(\sigma) \right|^2 \right)^{1/2} \\
&\quad - (1-\varepsilon)\sqrt{\delta N} m^{1/2} \sqrt{\frac{20 \log N}{\delta N}} - \frac{3\sqrt{\varepsilon}}{\sqrt{\delta N}} \delta \left(\sum_{j=1}^{m} |(v_p, W^{(j)})|^2 \right)^{1/2} \\
&\quad - \frac{3\sqrt{\varepsilon}}{\sqrt{\delta N}} m^{1/2} \sqrt{20\delta N \log N} \\
&\geq (1-\varepsilon)\sqrt{\delta N} \left(\sum_{j=1}^{m} \left| \left(\frac{v_p}{\sqrt{N}}, \frac{W^{(j)}}{\sqrt{N}} \right) - \nu_j^P(\sigma) \right|^2 \right)^{1/2} \\
&\quad - 3\sqrt{\frac{\varepsilon\delta}{N}} \left(\sum_{j=1}^{m} |(v_p, W^{(j)})|^2 \right)^{1/2} \\
&\quad - (1-\varepsilon)\varepsilon(20\delta N \log N)^{1/4} - 3\varepsilon^{3/2}(20\delta N \log N)^{1/4}
\end{aligned}$$

$$\geq (1-\varepsilon)\sqrt{\delta N}\Big(\sum_{j=1}^{m}\Big|\Big(\frac{v_p}{\sqrt{N}},\frac{W^{(j)}}{\sqrt{N}}\Big)-\nu_j^p(\sigma)\Big|^2\Big)^{1/2}$$
$$-3\sqrt{\frac{\varepsilon\delta}{N}}\Big(\sum_{j=1}^{m}|(v_p,W^{(j)})|^2\Big)^{1/2}-4\varepsilon(20\delta N\log N)^{1/4}.$$

By remark (ii) above and Lemma 1,

$$\Big(\sum_{j=1}^{m}\Big|\Big(\frac{v_p}{\sqrt{N}},\frac{W^{(j)}}{\sqrt{N}}\Big)-\nu_j^p(\sigma)\Big|^2\Big)^{1/2}\geq\frac{1}{16}\Big[\frac{1}{2^{16U(\sigma)}}-\frac{1}{N^{1/3}}\Big]$$

provided that $U(\sigma) \leq (r-2)/32$, which lets us use Lemma 1. Now if $U(\sigma) > (r-2)/32$, then the last inequality obviously holds since the right side becomes negative. In addition, by Bessel's inequality,

$$\Big(\sum_{j=1}^{m}|(v_p,W^{(j)})|^2\Big)^{1/2}\leq N.$$

Hence with probability $\geq 1 - 5/N^3$

$$\Big(\frac{2}{\varepsilon^2}+U(\sigma)\Big)(20\delta N\log N)^{1/4}\geq\Big(\frac{2}{\varepsilon^2}+U(\sigma)\Big)^{1/2}U(\sigma)^{1/2}(20\delta N\log N)^{1/4}$$
$$\geq\frac{(1-\varepsilon)(\delta N)^{1/2}}{16\cdot 2^{16U(\sigma)}}-\frac{(1-\varepsilon)(\delta N)^{1/2}}{16N^{1/3}}-3(\varepsilon\delta N)^{1/2}-4\varepsilon(20\delta N\log N)^{1/4}$$
$$\geq\frac{(\delta N)^{1/2}}{30\cdot 2^{16U(\sigma)}}-4(\varepsilon\delta N)^{1/2}.$$

To finish the proof of (*) and hence of Theorem 1 it is enough to verify that the inequalities given above imply that

$$U(\sigma)\geq c\log\Big(\frac{\delta N}{\log N}\Big)$$

for some absolute positive constant c. Indeed if $\frac{1}{30\cdot 2^{16U(\sigma)}}\leq 10\varepsilon^{1/2}$, then $30\cdot 2^{16U(\sigma)}\geq\frac{1}{10}\Big(\frac{\delta N}{20\log N}\Big)^{1/32}$, i.e.,

$$\log 30+16U(\sigma)\log 2\geq\frac{1}{32}\log\Big(\frac{\delta N}{20\log N}\Big)-\log 10.$$

Consequently,

$$U(\sigma)\geq c\log\Big(\frac{\delta N}{\log N}\Big),$$

where $c > 0$ is an absolute constant. On the other hand, if $\frac{1}{30 2^{16U(\sigma)}}\geq 10\varepsilon^{1/2}$, then since

$$\frac{2}{\varepsilon^2}=2\Big(\frac{\delta N}{20\log N}\Big)^{1/8}\gg U(\sigma),$$

we get that

$$3\cdot\Big(\frac{\delta N}{20\log N}\Big)^{1/8}(20\delta N\log N)^{1/4}\geq\frac{(\delta N)^{1/2}}{60\cdot 2^{16U(\sigma)}}$$

or

$$180\cdot 2^{16U(\sigma)}\geq\Big(\frac{\delta N}{20\log N}\Big)^{1/8},$$

i.e., again

$$U(\sigma)\geq c\log\Big(\frac{\delta N}{\log N}\Big).$$

Now we consider the case of a trigonometric system. For $U(\sigma)$ defined at the beginning of the paper one has an estimate analogous to Theorem 1.

THEOREM 2. There exists an absolute constant $b > 0$ such that for $N = 2, 3, \ldots$ and $1 \leq q \leq N$

$$\nu\left\{\sigma \in S^q_{2N+1};\ U(\{\sigma - N - 1\}) \leq b \log\left(2 + \frac{q}{\log N}\right)\right\} < \frac{1}{N^2}.$$

We have used the usual notation above: the set

$$\sigma - N - 1 = \{k - N - 1;\quad k \in \sigma\}$$

is a subset of the set

$$\{-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N\}.$$

Proof of Theorem 2. The first step in the proof of Theorem 2 is to consider for given N the discrete trigonometric system

$$\varphi_k = \left\{\varphi_{k,j} = e^{2\pi i k j/(2N+1)}\right\}_{j=0}^{2N} \in \mathbb{C}^{2N+1}, \qquad -N \leq k \leq N.$$

As in the previous case we denote by $\varphi^{(j)}$, $0 \leq j \leq 2N$, the columns of the matrix $\{\varphi_{k,j}\}_{k=-N,j=0}^{N\ \ 2N}$.

Repeating the proof of Theorem 1 for the system $\{\varphi_k\}^N_{k=-N}$ in place of the Walsh system $\{W_k\}^N_{k=1}$ and for $p = [N/2]$ instead of $p = [N/3]$, we find that $\nu(A) \geq 1 - 1/N^2$, where A is the set of all subsets $\sigma \subset S^q_{2N+1}$ for which one can find a sequence $a^\sigma = (a_k^\sigma)^N_{k=-N}$ with support in $\sigma - N - 1$ such that

$$\left\|\sum_{k=-N}^{N} a_k^\sigma \varphi_k\right\|_\infty = 1 \quad \text{and} \quad \left\|\sum_{k=-p}^{p} a_k^\sigma \varphi_k\right\|_\infty \geq c \log\left(2 + \frac{q}{\log N}\right),$$

where $c > 0$ is an absolute constant. Here, in contrast with the Walsh system, we consider the symmetric vectors

$$v_p = (\overbrace{0, 0, \ldots, 0}^{N-p\text{ times}}, \overbrace{1, 1, \ldots, 1}^{2p+1\text{ times}}, \overbrace{0, 0, \ldots, 0}^{N-p\text{ times}}) \in \mathbb{R}^{2n+1},$$

which obviously satisfy the relation

$$(v_p, \varphi^{(j)}) = D_p(2\pi j/(2N+1))$$

for all $0 \leq j \leq 2N$, where D_p denotes the usual complex Dirichlet kernel.

To derive Theorem 2 from the discrete case analyzed above, for any $\sigma \in A$ we consider the polynomial $t^\sigma(x) = \sum_{k=-N}^{N} a_k^\sigma e^{2\pi i k z}$ and its de la Vallee Poussin mean:

$$T^\sigma(x) = \frac{1}{p} \sum_{n=p}^{2p-1} \sum_{k=-n}^{n} a_k^\sigma e^{2\pi i k x}.$$

Then, using standard properties of the de la Vallee Poussin kernel, we easily get that $\|T_\sigma\|_\infty \leq 10$. On the other hand,

$$\left\|\sum_{k=-p}^{p} \widehat{T}^\sigma(n) e^{2\pi i k x}\right\|_\infty = \left\|\sum_{k=-p}^{p} a_k^\sigma e^{2\pi i k x}\right\|_\infty \geq \left\|\sum_{k=-p}^{p} a_k^\sigma \varphi_k\right\|_\infty \geq c \log\left(2 + \frac{q}{\log N}\right).$$

Consequently, for any $\sigma \in A$

$$U(\sigma) \geq \frac{c}{10} \log\left(2 + \frac{q}{\log N}\right).$$

Remark. The approach used in the proof of Theorem 1 can also be applied for other orthogonal systems. In particular, without changing the proof for each permutation of a discrete trigonometric system.

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