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ON BEST m-TERM APPROXIMATIONS AND THE ENTROPY OF SETS IN THE SPACE L^1

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In this paper, using results on the geometric properties of finite-dimensional convex bodies, we investigate approximate characteristics of sets of the spaces $L^1(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$, $p > 1$. The paper contains two sections. In Sec. 1, for a wide class of systems $\Phi = \{\varphi_n(x)\}$ we establish lower bounds for best approximations of functions of Sobolev classes by polynomials of the form

$$\sum_{i=1}^{m} a_{n_i}\varphi_{n_i}(x), \qquad 1 \le n_1 < n_2 < \cdots < n_m;$$

here the coefficients a_{n_i} and indices n_i depend, in general, on the function approximated.

In Sec. 2, we establish lower bounds for the ε-entropy, widths, and best m-term triconometric approximations in classes of functions of many variables with bounded mixed derivative or difference. In particular, the method developed gives the possibility of obtaining order-precise lower bound for entropy numbers of the class W_p^r in the space L^q for $p = \infty$, $q = 1$, and even r. As is known, for the problems considered, obtaining exact results in "extreme cases," i.e., where the parameters p and q take the extreme values 1 and ∞, is usually the most complicated.

1. On best m-term approximations

Let D be a bounded domain in $\mathbb{R}^d$, $d = 1, 2, \ldots$, let $1 \le p \le \infty$, and let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of functions of the space $L^p(D)$. Given a function $f \in L^p(D)$, we put

$$\sigma_m(f,\Phi)_p = \inf_{\substack{\{n_i\}=\Lambda\subset\mathbb{Z}^+,|\Lambda|=m,\\ \{c_i\}\in\mathbb{R}^m}} \left\| f(x) - \sum_{i=1}^{m} c_i\varphi_{n_i}(x) \right\|_{L^p(D)}, \tag{1}$$

where $|\Lambda|$ is the number of elements of the finite set Λ. Next, if $K \subset L^p(D)$ is some class of functions, then we put

$$\sigma_m(K,\Phi)_p = \sup_{f\in K} \sigma_m(f,\Phi). \tag{2}$$

Quantity (1) is called the best m-term approximation of the function f, relative to the system Φ, in $L^p(D)$. The difference between this quantity and the quantity of the usual best approximation, given by

$$E_m(f,\Phi) = \inf_{\varphi\in\Phi(N)} \|f - \varphi\|_{L^p(D)},$$

where $\Phi(N) = \left\{\sum_{i=1}^{m} c_i\varphi_i(x)\right\}$ is the space of polynomials of order $\le m$ relative to the system Φ, consists in the possibility of choosing the spectrum of the approximating polynomial, i.e., the set Λ (see (1)), depending on the approximated function f. It is clear that $\sigma(f,\Phi)_p \le E_m(f,\Phi)_p$. Yet, in the seventiees (see [1, 2]),

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it had been established that for many natural functional compacta K and systems Φ the quantities (2) decrease as $m \to \infty$ much more rapidly than

$$E_m(K, \Phi) = \sup_{f \in K} E_m(f, \Phi).$$

In recent years, interest in the study of m-term approximations has been renewed, in particular, in connection with their applications to "image processing" problems. Moreover, in the opinion of some authors (see [3, 4]), the case $p = 1$ is particularly interesting for applications. It is this case that we consider below. Under certain assumptions on the properties of the system Φ, we establish, in particular, the lower bound of quantity (2) in the case where $p = 1$ and

$$K = \operatorname{Lip} \alpha = \{f : \|f\|_\infty \le 1,\ |f(x) - f(y)| \le |x - y|^\alpha,\ 0 \le x, y \le 1\}, \qquad 0 < \alpha \le 1;$$

this bound shows that in the case considered the use of m-term approximations has no essential advantages in comparison with the usual approximations. We mention at once that the study of quantities (1) and (2) makes sense only in the case where the system Φ has some "minimality property" (otherwise, the functions $\{\varphi_n\}$ form a dense set on a sphere in $L^p(D)$ and then $\sigma_1(f, \Phi) = 0$ for any $f \in L^p(D)$). A usual property of such a type is the orthogonality of the system Φ. As shown in [5], in considering approximations in $L^2(D)$ the assumption of the orthogonality of the system Φ is sufficient for obtaining nontrivial lower bounds of $\sigma_m(K, \Phi)$, depending on the geometric properties of the functional class K. For instance, we established [5] that

$$\sigma_m(\operatorname{Lip} \alpha, \Phi)_2 \ge c m^{-\alpha}, \qquad c > 0, \qquad m = 1, 2, \dots$$

for any orthonormalized system Φ. In passing to approximations in $L^1(D)$ the situation is changed. This is illustrated clearly by the following statement (the proof is given at the end of the section).

Proposition 1. *There exists a complete orthonormal system $\Phi = \{\varphi_n \in L^2(0,1)\}$ such that the set $A(\Phi) = \{\lambda\varphi_n,\ \lambda > 0,\ n = 1, 2, \dots\}$ is dense in $L^1(0,1)$ and, consequently, $\sigma_1(f, \Phi)_1 = 0$ for any function $f \in L^1(0,1)$.*

Thus, to obtain nontrivial lower bounds for $\sigma_m(K, \Phi)$ we need to restrict the class of the systems Φ under consideration. We assume that the system $\Phi = \{\varphi_n\}$ satisfies the following two conditions:

(I) There exist positive constants K_1, K_2, K_3, such that for each $N = 1, 2, \dots$ there exists a finite set $\Omega_N \subset D$ such that $|\Omega_N| \le K_1 \cdot N$, and for any function $\varphi \in \Phi(N)$ and $1 \le p \le \infty$ the following inequalities hold:

$$K_2 \left(\frac{1}{|\Omega_N|} \sum_{x \in \Omega_N} |\varphi(x)|^p \right)^{1/p} \le \|\varphi\|_p \le K_3 \left(\frac{1}{|\Omega_N|} \sum_{x \in \Omega_N} |\varphi(x)|^p \right)^{1/p}. \tag{3}$$

(II) There exist constants K_4, K_5, such that for $N = 1, 2, \dots$, $m = 1, 2, \dots$, $1 \le p \le \infty$, and for any function $\varphi \in \Phi(N)$, the following inequality holds:

$$\sigma_m\left(\varphi, \{\varphi_n\}_{n=1}^{K_4 N}\right) \le K_5 \sigma_m(\varphi, \Phi).$$

Note that conditions (I) and (II) are valid for the known classical orthonormal systems, the trigonometric, Haar, Walsh, and Francklin systems, as well as for systems of the "wavelet" type.

Systems possessing property (3) for $p = \infty$ (when the norms $\|\varphi\|_p$ and $\sup_{x \in \Omega_N} |\varphi(x)|$ are considered to be equivalent) are called [6] quasi-matrix systems. Conditions of type (I) arise naturally in many problems of function theory. For a triconometric system, condition (I) was established by Martsinkevich (see [7]). For a system Φ of linearly independent functions, condition (II) is satisfied whenever the following condition holds:

(II′) For any $n = 1, 2, \dots$ there exists a multiplier $\Lambda_N = \{\lambda_{N,n}\}$ (i.e., a linear operator defined by $\Lambda_N(\varphi_n) = \lambda_n \varphi_n$, $n = 1, 2, \dots$, on polynomials relative to Φ) such that $\lambda_{N,n} = 1$ for $n \le N$, $\lambda_{N,n} = 0$ for $n > K_4 N$, and

$$\|\Lambda_N(\varphi)\|_p \le K_5 \|\varphi\|_p, \qquad 1 \le p \le \infty,$$

for any polynomial relative to Φ.

For $p \in [1, \infty]$, $N = 1, 2, \ldots$, we put

$$\Phi(N)_p = \{f \in \Phi(N) : \|f\|_{L^p(D)} \le 1\}.$$

The following lemma holds.

Lemma 1. *Let a system Φ of linearly independent functions satisfy conditions* (I) *and* (II). *Then there exist constants $\gamma > 0$, $c > 0$, such that*

$$\sigma_m(\Phi(n)_\infty, \Phi)_1 \ge c \tag{4}$$

for $n = 1, 2, \ldots$ and $m < \gamma n$.

Proof. Using conditions (I), (II), and the linear independence of functions of Φ, we reduce the problem of estimating the left-hand side of (4) to the corresponding problem in the finite-dimensional space $\mathbb{R}^M$. Below we denote by B_p^M the unit ball in ℓ_p^M, and by $B_{p,r}^M(x)$ the set $x + r \cdot B_p^M$. We introduce a discrete analog of quantitites (1), (2). Let $U = \{u_i\}_{i=1}^n$ be a set of vectors of $\mathbb{R}^M$. For $m \le n$ we put

$$\sigma_m(x, U)_1 = \inf_{\{c_i\}, \{k_i\}} \left\| x - \sum_{i=1}^m c_i u_{k_i} \right\|_{\ell_1^M}, \tag{5}$$

$$\sigma_m(G, U)_1 = \sup_{x \in G} \sigma_m(x, U)_1, \tag{6}$$

where the lower bound in (5) is taken over all m-element subsets $\{k_i\}_{i=1}^m$ of the set $\{1, \ldots, n\}$, and over all the coefficients $\{c_i\}$.

According to condition (II), to prove Lemma 1 it is sufficient to establish the inequality

$$\sigma_m(\Phi(n)_\infty, \{\varphi_i\}_{i=1}^{K_4 n})_1 > c' > 0 \tag{7}$$

under assumption that $m \le \gamma n$.

Given a number n, we denote by P the operator of restriction of φ to the finite set $\Omega_{K_4 n} = \{x_i\}$ (see condition (I)),

$$P(\varphi) = \{\varphi(x_i)\}_{i=1}^M \in \mathbb{R}^M, \qquad M = |\Omega_{K_4 n}| \le K_1 \cdot K_4 \cdot n.$$

Using now condition (I), we see that inequality (7) is valid in turn, if for any set $U = \{u_i\}_{i=1}^M \in \mathbb{R}^M$ and for $m \le n/2$ the following inequality holds:

$$\sigma_m(P(n), U)_1 \ge cM > 0, \tag{8}$$

where $P(n) = \{P(\varphi),\ \varphi \in \Phi(n)_\infty\}$, and the constant c depends only on K_i, $1 \le i \le 5$.

Since the functions of Φ are linearly independent and condition (I) holds for $p = \infty$, we see that $\{P(\varphi),\ \varphi \in \Phi(n)\}$ is an n-dimensional subspace of $\mathbb{R}^M$, and $P(n)$ contains the n-dimensional cross section of the cube $K_3^{-1} B_\infty^M$. Thus, to prove Lemma 1 it is sufficient to prove the following statement.

Lemma 2. *Let n and M be positive integers connected by the inequalities $\alpha M < n \le M$. Also let $m \le n/2$, let G_n be an arbitrary n-dimensional cross section of the cube B_∞^M, and let $U = \{u_i\}$ be any system of vectors of $\mathbb{R}^M$. Then the following inequality is valid:*

$$\sigma_m(G_n, U)_1 \ge cM, \qquad c = c(\alpha) > 0.$$

Taking into account the known estimates of the volume of the cube cross sections (see [8] and also [9, 10]), we see that Lemma 2 follows in turn from a more general statement.

Theorem 1. *Let n and M be positive integers connected by the inequalities $\alpha M \le n \le M$, and let Ω be a family of systems $U^j = \{u_i^j\}_{i=1}^M \subset \mathbb{R}^M$, $j = 1, \dots, s$, $s \le K^M$. Also let $L_n \subset \mathbb{R}^M$ be an n-dimensional subspace, $G \subset L_n \cap B_2^M$, and*

$$\operatorname{Vol}_n G \ge \beta^n \operatorname{Vol}_n B_2^n, \qquad \beta > 0.$$

Then for $m \le n/2$ we have

$$\rho_m \equiv \sup_{x \in G} \inf_j \sigma_m(x, U^j)_1 > cM^{1/2}, \qquad c = c(\alpha, \beta, K) > 0.$$

Remark 1. We have denoted by $\operatorname{Vol}_n G$ the n-dimensional Lebesgue measure (the volume) of the set $G \subset L \subset \mathbb{R}^M$, $\dim L = n$. If G is an n-dimensional cross section of the cube B_∞^M, then, applying Theorem 1 and using the inclusion $G \subset M^{1/2} \cdot (B_2^M \cap L)$ and the estimate $\operatorname{Vol}_n G \ge 1 \ge c^n M^{n/2} \operatorname{Vol} B_2^n$ (see [8]), we arrive at the conclusion of Lemma 2.

Remark 2. Theorem 1 shows that under its hypotheses, even if we fix a wide class of systems and choose in this class a system with respect to which the m-term approximation is taken, depending on the approximated element, we do not obtain an essential improvement of the approximation in comparison with the trivial method of approximation of any element of G by the zero element.

Remark 3. Under the hypotheses of Theorem 1, instead of systems U^j of vectors consisting of M elements one can consider systems of vectors consisting of $b \cdot M$ elements (b is a fixed constant). This case is reduced to the previous one if instead of $U^j = \{u_i^j\}_{i=1}^{bM}$ we consider all M-term sets of u_i^j as separate systems of vectors. Then their number increases $\le [c(b)]^M$ times, which influences only the value of the constant K.

Proof Theorem 1. Evidently, it is sufficient to consider the case $m = [n/2]$. Put $\rho = \rho_{[n/2]}$. Given a number j, we denote by U_m^j the set of all subspaces $X \subset \mathbb{R}^m$ generated by m elements of the system U^j,

$$U_m^j = \{X : X = \operatorname{span}(\{u_{i_k}^j\}_{k=1}^m)\}.$$

By the definition of ρ (see the formulation of Theorem 1), the following inclusion holds:

$$G \subset \bigcup_{j=1}^s \bigcup_{X \subset U_m^j} (X + B_{1,\rho}^M);$$

moreover, by the assumption, $G \subset L_n \cap B_2^M$. Thus,

$$G \subset \bigcup_{j=1}^s \bigcup_{X \subset U_m^j} \{(X + B_{1,\rho}^M) \cap L_n \cap B_2^n\}. \tag{9}$$

For fixed j and X, we estimate the n-dimensional volume of the set

$$H = (X + B_{1,\rho}^M) \cap L_n \cap B_2^M.$$

It is clear that

$$H \subset F = P_{L_n}\{(X + B_{1,\rho}^M)\} \cap B_2^M, \tag{10}$$

where P_L is the operator of orthogonal projection onto $L \subset \mathbb{R}^M$. Further,

$$P_{L_n}(X + B_{1,\rho}^M) = Q + B'_\rho,$$

where $Q = P_{L_n}(X)$, $B'_\rho = P_{L_n}(B_{1,\rho}^M)$. Moreover, $l = \dim Q \le \dim X \le m$ and each element $x \in F$ can be represented in the form

$$x = q + y, \qquad q \in Q, \quad y \in B'_\rho, \quad \|x\|_2 \le 1. \tag{11}$$

Let $Q^\perp$ be the orthogonal complement of Q to L_n. (Note that $\dim Q^\perp \geq n - m$.) Let us represent the element y in (11) in the form $y = y - P_{Q^\perp} y + P_{Q^\perp} y$. Then we have

$$x = (q + y - P_{Q^\perp}(y)) + P_{Q^\perp}(y)$$

and

$$q + y - P_{Q^\perp}(y) \in Q \cap B_2^M, \qquad P_{Q^\perp}(y) \in P_{Q^\perp}(B'_\rho).$$

Therefore, the n-dimensional volume of F is estimated as

$$\operatorname{Vol}_n F \leq \operatorname{Vol}_l B_2^l \cdot \operatorname{Vol}_{n-l} P_{Q^\perp}(B'_\rho). \tag{12}$$

Since $Q^\perp \subset L_n$, by the definition B'_ρ we have

$$P_{Q^\perp}(B'_\rho) = P_{Q^\perp} P_{L_n}(B_{1,\rho}^M) = P_{Q^\perp}(B_{1,\rho}^M).$$

We now use the following known estimate:

$$\operatorname{Vol}_r P_Y(B_1^M) \leq c_\gamma^r \cdot r^{-r}, \qquad Y \subset \mathbb{R}^M, \quad \dim Y = r \geq \gamma M. \tag{13}$$

Taking into account that $P_Y(B_1^M)$ is a convex polyhedron of $\leq 2M$ vertices, inscribed in an r-dimensional Euclidean ball (see [11, 12]), inequality (13) follows from the known estimates for volumes of polyhedrons. We also note that (13) follows directly from the simplest estimate of cardinality of the covering of the octahedron B_1^M by Euclidean balls, and that there exist an absolute constant K and points $x_1, \ldots, x_k \in \mathbb{R}^M$, $k \leq K^M$, such that

$$B_1^M \subset \bigcup_{i=1}^{K^M} B_{2,M^{-1/2}}^M(x_i)$$

(see, e.g., [13, p. 56]).

By inequalities (12) and (13), we obtain

$$\operatorname{Vol}_n F \leq c^n l^{-l/2} \rho^{n-l} (n-l)^{-(n-l)}. \tag{14}$$

The conditions of Theorem 1, inclusion (9), and estimate (14) imply that

$$(n-l)^{-n/2} < 2^{n/2} n^{-n/2} < (c')^n l^{-l/2} (n-l)^{-(n-l)} \rho^{n-l} \tag{15}$$

(we also used the condition $l \leq n/2$). Comparing the left-hand and right-hand sides of (15) and taking into account that the function y^y is bounded below by a positive constant for $y > 0$, we obtain

$$\rho \geq c'' n^{1/2} \geq c M^{1/2}.$$

Theorem 1 has been proved.

Remark 4. It is easy to see that in the formulation and in the proof of Theorem 1 the norm ℓ_1^M can be changed to any norm defined on $\mathbb{R}^M$ such that an analog of inequality (13) holds,

$$\operatorname{Vol}_r(P_Y(B)) \leq c_\gamma^M \operatorname{Vol}_r(B_2^M \cdot M^{-1/2} \cap Y), \qquad \dim Y = r \geq \gamma M, \tag{$\star$}$$

where B is the unit ball in $\mathbb{R}^M$. In turn, the last estimate is valid if, for example, the Euclidean ball $M^{-1/2} \cdot B_2^M$ is an ellipsoid of maximal volume, inscribed in B, and $\operatorname{Vol}(B) \leq C^M \cdot M^{-M/2} \cdot \operatorname{Vol}(B_2^M)$.

Now we give consequences of Theorem 1 and Lemma 1 for approximations of functional classes. We begin with the one-dimensional case.

Theorem 2. *Let a system Φ of linearly independent functions on $D = (0,1)$ satisfy conditions* (I) *and* (II). *Suppose that for some $r > 0$ the functions of Φ satisfy the "Bernstein inequality," i.e., there exists a constant $c = c(r)$ such that for any $N = 1, 2, \ldots$ and $\varphi \in \Phi(N)$,*

$$\|\varphi^{(r)}\|_\infty \le c \cdot N^r \|\varphi\|_\infty \tag{16}$$

(for fractional r the derivative is considered as the Weyl derivative). Then for the class

$$W_\infty^r = \{f : \|f\|_\infty + \|f^{(r)}\|_\infty \le 1\}$$

the following lower bound holds:

$$\sigma_m(W_\infty^r, \Phi)_1 \ge c_1 \cdot m^{-r}, \qquad m = 1, 2, \ldots . \tag{17}$$

Proof. By inequality (16), we have

$$\Phi(N)_\infty \subset c_2 n^r \cdot W_\infty^r. \tag{18}$$

Let γ be the constant defined in Lemma 1. Given a number m, we put $n = [m/\gamma] + 1$ and apply Lemma 1. Then, by (4) and (18), we arrive at estimate (17).

As a consequence of Theorem 2, we get the lower bound for best m-term triconometric approximations, established in [14]

$$\sigma_m(W_\infty^r, \mathbb{T})_1 \ge c(r) m^{-r}, \qquad m = 1, 2, \ldots .$$

In the multidimensional (d-dimensional) case, the following analog of Theorem 2 is valid.

Theorem 3. *Let a system Φ of linearly independent functions on a domain $D \subset \mathbb{R}^d$ satisfy conditions* (I) *and* (II). *Suppose that for some positive integer $r > 0$ and for a vector $l = (l_1, \ldots, l_d)$, such that $l_1 + \cdots + l_d \le r$ and $l_i \ge 0$ are integers, the following inequalities hold:*

$$\|D^l \varphi\|_\infty \equiv \left\| \frac{\partial^{l_1 + \cdots + l_d}}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} \right\|_\infty \le c N^{r/d} \|\varphi\|_\infty, \qquad \varphi \in \Phi(N), \quad N = 1, 2, \ldots .$$

Then for the isotropic Sobolev class

$$SW_\infty^r = \left\{ f : \sum_{\|l\|_1 \le r} \|D^l f\|_\infty \le 1 \right\}$$

the following lower bound holds:

$$\sigma_m(SW_\infty^r, \Phi)_1 \ge c_1 m^{-r/d}.$$

The proof of this theorem repeats the proof of Theorem 2.

We now give an application of Theorem 1. As above, we assume that D is a bounded domain in $\mathbb{R}^d$, M is a positive integer, $\{D_k\}_{k=1}^M$ is a partition of D (up to a set of measure zero) into subdomains such that

$$\alpha \frac{|D|}{M} \le |D_k| \le \beta \frac{|D|}{M},$$

where $|E|$ is the Lebesgue measure of E. Further, we denote the space of functions constant on subdomains D_k, $k = 1, \ldots, M$, by $S^M = S(\{D_k\}_{k=1}^M)$, and the unit L^∞-ball in this space by $S_\infty^M = \{f \in S^M : \|f\|_\infty \le 1\}$.

Theorem 4. *Let $U^j = \{u_i^j\}_{i=1}^M$, $j = 1, \dots, s$, $s \le K^M$, be a family of systems of functions of S^M. Then for all $m \le M/2$ we have*

$$\sup_{f \in S_\infty^M} \inf_j \sigma(f, U^j)_1 \ge c, \qquad c = c(\alpha, \beta, K) > 0. \tag{19}$$

Proof. Let $\varphi \in S^M$ and $\varphi(x) = \varphi_k$ for $x \in D_k$, $k = 1, 2, \dots, M$. Then for $1 \le p \le \infty$ we have

$$\|\varphi\|_p = \left(\sum_{k=1}^M |\varphi_k|^p |D_k| \right)^{1/p} \asymp \left(\frac{1}{M} \sum_{k=1}^M |\varphi_k|^p \right)^{1/p}. \tag{20}$$

Associating with each function $\varphi \in S^M$ the vector $(\varphi_1, \dots, \varphi_M) \in \mathbb{R}^M$, we reduce the problem to the "finite-dimensional case." Then, by Theorem 1 (with $n = M$ and $G = B_\infty^M$) and relation (20), we obtain estimate (19).

We conclude this section with the proof of Proposition 1.

Proof Proposition 1. Obviously, the completeness of the system Φ should not be given consideration; an incomplete system Φ with dense $A(\Phi)$ can always be completed. It is also easy to see that it is sufficient to construct an orthonormal system Φ for which $A(\Phi)$ is dense on the unit sphere of $L^2(0,1)$ in the norm of L^1.

In fact, for any $\varepsilon > 0$ and for any function $f \in L^1(0,1)$ there exists $g \in L^2(0,1)$ such that $\|g\|_2 > 0$ and $\|f - g\|_{L_1} < \varepsilon/2$. If, in turn, for a given g there exist n and $\lambda > 0$ such that

$$\left\| \frac{g}{\|g\|_2} - \lambda \varphi_n \right\|_{L^1} < \frac{\varepsilon}{2\|g\|_2},$$

then $\left\| f - \lambda \|g\|_2 \varphi_n \right\|_{L^1} < \varepsilon$.

Let $g_n(x)$, $\|g_n\|_2 = 1$, $n = 1, 2, \dots$, be a sequence of functions, dense on the unit sphere of $L^2(0,1)$ in the norm of $L^2(0,1)$, such that

$$g_n(x) = 0 \quad \text{for} \quad x \in \left(0, \tfrac{1}{n+1}\right), \quad n = 1, 2, \dots, \tag{21}$$

(the existence of such a sequence can easily be established). We put

$$f_n(x) = g_n(x) \cdot 10^{-n}, \qquad n = 1, 2, \dots. \tag{22}$$

It is easy to see that

$$\left\| \{f_n\}_{n=1}^\infty \right\|_{(0,1)} \le \tfrac{2}{10},$$

where for the sequence $\{f_n\}_{n=1}^\infty \subset L^2(\Omega)$, we have

$$\left\| \{f_n\}_{n=1}^\infty \right\|_\Omega \equiv \sup_{\sum a_n^2 \le 1} \left\| \sum_{n=1}^\infty a_n f_n \right\|_{L^2(\Omega)}.$$

We begin by constructing an auxiliary system $\{\psi_n\}$ such that ψ_n differ from φ_n only by a multiplier,

$$\varphi_n(x) = \rho_n \psi_n(x), \qquad \rho_n = \text{const}, \quad n = 1, 2, \dots.$$

In the interval $\left(\frac{1}{n+1}, 1\right)$ the function ψ_n is defined by

$$\psi_n(x) = f_n(x), \qquad n = 1, 2, \dots, \quad x \in \left(\tfrac{1}{n+1}, 1\right).$$

To complete the definition of the functions ψ_n, it is sufficient to define the set $\{\psi_n\}_{n=1}^N$ for $N = 1, 2, \dots$ in the interval $\left(\frac{1}{N+2}, \frac{1}{N+1}\right)$. In this case we also control the validity of the relations

$$\frac{1}{10} \le \left\|\{\psi_n\}_{n=1}^{\infty}\right\|_{\left(\frac{1}{s+2},1\right)} \le \frac{2}{10} + \sum_{n=1}^{s} 10^{-n}, \qquad s = 1, 2, \dots . \tag{23}$$

We fix $N \ge 1$ and (considering the functions ψ_n, $n = 1, 2, \dots$ to be already defined in the interval $\left(\frac{1}{N+1}, 1\right)$ in such a way that (23) holds for $s = N - 1$) we define the set $\{\psi_n(x)\}_{n=1}^N$, $x \in \left(\frac{1}{N+2}, \frac{1}{N+1}\right)$.

By the Schur theorem (see [15, p. 256]), we can realize this procedure so that the functions $\{\psi_n(x)\}_{n=1}^N$ be pairwise orthogonal on $\left(\frac{1}{N+2}, 1\right)$ and (see (22), (23))

$$\|\psi_n\|_{L^2\left(\frac{1}{N+2},1\right)} = \left\|\{\psi_n\}_{n=1}^{N}\right\|_{\left(\frac{1}{N+1},1\right)} \le \left\|\{\psi_n\}_{n=1}^{N-1}\right\|_{\left(\frac{1}{N+1},1\right)} + \|\psi_N\|_{L^2(0,1)} \le \frac{2}{10} + \sum_{n=1}^{N-1} 10^{-n} + 10^{N}$$

for $n = 1, 2, \dots, N$ Hence, inequality (23) is also valid for $s = N$. Moreover, by means of the corresponding contraction and multiplication by a constant (without change of the norm in $L^2\left(\frac{1}{N+2}, \frac{1}{N+1}\right)$) one can ensure the smallness of the norms $\|\psi_n\|_{L^1\left(\frac{1}{N+2}, \frac{1}{N+1}\right)}$, $n = 1, 2, \dots, N$ (see also the remark on p. 258 in [15]). More exactly, one can assume that

$$\|\psi_n\|_{L^1\left(\frac{1}{N+2}, \frac{1}{N+1}\right)} \le \frac{\|f_n\|_{L^1}}{4^N}, \qquad n = 1, 2, \dots .$$

Carrying out the mentioned constructions for all N, we obtain the system $\{\psi_n(x)\}_{n=1}^{\infty}$, $x \in (0, 1)$ of functions such that

(1) $\{\psi_n\}$ is an orthogonal system;

(2) $\frac{1}{10} \le \|\psi_n\|_{L^2(0,1)} \le \frac{4}{10}$, $n = 1, 2, \dots$;

(3) $\|f_n - \psi_n\|_{L^1} \le \|f_n\|_{L^1} \cdot \sum_{s=n}^{\infty} 4^{-s} \le \frac{\|f\|_{L^1}}{3^n}$, $n = 1, 2, \dots$.

By the last inequality, we also have

$$\|g_n - 10^n \psi_n\|_{L^1} \le \frac{\|g_n\|_{L^1}}{3^n} \le \frac{1}{3^n}, \qquad n = 1, 2, \dots;$$

hence, for any function $f \in L^2$, $\|f\|_{L^2(0,1)} = 1$, and for any $\varepsilon > 0$,

$$\|f - 10^{n^*} \cdot \psi_{n^*}\|_{L^1} < \varepsilon, \tag{24}$$

if n^* is sufficiently large and such that $\|f - g_{n^*}\|_{L^1} < \varepsilon/2$.

Finally, we put $\varphi_n(x) = \frac{\psi_n(x)}{\|\psi_n\|_{L^2(0,1)}}$, $n = 1, 2, \dots$. By (24) (see also (23)), $\Phi = \{\varphi_n\}$ is an orthonormal system, and on the unit sphere in $L^2(0, 1)$ the set $A(\Phi)$ is dense in the norm of $L^1(0, 1)$. Proposition 1 has been proved.

2. Lower bounds for approximate characteristics of classes of functions with bounded mixed derivative or difference

In this section, we establish order-precis lower bounds for best m-term triconometric approximations in the metric of L^p of classes of functions with bounded mixed difference H_{∞}^{r} or derivative $W_{q,\alpha}^{r}$ for $1 < p \le q < \infty$ (the definition of the classes is given below). In this case, the study of the properties of the functional class is reduced to a discrete problem (i.e., to the study of properties of a set in a finite-dimensional space) by using the "quasi-matrix property" of the space of triconometric polynomials with parallelepiped harmonics. The corresponding upper bounds for m-term approximations are known and

are attained when approximating by polynomials with hyperbolic cross harmonics (see [16–18], where the history of the problem considered in this section is also discussed).

The second part of the section is devoted to estimates of the ε-entropy and of the Kolmogorov widths for the class $W^r_{\infty,\alpha}$. In this case, it is difficult to apply directly results of the finite-dimensional geometry, because we do not know if a space of polynomials with hyperbolic cross harmonics is "quasi-matrix." However, in a number of cases one can avoid this difficulty and obtain new order-precise results. In addition to modern results of the geometry of convex sets, our approach is based essentially on the uniqueness of the maximal volume ellipsoid inscribed in a centrally symmetric convex body.

Let us introduce the notations and definitions we use below. We denote by $f * g$ the convolution of functions f and g. In this section, we assume that functions considered are 2π-periodic with respect to each variable and such that

$$(f * g)(x) = (2\pi)^{-d} \int_{[0,2\pi]^d} f(x-y)g(y)\,dy.$$

For our purposes it is more convenient to use the following definition of the L^p-norm:

$$\|f\|_p = \left((2\pi)^{-d} \int_{[0,2\pi]^d} |f(x)|^p\,dx\right)^{1/p}, \qquad 1 \le p < \infty.$$

For $r > 0$ we define Bernoulli kernels

$$F_r(x,\alpha) = 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos\left(kx - \frac{\alpha\pi}{2}\right), \qquad x \in \mathbb{R}, \quad \alpha \in \mathbb{R};$$

$$F_r(x,\alpha) = \prod_{j=1}^{d} F_r(x_j,\alpha_j), \quad x = (x_1,\dots,x_d) \in \mathbb{R}^d, \ \alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{R}^d.$$

We denote by I^r_α the convolution operator with the kernel $F_r(x,\alpha)$, $I^r_\alpha\varphi = F_r(x,\alpha) * \varphi(x)$. For $r = 0$ this operator can be well defined on the set of triconometric polynomials. It is convenient for us to use the notation $I_\alpha = I^0_\alpha$. The class $W^r_{q,\alpha}$ is defined as follows:

$$W^r_{q,\alpha} = \{f : f = I^r_\alpha\varphi, \ \|\varphi\|_q \le 1\},$$

where $r > 0$, $1 \le q \le \infty$, $\alpha \in \mathbb{R}^d$.

We now define classes H^r_∞. We denote by $\Delta^l_{t_j}$ the operator of the l-multiple difference with step t_j in variable x_j, $j = 1,\dots,d$. For a set of positive integers a of $[1,d]$ we denote $\Delta^l_t(a) = \prod_{j\in a} \Delta^l_{t_j}$. Put $l = [r] + 1$, where $[r]$ is the integer part of r. The class H^r_∞ is defined as follows:

$$H^r_\infty = \left\{f : \text{ for all } a \subset [1,d] \text{ we have} \|\Delta^l_t(a)f\|_\infty \le \prod_{j\in a} |t_j|^r\right\}.$$

We need the following well-known triconometric polynomials:

(a) the one-dimensional Fejer kernel,

$$\mathcal{K}_n(x) = \sum_{|k|<n} \left(1 - \frac{|k|}{n}\right) e^{ikx};$$

(b) the multidimensional Fejer kernel,

$$\mathcal{K}_{\mathbb{N}}(x) = \prod_{j=1}^{d} \mathcal{K}_{N_j}(x_j), \qquad \mathbb{N} = (N_1,\dots,N_d), \quad x = (x_1,\dots,x_d);$$

(c) the one-dimensional de la Vallee-Poussin kernel,

$$V_m(x) = 2\mathcal{K}_{2m}(x) - \mathcal{K}_m(x);$$

(d) the multidimensional de la Vallee-Poussin kernel,

$$V_{\mathbb{N}}(x) = \prod_{j=1}^{d} V_{N_j}(x_j).$$

We also need the following kernels:

$$\begin{aligned} &A_m(x) = V_{2^{m-1}}(x) - V_{2^{m-2}}(x), \qquad m \geq 2, \\ &A_1(x) = V_1(x) - 1, \qquad \mathcal{A}_0(x) \equiv 1, \qquad x \in \mathbb{R}, \\ &A_s(x) = \prod_{j=1}^{d} \mathcal{A}_{s_j}(x_j), \qquad s = (s_1, \dots, s_d), \quad s_j \geq 0, \quad j = 1, \dots, d, \quad x \in \mathbb{R}^d. \end{aligned}$$

We denote by A_s the convolution operator with the kernel $\mathcal{A}_s(x)$.

In addition to the L^p-norm, we consider the norm $B_{q,\theta}$, analogous to the norm of the Besov space, defined for triconometric polynomials by

$$\|t\|_{B_{q,\theta}} = \left(\sum_s \|A_s(t)\|_q^\theta \right)^{1/\theta}, \qquad 1 \leq q \leq \infty, \quad 1 \leq \theta \leq \infty;$$

this definition is naturally modified for $\theta = \infty$.

By analogy, we define the norm $\|f\|_{B_{q,\theta}}$ for functions $f \in L^1$ such that the series $\sum_s \|A_s(f)\|_q^\theta$ is convergent. Below, the norms $B_{q,\theta}$ play an auxiliary role.

Let us define spaces of triconometric polynomials with harmonics of sets related to hyperbolic crosses. For $s = (s_1, \dots, s_d) \in \mathbb{Z}_+^d$ we put

$$\begin{aligned} \rho(s) &= \left\{ k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d : [2^{s_j - 1}] \leq k_j < 2^{s_j},\ j = 1, \dots, d \right\}, \\ \bar{\rho}(s) &= \left\{ k \in \mathbb{Z}_+^d : 2^{s_j - 1} < k_j < 2^{s_j},\ j = 1, \dots, d \right\}, \\ D_n &= \bigcup_{\|s\|_1 = n} \rho(s), \qquad |D_n| \asymp 2^n \cdot n^{d-1}, \\ \theta_n &= \left\{ s : \|s\|_1 = 2[n/2],\ s_j \text{ is even},\ s_j > 0,\ j = 1, \dots, d \right\}, \end{aligned}$$

and, finally,

$$Y_n = \bigcup_{s \in \theta_n} \bar{\rho}(s).$$

For a set $\Lambda \subset \mathbb{Z}_+^d$ we denote

$$\mathcal{T}(\Lambda) = \left\{ t : t(x) = \sum_{|k| \in \Lambda} c_k e^{i(k,x)} \right\}.$$

Given a normed function space X, we denote by $\mathcal{T}(\Lambda)_X$ the unit X-ball in $\mathcal{T}(\Lambda)$.

We now prove a statement on the approximation of polynomials of $\mathcal{T}(Y_n)_{B_{\infty,\infty}}$; below we use this statement to establish lower bounds for approximations of the classes H_∞^r and $W_{q,\alpha}^r$ by m-term triconometric polynomials.

Theorem 2.1. *There exists a constant $c(d) > 0$ such that for any set $\Phi = \{\varphi_j\}_{j=1}^s \subset B_{1,1}$, $s \le c'|Y_n|$, of functions the estimate*

$$\sigma_m(T(Y_n)_{B_{\infty,\infty}}, \Phi)_{B_{1,1}} \ge c_1 n^{d-1}, \qquad c_1 = c_1(d, c') > 0,$$

holds for all $m \le c(d)|Y_n|$.

Proof. Instead of $T(Y_n)$ it is convenient for us to consider a subspace $T'(Y_n) \subset T(Y_n)$ of dimension $\ge c(d)|Y_n|$. Let us describe this subspace. For $\mathbb{N} = (N_1, \dots, N_d)$ we denote by $RT(\mathbb{N})$ the space of real triconometric polynomials of d variables of degree $\le N_j$ in variable x_j, $j = 1, \dots, d$. We define the subspace $T'(Y_n)$ by

$$T'(Y_n) = \left\{ t : t(x) = \sum_{s \in \theta_n} e^{i(k^s, x)} t_s^1(x),\ t_s^1 \in RT(2^{s-2} - \mathbb{I}) \right\},$$

where $k^s = (k_1^s, \dots, k_d^s)$, $k_j^s = 2^{s_j - 1} + 2^{s_j - 2}$, $s_j \ge 2$, $2^{s-2} = (2^{s_1 - 2}, \dots, 2^{s_d - 2})$, $\mathbb{I} = (1, \dots, 1)$. It is clear that $T'(Y_n)_{B_{\infty,\infty}} \subset T(Y_n)_{B_{\infty,\infty}}$.

In this space, the discretization is carried out in the following way. Put

$$\Omega(\mathbb{N}) = \left\{ x^k = \left(\frac{2\pi k_1}{2N_1 + 1}, \dots, \frac{2\pi k_d}{2N_d + 1} \right),\ k_j = 0, 1, \dots, 2N_j,\ j = 1, \dots, d \right\}.$$

With a polynomial $t \in T'(Y_n)$ we associate the vector $J(t) \in \mathbb{R}^M$,

$$J(t) = \{t_s^1(x^k)\}_{x^k \in \Omega(2^{s-2} - \mathbb{I}), s \in \theta_n}$$

where $M = \sum_{s \in \theta_n} v(2^{s-2} - \mathbb{I})$, $v(\mathbb{N}) = \prod_{j=1}^d (2N_j + 1)$.

Conversely, with a vector $y = \{y(x^k)\}_{x^k \in \Omega(2^{s-2} - \mathbb{I}), s \in \theta_n} \in \mathbb{R}^M$ we associate the polynomial

$$J^{-1}(y) = \sum_{s \in \theta_n} e^{i(k^s, x)} \{v(2^{s-2} - \mathbb{I})\}^{-1} \sum_{x^k \in \Omega(2^{s-2} - \mathbb{I})} y(x^k) D_{2^{s-2} - \mathbb{I}}(x - x_k),$$

where

$$D_{\mathbb{N}}(x) = \prod_{j=1}^d D_{N_j}(x_j)$$

is a multidimensional Dirichlet kernel normalized by $D_{\mathbb{N}}(0) = v(\mathbb{N})$.

In the space $\mathbb{R}^M$ let us consider the set

$$H = \bigtimes_{s \in \theta_n} S_\infty(2^{s-2} - \mathbb{I}),$$

where the symbol $\bigtimes$ means the direct product, and

$$S_\infty(\mathbb{N}) = \left\{ y = \{y(x^k)\}_{x^k \in \Omega(\mathbb{N})} \in \mathbb{R}^{v(\mathbb{N})} : \|t(x, y)\|_\infty \le 1 \ \ t(x, y) = v(\mathbb{N})^{-1} \sum_{x^k \in \Omega(\mathbb{N})} y(x^k) D_{\mathbb{N}}(x - x^k) \right\}.$$

It is known (see Lemma 1.1 of [18]) that

$$\mathrm{Vol}(S_\infty(\mathbb{N})) \ge c(d)^{-v(\mathbb{N})},$$

and therefore

$$\operatorname{Vol} H \geq \prod_{s \in \theta_n} c(d)^{-v(2^{s-2}-\mathbb{I})} \geq c(d)^{-|Y_n|}. \tag{2.1}$$

We need a special operator K^n that maps L^1 into $T(Y_n)$. Introducing the polynomial

$$\mathcal{K}^n(x) = \sum_{s \in \theta_n} e^{i(k^s,x)} \mathcal{K}_{2^{s-2}}(x),$$

we define the operator K^n as the convolution operator with the kernel $\mathcal{K}^n(x)$. It is clear that the range of K^n lies in $T(Y_n)$. Moreover, this operator is bounded as an operator from $B_{1,1}$ to $B_{1,1}$. In fact, we have

$$\|A_\mu(K^n f)\|_1 = \|K^n A_\mu(f)\|_1 = \left\| \sum_{\mu - \mathbb{I} \leq s \leq \mu} e^{i(k^s,x)} \mathcal{K}_{2^{s-2}} * A_\mu(f) \right\|_1 \leq c(d) \|A_\mu(f)\|_1.$$

Therefore,

$$\|K^n(f)\|_{B_{1,1}} \leq c(d) \|f\|_{B_{1,1}}.$$

The operator JK^nJ^{-1} maps the set H to a set H' which is convex, centrally symmetric, and such that

$$\operatorname{Vol} H' \geq c(d)^{-|Y_n|}. \tag{2.2}$$

Let us clarify relation (2.2). Consider the image of the set $S_\infty(2^{s-2} - \mathbb{I})$ under the action of JK^nJ^{-1}. We first note that the action K^n can be easily described in terms of Fourier coefficients of the polynomials t_s^1. Namely, if

$$t(x) = \sum_{s \in \theta_n} e^{i(k^s,x)} t_s^1(x),$$

then

$$\widehat{K^n t}(k) = \widehat{\mathcal{K}}_{2^{s-2}}(k - k^s) \hat{t}_s^1(k - k^s), \qquad k \in \bar{\rho}(s).$$

Let us consider the operator which associates the vector

$$\{v(\mathbb{N})^{-1/2} t(x^k)\}_{x^k \in \Omega(\mathbb{N})}$$

with the set of Fourier coefficients in the expansion (by sines and cosines) of a polynomial $t \in RT(\mathbb{N})$. This operator is orthogonal in $\mathbb{R}^{v(\mathbb{N})}$. Moreover,

$$\prod_{k \in \bar{\rho}(s)} \widehat{\mathcal{K}}_{2^{s-2}}(k - k^s) \geq c(d)^{|\bar{\rho}(s)|}. \tag{2.3}$$

Relation (2.3) and the above-mentioned remarks yield estimate (2.2).

It is clear that $H' \subset B_\infty^{|Y_n|} \subset |Y_n|^{1/2} B_2^{|Y_n|}$.

Functions $K^n f$ have the form

$$K^n f = t = \sum_{s \in \theta_n} e^{i(k^s,x)} t_s(x), \qquad t_s \in T(2^{s-2} - \mathbb{I}).$$

On functions of such a form we define the operator R,

$$R(t) = \sum_{s \in \theta_n} e^{i(k^s,x)} t_s^1(x), \qquad t_s^1 = \operatorname{Re} t_s.$$

It is easy to see that R is bounded as an operator from $B_{1,1}$ to $B_{1,1}$; more exactly,

$$\|R\|_{B_{1,1}\to B_{1,1}} \le 1.$$

Now instead of the set $\Phi = \{\varphi_j\}_{j=1}^s$ we consider the set $\Psi = \{\psi_j\}_{j=1}^s$ of functions, $\psi_j = RK^n\varphi_j$. Then $\psi_j \in \mathcal{T}'(Y_n)$, and for any $t \in \mathcal{T}'(Y_n)$ and $\varphi \in \operatorname{span}(\{\varphi_j\}_{j=1}^s)$ we have the following estimate for $\psi = RK^n\varphi$:

$$\|K^n t - \psi\|_{B_{1,1}} = \|RK^n(t-\varphi)\|_{B_{1,1}} \le c(d)\|t-\varphi\|_{B_{1,1}}. \tag{2.4}$$

Let us consider a system $U = \{u_j\}_{j=1}^s$, $u_j = J\psi_j \in \mathbb{R}^M$. Then, by Remark 3 to Theorem 1.1, estimate (2.2) implies the inequality

$$\sigma_m(H', U)_1 \ge c_2(d, c')|Y_n|. \tag{2.5}$$

Further, for an arbitrary $t \in \mathcal{T}'(Y_n)$,

$$e^{i(k^s,x)}t_s^1(x) = \sum_{s\le\mu\le s+\mathbb{I}} A_\mu(t);$$

hence,

$$\|t_s^1\|_1 \le \sum_{s\le\mu\le s+\mathbb{I}} \|A_\mu(t)\|_1.$$

Therefore, for $t \in \mathcal{T}'(Y_n)$ we obtain

$$\begin{aligned}\|t\|_{B_{1,1}} &= \sum_{s\in\theta_n}\sum_{s\le\mu\le s+\mathbb{I}} \|A_\mu(t)\|_1 \ge \sum_{s\in\theta_n} \|t_s^1\|_1 \\ &\ge c(d)\sum_{s\in\theta_n}\{v(2^{s-2}-\mathbb{I})\}^{-1} \sum_{x^k\in\Omega(2^{s-2}-\mathbb{I})} |t_s^1(x^k)| \ge c(d)2^{-n}\|Jt\|_1.\end{aligned} \tag{2.6}$$

Comparing relations (2.6), (2.5), and (2.4), we complete the proof of Theorem 2.1.

Theorem 2.1 can be applied for the study of m-term triconometric approximations, i.e., approximations relative to the system $\mathbb{T} = \{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$.

Corollary 1. *For $m = 1, 2, \ldots$ the following lower bounds hold:*

$$\sigma_m(H_\infty^r, \mathbb{T})_p \ge c(r,d,p)m^{-r}(\log m)^{(d-1)(r+1/2)}, \qquad 1 < p \le \infty, \tag{$*$}$$

$$\sigma_m(W_{q,\alpha}^r, \mathbb{T})_p \ge c(r,d,q,p)m^{-r}(\log m)^{(d-1)r}, \qquad 1 < p \le q < \infty. \tag{$**$}$$

Proof of these estimates is based on Theorem 2.1 and on the following well-known inequalities. The inequality (an analog is in [16, p. 36])

$$\|f\|_p \le c(d,p)\|f\|_{B_{p,2}}, \qquad 2 \le p < \infty, \tag{2.7}$$

is a simple consequence of the Littlewood–Paley theorem. The inequality dual to (2.7) takes the form

$$\|f\|_p \ge c(d,p)\|f\|_{B_{p,2}}, \qquad 1 < p \le 2. \tag{2.8}$$

First, we prove relation ($*$). By the characterizing theorem (see [16, p. 32]), for the classes H_∞^r there exists a positive number $a(r,d)$ such that

$$a(r,d)2^{-rn}\mathcal{T}(Y_n)_{B_{\infty,\infty}} \subset H_\infty^r. \tag{2.9}$$

Further, let us denote by P_{Y_n} the orthogonal projector onto $\mathcal{T}(Y_n)$. It is known (see, e.g., [16, p. 7]) that P_{Y_n} is bounded as an operator from L^p to L^p, $1 < p < \infty$. Therefore,

$$\sigma_m(\mathcal{T}(Y_n)_{B_{\infty,\infty}}, \mathbb{T})_p \geq c(p,d)\sigma_m(\mathcal{T}(Y_n)_{B_{\infty,\infty}}, \{e^{i(k,x)}\}_{k\in Y_n})_p. \tag{2.10}$$

To estimate the right-hand side of (2.10), we use Theorem 2.1 for $\Phi = \{e^{i(k,x)}\}_{k\in Y_n}$ and $s = |Y_n|$.

The right-hand side of relation $(*)$ is independent of p (with the exception of the constant); therefore, it is sufficient to prove $(*)$ for $1 < p \leq 2$. In this case the lower bound of the norm in L^p is given by inequality (2.8). We note that for polynomials of $\mathcal{T}(Y_n)$ we have

$$\|t\|_{B_{1,1}} \leq c(d)n^{(d-1)/2}\|t\|_{B_{1,2}} \leq c(d)n^{(d-1)/2}\|t\|_{B_{p,2}}, \qquad 1 \leq p \leq \infty. \tag{2.11}$$

Combining (2.8)–(2.11) and applying Theorem 2.1, we obtain estimate $(*)$.

Relation $(**)$ is proved analogously if instead of (2.9) we use the inclusion

$$a(r,d,q)n^{-(d-1)/2}2^{-rn}\mathcal{T}(Y_n)_{B_{\infty,\infty}} \subset W^r_{q,\alpha}. \tag{2.12}$$

Let us prove inclusion (2.12). Given a polynomial $t \in \mathcal{T}(Y_n)_{B_{\infty,\infty}}$, by (2.7) we have for $2 \leq q < \infty$,

$$\|t\|_q \leq c(d,q)\|t\|_{B_{q,2}} \leq c(d,q)|\theta_n|^{1/2}\|t\|_{B_{\infty,\infty}} \leq c(d,q)n^{(d-1)/2}.$$

Then, by the Bernstein inequality (see [16]), we obtain

$$\|(I^r_\alpha)^{-1}t\|_q \leq c(d,r,q)n^{(d-1)/2}2^{rn}. \tag{2.13}$$

Relation (2.13) implies inclusion (2.12).

This completes the proof of Corollary 1.

Now let Λ be a finite subset of $\mathbb{Z}^d$, and put

$$T(\Lambda) = \left\{t : t(x) = \sum_{k\in\Lambda} \hat{t}(k)e^{i(k,x)}\right\}.$$

We establish a number of properties of the spaces $T(\Lambda)$ and then apply these properties to estimate approximate characteristics of the classes $W^r_{q,\alpha}$ and H^r_∞. We use the profound results of Bourgain and Milman [19] for finite-dimensional convex bodies. In addition, we need the following classical result (see [20]).

Theorem A. *Let B be a centrally symmetric convex body in $\mathbb{R}^n$. Then there exists a unique ellipsoid of maximal volume, contained in B.*

We now recall the following definition.

Definition. For a normed space X, a constant $C_2(X)$ of cotype 2 is called the least of the constants C such that for any finite set $\{x_1, \ldots, x_m\} \subset X$,

$$C \cdot \int_0^1 \left\|\sum_{i=1}^m r_i(t)x_i\right\|_X dt \geq \left(\sum \|x_i\|^2_X\right)^{1/2};$$

here $r_i(t)$, $i = 1, \ldots, m$ are Rademacher functions.

A constant of cotype 2 is well defined for any finite-dimensional space X and, what is important for us, for the spaces L^p, $1 \leq p \leq 2$. Moreover, by the Khinchin inequalities, we have

$$C_2(X) \leq 10, \qquad X = L^1(\mathbb{R}^d). \tag{2.14}$$

Theorem [19]. *Let X be an n-dimensional real normed space with the unit ball B, and let $\mathcal{E}$ be an ellipsoid of maximal volume, contained in B. Then*

$$\left(\frac{\operatorname{Vol}_n B}{\operatorname{Vol}_n \mathcal{E}}\right)^{1/n} \le KC_2(X)\cdot \log^4 C_2(X).$$

Given a finite subset $\Lambda \subset \mathbb{Z}^d$, we define the operator

$$A = A(\Lambda)\colon T(\Lambda) \to \mathbb{R}^{2|\Lambda|}$$

by

$$A(t) = \{\operatorname{Re}\hat{t}(k),\ \operatorname{Im}\hat{t}(k),\ k \in \Lambda\} \in \mathbb{R}^{2|\Lambda|}, \qquad t \in T(\Lambda); \tag{2.15}$$

here the coordinate order is fixed in an arbitrary way. Put

$$B_\Lambda = \{A(t) : t \in T(\Lambda),\ \|t\|_1 \le 1\}. \tag{2.16}$$

It is clear that B_Λ is a convex centrally symmetric body in $\mathbb{R}^{2|\Lambda|}$. The following lemma is valid.

Lemma 2.1. *The unit ball $B_2^{2|\Lambda|}$ is an ellipsoid of maximal volume, lying in B_Λ.*

Proof. The obvious relations

$$\|t\|_1 \le \|t\|_2, \qquad \|t\|_2 = \|A(t)\|_{\ell_2^n}, \qquad t \in T(\Lambda),$$

yield the inclusion $B_2^{2|\Lambda|} \subset B_\Lambda$. Let us show that B_Λ cannot contain an ellipsoid of volume greater than $\operatorname{Vol} B_2^{2|\Lambda|}$. In fact, let

$$\mathcal{E} = \mathcal{E}_\Lambda = \left\{a \in \mathbb{R}^{2|\Lambda|} : \sum_{j=1}^{2|\Lambda|} \frac{(a,\tilde{c}_j)^2}{\tilde{\mu}_j^2} \le 1\right\}$$

be an ellipsoid of maximal volume inscribed in B_Λ, and let $\{\tilde{c}_j, \tilde{\mu}_j\}_{j=1}^{2|\Lambda|}$ be a set of directions of the semi-axes $\mathcal{E}_\Lambda$ and of their lengths, $\|\tilde{c}_j\|_2 = 1$, $1 \le j \le 2|\Lambda|$. We define the operator

$$J\colon \mathbb{R}^{2|\Lambda|} \to \mathbb{R}^{2|\Lambda|}$$

by

$$J\big(\{a_k, b_k\}_{k\in\Lambda}\big) = \big(\{-b_k, a_k\}_{k\in\Lambda}\big).$$

It is clear that J is orthogonal in $\mathbb{R}^{2|\Lambda|}$, $J^2 = -\operatorname{Id}$, and for $t \in T(\Lambda)$,

$$J(A(t)) = A(i\cdot t), \qquad \|it\|_1 = \|t\|_1.$$

By the last relations and Theorem A, we have

$$J(\mathcal{E}_\Lambda) = \mathcal{E}_\Lambda.$$

Moreover, $(a, J(a)) = 0$ for any $a \in \mathbb{R}^{2|\Lambda|}$, and the two-dimensional subspace generated by the vectors a and $J(a)$ is J-invariant. Therefore, it is easy to see that the ellipsoid $\mathcal{E}_\Lambda$ can be rewritten in the form

$$\mathcal{E}_\Lambda = \left(a \in \mathbb{R}^{2|\Lambda|} : \sum_{j=1}^{|\Lambda|} \frac{(a,c_j)^2 + (a, J(c_j))^2}{\lambda_j^2} \le 1\right\},$$

where $\|c_j\|_2 = \|J(c_j)\|_2 = 1$ for $j = 1, \dots, |\Lambda|$. With each pair $\{c_j, J(c_j)\}$ of vectors we associate a polynomial $t_j \in T(\Lambda)$ such that $c_j = A(t_j)$. Then, by the orthogonality of the system $\{c_j, J(c_j)\}$ of semi-axes, we see that the polynomials t_j are orthogonal and

$$\mathcal{E}_\Lambda = \left\{ a \in \mathbb{R}^{2|\Lambda|} : \sum_{j=1}^{|\Lambda|} \frac{|(A^{-1}(a), t_j)|^2}{\lambda_j^2} \le 1 \right\}. \tag{2.17}$$

Let $\widetilde{F}_h$ be an h-translation operator, $\widetilde{F}_h(f(x)) = f(x-h)$, and

$$F_h(a) = A\big[\widetilde{F}_h(A^{-1}(a))\big], \qquad a \in \mathbb{R}^{2|\Lambda|}.$$

It is clear that F_h is an orthogonal operator in $\mathbb{R}^{2|\Lambda|}$. Since the space $T(\Lambda)$ and the norm $\|\cdot\|_1$ are translation-invariant for any $h \in \mathbb{R}^d$, we have

$$F_h(\mathcal{E}_\Lambda) = \mathcal{E}_\Lambda, \qquad h \in \mathbb{R}^d. \tag{2.18}$$

In turn, property (2.18) implies that

$$\begin{aligned} &\text{for any } \lambda \text{ the linear hull } E_\lambda = \operatorname{span}\big(\{t_j\},\ j \in \{1, \dots, |\Lambda|\},\ \lambda_j = \lambda\big) \\ &\text{(over the field of complex numbers) is translation-invariant.} \end{aligned} \tag{2.19}$$

It is known that property (2.19) implies that there exists a basis of exponents in E_λ,

$$E_\lambda = \operatorname{span}\big\{e^{i(k_\lambda^1, x)}, \dots, e^{i(k_\lambda^s, x)}\big\}, \qquad s = \dim E_\lambda. \tag{2.20}$$

By (2.20) and (2.17), we conclude that

$$\mathcal{E}_\Lambda = \left\{ a \in \mathbb{R}^{2|\Lambda|} : \sum_{\lambda : E_\lambda \ne \varnothing} \sum_{k : e^{i(k,x)} \in E_\lambda} \frac{\big|(A^{-1}(a), e^{i(k,x)})\big|^2}{\lambda^2} \le 1 \right\}$$

and therefore the ellipsoid $\mathcal{E}_\Lambda$ can be represented in the form

$$\mathcal{E}_\Lambda = \left\{ a : \sum_{k \in \Lambda} \frac{\big|\widehat{A^{-1}(a)}(k)\big|^2}{\mu_k^2} \le 1 \right\}.$$

In other words, $\left\| \sum_{k \in \Lambda} c_k e^{i(k,x)} \right\|_1 \le 1$ for any complex numbers c_k such that

$$\sum_{k=1}^{|\Lambda|} |c_k|^2 \mu_k^{-2} \le 1.$$

Thus, $\mu_k^2 \le 1$ for any k, and, therefore, $\mathcal{E}_\Lambda \subset B_2^{2|\Lambda|}$. This completes the proof of Lemma 2.1.

To give a complete proof, we deduce (2.20) from (2.19). Let $\psi_1, \dots, \psi_s$ be an orthonormal basis in E_λ. By property (2.19), for any $y \in \mathbb{R}^d$ and $m = 1, \dots, s$,

$$\psi_m(x-y) = \sum_{j=1}^{s} c_j^m(y) \psi_j(x)$$

(the functions ψ_m and c_j^m are continuous) and therefore for any k and $m = 1, \dots, s$,

$$\hat{\psi}_m(k) \cdot e^{i(k,x)} = (2\pi)^{-d} \int_{[0,2\pi]^d} \psi_m(x-y) e^{i(k,y)} dy = \sum_{j=1}^{s} b_j^{k,m} \psi_j(x) \in E_\lambda,$$

whence (2.20) follows immediately.

Lemma 2.2. *There exists an absolute constant C such that for any finite set $\Lambda \subset \mathbb{Z}^d$,*

$$\mathrm{Vol}_s(B_\Lambda) \le C^s \cdot \mathrm{Vol}\, B_2^s, \qquad s = 2|\Lambda|. \tag{2.21}$$

Proof. Let X be a normed space for which B_Λ is the unit ball. Then, by (2.14) (see also (2.16)), we have $C_2(X) \le 10$. Applying Theorem B and Lemma 2.1, we arrive at the conclusion of the lemma.

In addition to $T(\Lambda)$, let us introduce the space of polynomials with real coefficients,

$$T_R(\Lambda) = \{t \in T(\Lambda) : \hat{t}(k) \in R\},$$

and the operator $A_R\colon T_R(\Lambda) \to \mathbb{R}^{|\Lambda|}$ defined by

$$A_R(t) = \{\hat{t}(k)\}_{k\in\Lambda}.$$

We also put

$$B_{R,1}(\Lambda) = \left\{\{\hat{t}(k)\} \in \mathbb{R}^{|\Lambda|} : \left\|\sum_{k\in\Lambda} \hat{t}(k) e^{i(k,x)}\right\|_1 \le 1\right\}.$$

Lemma 2.3. *For any finite set $\Lambda \subset \mathbb{Z}^d$, the following estimate is valid:*

$$\mathrm{Vol}_{|\Lambda|}\{B_{R,1}(\Lambda)\} \le C^{|\Lambda|} \cdot \mathrm{Vol}\, B_2^{|\Lambda|}$$

where C is an absolute constant.

Proof. We represent $R^{2|\Lambda|}$ in the form

$$R^{2|\Lambda|} = R^{|\Lambda|} \otimes R^{|\Lambda|} \equiv A(T_R(\Lambda)) \otimes A(iT_R(\Lambda)).$$

It is clear that for any $a \in \frac{1}{2}B_{R,1}(\Lambda)$, $b \in \frac{1}{2}B_{R,1}(\Lambda)$, we have

$$a \otimes b \in B_\Lambda$$

and therefore

$$\left[(\tfrac{1}{2})^{|\Lambda|} \cdot \mathrm{Vol}\{B_{R,1}(\Lambda)\}\right]^2 \le \mathrm{Vol}\, B_\Lambda.$$

By the last inequality and inequality (2.21), we arrive at the conclusion of Lemma 2.3.

Given a function f, a subset $\Lambda \subset \mathbb{Z}^d$, and $1 \le p \le \infty$, we put

$$E_\Lambda^\perp(f)_p = \inf_{u:\hat{u}(k)=0, k\in\Lambda} \|f - u\|_p,$$

and

$$B_{R,\infty}^\perp(\Lambda) = \left\{\{\hat{t}(k)\} \in \mathbb{R}^{|\Lambda|} : t \in T_R(\Lambda),\ E_\Lambda^\perp(t)_\infty \le 1\right\}.$$

Lemma 2.4. *For any finite set $\Lambda \subset \mathbb{Z}^d$, the following estimate is valid:*

$$\mathrm{Vol}_{|\Lambda|}\{B_{R,\infty}^\perp(\Lambda)\} \ge c^{|\Lambda|}\, \mathrm{Vol}\, B_2^{|\Lambda|}.$$

The proof is based on the Bourgain–Milman inequality [19], by which for any convex centrally symmetric body $K \subset \mathbb{R}^n$,

$$\mathrm{Vol}\, K \cdot \mathrm{Vol}\, K^0 \ge C^n (\mathrm{Vol}\, B_2^n)^2,$$

where $C > 0$ is an absolute constant and K^0 is a polar to K, i.e.,

$$K^0 = \left\{x \in \mathbb{R}^n : \sup_{y\in K}(x,y) \le 1\right\}.$$

Let $K = B_{R,1}(\Lambda)$. Taking into account Lemma 2.3, it is sufficient to verify that $\frac{1}{2}K^0 \subset B^{\perp}_{R,\infty}(\Lambda)$. The following equality is a consequence of the Hahn–Banach theorem:

$$E^{\perp}_{\Lambda}(t)_{\infty} = \sup_{\substack{g \in T(\Lambda) \\ \|g\|_1 \le 1}} |(t,g)|, \qquad t \in T(\Lambda). \tag{2.22}$$

Let us represent an arbitrary polynomial $g \in T(\Lambda)$, $\|g\|_1 \le 1$, in the form

$$g = g' + ig'' \equiv \sum_{k \in \Lambda} \operatorname{Re} \hat{g}(k) e^{i(k,x)} + i \sum_{k \in \Lambda} \operatorname{Im} \hat{g}(k) e^{i(k,x)}.$$

Then $g', g'' \in T_R(\Lambda)$ and

$$\|g'\|_1 = \left\| \tfrac{1}{2}(g(x) + \bar{g}(-x)) \right\|_1 \le 1, \qquad \|g''\|_1 = \left\| \tfrac{1}{2}(g(x) - \bar{g}(-x)) \right\|_1 \le 1.$$

Since

$$|(t,g)| \le 2 \max\{|(t,g')|, |(t,g'')|\},$$

it follows from (2.22) that

$$E^{\perp}_{\Lambda}(t)_{\infty} \le 2 \max_{\substack{g \in T_R(\Lambda) \\ \|g\|_1 \le 1}} |(t,g)|.$$

If $t = A_R^{-1}(a)$ and a is an arbitrary vector of $\mathbb{R}^{|\Lambda|}$, then the last inequality implies the inclusion $\frac{1}{2}K^0 \subset B^{\perp}_{R,\infty}(\Lambda)$.

Lemma 2.4 has been proved.

We now use Lemma 2.4 to establish new lower bounds for the Kolmogorov widths and the entropy numbers of the classes $W^r_{\infty,\alpha}$. We formulate these results as the following theorem.

Theorem 2.2. *For any $r > 0$ we have*

$$d_m(W^r_{\infty,\alpha}, L_p) \ge c(r,d,p) m^{-r} (\log m)^{r(d-1)}, \qquad p > 1,$$
$$\varepsilon_m(W^r_{\infty,0}, L_1) \ge c(r,d) m^{-r} (\log m)^{r(d-1)}.$$

Proof. We begin by proving the first relation. We use the following lemma.

Lemma A [9]. *Let a convex centrally symmetric body A be contained in the unit ball B_2^N of the Euclidean space $\mathbb{R}^N$, and let $\operatorname{Vol}(A) \ge c_1^{-N} \operatorname{Vol} B_2^N$, where $c_1 > 0$ is a constant. Then for any subspace $L \subset \mathbb{R}^N$ of dimension not less than $N/2$ there exists an element $a \in A \cap L$ such that $\|a\|_{\ell_2^N} \ge c_2 > 0$.*

Let $\Psi \subset L^p$ be an arbitrary subspace of dimension m, and let n be the least number that satisfies the condition $|D_n| \ge 4m$. We use Lemma A, where as the body A we take $B^{\perp}_{R,\infty}(D_n)$. By the definition of this set and the conclusion of Lemma 2.4, the body A satisfies the condition of Lemma A. As the subspace L we take

$$L = \{A_R(t),\ t \in T_R(D_n),\ (I_\alpha t, \psi) = 0 \text{ for all } \psi \in \Psi\}.$$

It is clear that $\dim L \ge \frac{1}{2}|D_n|$. Then, by Lemma A, we find an element

$$a \in B^{\perp}_{R,\infty}(D_n)$$

such that for $\varphi = A_R^{-1}(a) \in T_R(D_n)$, we have

$$E^{\perp}_{D_n}(\varphi)_{\infty} \le 1, \qquad \|\varphi\|_2 \ge c_2 > 0,$$

and

$$(I_\alpha\varphi,\psi)=0,\qquad \psi\in\Psi.$$

Suppose that the element $\varphi^\perp\in\{T(D_n)\}^\perp$ is such that

$$\|\varphi-\varphi^\perp\|_\infty\le 2. \tag{2.23}$$

We define

$$f=I_\alpha^r\big(\tfrac{1}{2}(\varphi-\varphi^\perp)\big);$$

then $f\in W^r_{\infty,\alpha}$. We take an arbitrary $\psi\in\Psi$ and estimate the quantity $\|f-\psi\|_p$ from below. Suppose that $q=\frac{p}{p-1}$. Then inequality (2.23) implies that

$$\|\varphi-\varphi^\perp\|_q\le 2. \tag{2.24}$$

Obviously, it is sufficient to consider the case $1<p\le 2$. Then $2\le q<\infty$, and, by the multidimensional Littlewood–Paley theorem and the Riesz theorem on the boundedness of the triconometric conjugation as an operator from L^q to L^q, we see that the orthoprojector on $T(D_n)$ is bounded as an operator from L^q to L^q. Therefore, taking into account (2.24), we obtain

$$\|\varphi\|_q\le c_1(q). \tag{2.25}$$

By the above-mentioned Riesz theorem, inequality (2.25) implies that

$$\|I_\alpha\varphi\|_q\le c_2(q). \tag{2.26}$$

Further, we have

$$\begin{aligned} b:&=(f-\psi,I_\alpha\varphi)=(f,I_\alpha\varphi)=\frac{1}{2}(I_\alpha^r\varphi-I_\alpha^r\varphi^\perp,I_\alpha\varphi)=\frac{1}{2}(I_\alpha^r\varphi,I_\alpha\varphi)\\ &=\frac{1}{2}\sum_{k\in D_n}\widehat{F}_r(k,0)|\hat\varphi(k)|^2\ge 2^{-rn-1}\|\varphi\|_2^2\ge 2^{-rn-1}c_2^2. \end{aligned} \tag{2.27}$$

On the other hand,

$$b\le\|f-\psi\|_p\|I_\alpha\varphi\|_q\le c_2(q)\|f-\psi\|_p. \tag{2.28}$$

The first relation of Theorem 2.2 follows now from (2.27), (2.28), the arbitrariness of $\psi\in\Psi$, and the arbitrariness of the space Ψ, $\dim\Psi=m$.

We now prove the second relation of the conclusion of Theorem 2.2. In the proof we use Lemma 2.4, where $\Lambda=D_n$ and n is the least number that satisfies the condition $|D_n|\ge m$. Then, for the number $N_{\varepsilon,n}$ of the ε-network elements of the set $B^\perp_{R,\infty}(D_n)$, the estimate of the volume of $B^\perp_{R,\infty}(D_n)$, given by Lemma 2.4, implies the following estimate in the metric of $\ell_2^{|D_n|}$:

$$N_{\varepsilon,n}\ge\Big(\frac{c}{\varepsilon}\Big)^{|D_n|}.$$

Therefore, for $\varepsilon=\varepsilon_0>0$ we have

$$N_{\varepsilon,n}>2^{|D_n|}.$$

Thus we see that there exist 2^m polynomials $\{t_j\in T_R(D_n)\}_{j=1}^{2^m}$ such that

$$E^\perp_{D_n}(t_j)_\infty\le 1,\quad j=1,\dots,2^m;\qquad \|t_i-t_j\|_2\ge\frac{1}{2}\varepsilon_0,\quad i\ne j.$$

Further, suppose that the elements $t_j^\perp$, $j = 1, \dots, 2^m$, are such that $t_j^\perp \in \{T(D_n)\}^\perp$ and

$$\|t_j - t_j^\perp\|_\infty \le 2. \tag{2.29}$$

We consider the set

$$\varphi_j = \frac{1}{2}(t_j - t_j^\perp), \quad f_j = \varphi_j * F_r(x,0), \qquad j = 1, 2, \dots, 2^m.$$

Then $f_j \in W^r_{\infty,0}$, $j = 1, \dots, 2^m$. Let us estimate $\|f_i - f_j\|_1$ from below for $i \ne j$. We consider quantities

$$\sigma_{ij} = (f_i - f_j, \varphi_i - \varphi_j).$$

On the one hand, by (2.29) we have

$$\sigma_{ij} \le 2\|f_i - f_j\|_1.$$

On the other hand,

$$\sigma_{ij} = \sum_k \widehat{F}_r(k,0)|\hat\varphi_i(k) - \hat\varphi_j(k)|^2 \ge \sum_{k \in D_n} F_r(k,0)|\hat\varphi_i(k) - \hat\varphi_j(k)|^2$$
$$\ge 2^{-rn-2} \sum_{k \in D_n} |\hat t_i(k) - \hat t_j(k)|^2 = 2^{-rn-2}\|t_i - t_j\|_2^2.$$

Thus,

$$\|f_i - f_j\|_1 \ge 2^{-rn-5}\varepsilon_0^2, \qquad i \ne j.$$

Taking into account that the set $\{f_i\}$ consists of 2^m elements and $2^n n^{d-1} \le c(d)m$, we arrive at the second conclusion of Theorem 2.2.

Going back to m-term approximations, we now use Lemma 2.2 to prove a statement similar to Theorem 2.1 for approximations in L^1.

Theorem 2.3. *There exists a constant $c(d)$ such that for any set $\Phi = \{\varphi_j \in \mathcal{T}(Y_n)\}_{j=1}^N$, $N \le K|Y_n|$, the following estimate holds for any all $m < c(d)|Y_n|$:*

$$\sigma_m\big(\mathcal{T}(Y_n)_{B_{\infty,\infty}}, \Phi\big)_1 \ge c(d,k)n^{(d-1)/2}.$$

Proof. We reduce the problem to the problem in $\mathbb{R}^{2|Y_n|}$. We carry out the discretization by using coefficients of the Fourier polynomials of $\mathcal{T}(Y_n)$. We consider the operator A defined by (see (2.15) for $\Lambda = Y_n$)

$$A(t) = \{\operatorname{Re}\hat t(k),\ \operatorname{Im}\hat t(k),\ k \in Y_n\}.$$

Put

$$A_\infty(Y_n) = \left\{A(t) : t \in \mathcal{T}(Y_n) \ \ \Big\| \sum_{k \in \bar\rho(s)} \hat t(k) e^{i(k,x)}\Big\|_\infty \le 1,\ s \in \theta_n\right\}.$$

The set $A_\infty(Y_n)$ is convex and centrally symmetric. The following estimate is known (see [17, p. 206]):

$$\operatorname{Vol}(A_\infty(Y_n)) \ge 2^{-n|Y_n|}c(d)^{-|Y_n|}, \qquad c(d) > 0. \tag{2.30}$$

To use Remark 4 to Theorem 1.1, we take as G the following set:

$$G = \{a \in \mathbb{R}^{2|Y_n|} : a|\theta_n|^{1/2} \in A_\infty(Y_n)\}.$$

Then $G \subset B_2^{2|Y_n|}$ and, by (2.30), we have

$$\operatorname{Vol} G \ge c(d)^{|Y_n|} \operatorname{Vol}(B_2^{2|Y_n|}). \tag{2.31}$$

Let us introduce the following norm in $\mathbb{R}^{2|Y_n|}$, induced by the L^1-norm in $\mathcal{T}(Y_n)$:

$$\|a\|_B = \|A^{-1}(a)\|_1 \qquad \text{for} \quad a \in \mathbb{R}^{2|Y_n|}.$$

To prove that this norm satisfies condition $(\star)$, we use Lemma 2.2 and the following well-known result (see, for example, [13, p. 56]).

Let B be a convex centrally symmetric body in $\mathbb{R}^N$, let $B_2^N \subset B$, and let $\operatorname{Vol} B \le C_1^N \operatorname{Vol} B_2^N$. Then there exist a constant $K = K(c_1)$ and points $\{x_1, \dots, x_M\} \subset \mathbb{R}^N$, $M \le K^N$, such that

$$B \subset \bigcup_{i=1}^{M} B_2^N(x_i). \tag{2.32}$$

By Lemma 2.2 for $\Lambda = Y_n$ and inclusion (2.32) for $N = 2|Y_n|$, we see that the ball $B = \{a : \|a\|_B \le 1\}$ satisfies the condition $(\star)$.

To complete the proof of Theorem 2.3, it remains to use estimate (2.31) and Remark 4 to Theorem 1.1.

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