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Logarithmic Growth of the L^1-Norm of the Majorant of Partial Sums of an Orthogonal Series

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ABSTRACT. It is proved that for any $N \times N$ orthogonal matrix $A = \{a_{ij}\}$ we have

$$\sum_{i=1}^{N} \max_{1 \le n \le N} \left| \sum_{j=1}^{n} a_{ij} \right| \ge \frac{1}{30} N^{1/2} \log N.$$

A multidimensional analog of this result is also established.

Let $\{\varphi_k\}_{k\ge1}$ be an orthonormal system (o.n.s.) of functions on some measure space. For the (possibly formal) orthogonal expansion

$$f(x) \sim \sum_k c_k \varphi_k(x),$$

consider the majorant

$$S^* f(x) = \sup_n \left| \sum_{k=1}^{n} c_k \varphi_k(x) \right|$$

of partial sums. We also set

$$S_N^* f(x) = \max_{n \le N} \left| \sum_{k=1}^{n} c_k \varphi_k(x) \right|.$$

The study of pointwise convergence and of convergence almost everywhere of orthogonal series is mainly reduced to the investigation of properties of the operators S^* and S_N^*. In this paper we study the behavior of the quantities $\|S^* f\|_{L^1}$, $\|S_N^* f\|_{L^1}$, $f \in L^1$, for discrete orthonormal systems $\{\varphi_k\}$. In particular, the following cases are of interest

$$\text{a)} \quad f(x) = \sum_{k=1}^{N} \varphi_k(x);$$
$$\text{b)} \quad f_y(x) = \sum_{k=1}^{N} \varphi_k(y)\varphi_k(x), \qquad N = 1, 2, \dots, \tag{1}$$

is the Nth-order kernel of the system $\{\varphi_k\}$ considered for some fixed value of the variable y.

If the functions φ_k are uniformly bounded, we have $\|\varphi_k\|_\infty \le M$, $k = 1, 2, \dots$; then, by a well-known result of A. M. Olevskii (see [1] and also [2, Ch. 9]), we have

$$\sup_y \max_{1 \le n \le N} \left\| \sum_{k=1}^{n} \varphi_k(y)\varphi_k(x) \right\|_{L^1} \ge C(M) \log N,$$
$$\max_{1 \le n \le N} \left\| \sum_{k=1}^{n} \varphi_k(x) \right\|_{L^1} \ge C(M) \log N, \qquad N = 1, 2, \dots, \quad C(M) > 0. \tag{2}$$

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Obviously, (2) implies that for uniformly bounded systems $\{\varphi_k\}$ we have

$$\|S_N^* f\|_{L^1} \geq C(M)\log N \tag{3}$$

if f has the form (1) a) and

$$\sup_y \|S_N^* f_y\|_{L^1} \geq C(M)\log N, \tag{3'}$$

where f_y is defined in (1) b).

It is well known that the validity of inequalities (2) substantially depends on the uniform boundedness of the functions $\{\varphi_k\}$ (2). However, it turns out (in particular, see Theorem 1 below), that the weaker inequalities (3) and (3′) remain valid in a more general situation.

In what follows, we restrict ourselves to the consideration of discrete o.n.s. (i.e., orthonormal bases in finite-dimensional Euclidean spaces). In this situation, our results can be viewed as assertions concerning the properties of orthogonal matrices, namely, of elements of the orthogonal group O^N.

Theorem 1. *Let $A = \{a_{ij}\}_{i,j=1}^N \in O^N$. Then*

$$\sum_{i=1}^N \max_{1\leq n\leq N}\left|\sum_{j=1}^n a_{ij}\right| \geq \frac{1}{30} N^{1/2}\log N. \tag{4}$$

Remark. a) The example of discrete trigonometric systems shows that the bound (4) is order exact. We do not know any examples of systems $\{\varphi_k\}_{k=1}^N$, $N = 1, 2, \dots$, for which the values $\left\|\sum_{k=1}^n \varphi_k\right\|_{L^1}$, $n = 1, \dots, N$, are uniformly bounded and the L^1 norm of the majorant

$$\left\|S^*\Big(\sum_{k\leq N}\varphi_k\Big)\right\|_{L^1}$$

is of the order of $\log N$.

b) The argument applied below in the proof of inequality (4) permits us to obtain the more general bound

$$\sum_{i=1}^N \max_{1\leq n\leq m}\left|\sum_{j=1}^n a_{ij}\right| \geq C m^{1/2}\log m, \qquad m = 1, \dots, N,$$

where $C > 0$ is some absolute constant. Similar generalizations can be obtained for a majority of the results of this paper.

c) In the proof of the bound (4) we only use (not to its full extent) the Bessel estimate for the matrix A.

Lemma 1 (S. B. Stechkin [3]). *For any (finite or infinite) sequence $\{a_n\}$ with $a_1 \geq a_2 \geq \dots \geq 0$ we have*

$$\sum_{n\geq 1} a_n \geq \frac{1}{2}\sum_{m\geq 1}\Big(\frac{1}{m}\sum_{j\geq m} a_j^2\Big)^{1/2}. \tag{5}$$

Sketch of the proof. Lemma 1 can easily be derived from its continuous counterpart

$$\int_0^\infty f(x)\,dx \geq \frac{2}{\pi}\int_0^\infty \Big(x^{-1}\int_x^\infty f^2(t)\,dt\Big)^{1/2} dx, \qquad f \in S, \tag{6}$$

where S is the class of nonnegative nonincreasing functions f on $(0, \infty)$. Since the right-hand side of (6) is a convex function of f, it suffices to verify (6) for f of the form $a^{-1}\chi(0, a)$, $a > 0$; these functions form the set of extremal points of the class $S \cap \{f\colon \int_0^\infty f(x)\,dx = 1\}$. Finally, for $f = a^{-1}\chi(0, a)$ a straightforward computation shows that inequality (6) is actually an equality.

Remark. a) It is conjectured in [3] that the best possible constant in (5) is $2/\pi$, but we know neither a suitable reference nor a proof of this fact.

b) The inequalities converse to (5) and (6) are valid without the monotonicity assumption (see [4, Theorems 337 and 338]).

Proof of Theorem 1. Let us restate the result in geometric terms. For a given orthonormal basis $\{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^N$ and a parallelepiped

$$P = \left\{x = \{x_j\}_{j=1}^N \in \mathbb{R}^N : |x_j| \le \alpha_j,\ j = 1, \dots, N\right\}$$

such that

$$\sum_{i=1}^n \varphi_i \in P, \qquad n = 1, 2, \dots, N,$$

we must show that

$$\sum_{j=1}^N \alpha_j \ge \frac{1}{30} N^{1/2} \log N. \tag{7}$$

To this end, let us find upper and lower bounds for the Kolmogorov widths (in the Euclidean metric) of the set P:

$$d_k(P) = \inf_{\substack{E \subset \mathbb{R}^N \\ \dim E \le k}} \max_{x \in P} \min_{y \in E} \|x - y\|_{\ell_2^N}. \tag{8}$$

Without loss of generality, we assume $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_N$; by setting $E = \{x : x_i = 0,\ i \ge k\}$, we obtain

$$d_{k-1}(P) \le \left(\sum_{j=k}^N \alpha_j^2\right)^{1/2}, \qquad k = 1, 2, \dots, N. \tag{9}$$

On the other hand, let $1 \le k \le N/2$, $s = [N/(2k)]$, and

$$\psi_q = \sum_{i=(q-1)s+1}^{qs} \varphi_i$$

for $q = 1, 2, \dots, 2k$. Then

1) ψ_q are pairwise orthogonal;
2) $\|\psi_q\|_{\ell_2^N} = s^{1/2}$, $q = 1, \dots, 2k$;
3) $\psi_q \in 2P$, $q = 1, \dots, 2k$.

Taking into account 1) and 2), from the standard estimate of the width of the octahedron (first stated in [5]) we obtain

$$d_k(2P) \ge d_k\big(\{\psi_q\}_{q=1}^{2k}\big) \ge \frac{1}{\sqrt{2}} s^{1/2},$$

and hence,

$$d_k(P) \ge \frac{1}{4\sqrt{2}} \left(\frac{N}{k}\right)^{1/2}, \qquad 1 \le k \le \frac{N}{2}. \tag{10}$$

By comparing (8) with (10), we obtain

$$\left(\sum_{j=k}^N \alpha_j^2\right)^{1/2} \ge \frac{1}{4\sqrt{2}} \left(\frac{N}{k}\right)^{1/2}, \qquad 1 \le k \le \frac{N}{2}.$$

Now let us apply the Stechkin lemma:

$$\sum_{j=1}^N \alpha_j \ge \frac{1}{8\sqrt{2}} \sum_{1 \le k \le N/2} \left(\frac{N}{k^2}\right)^{1/2} \ge \frac{1}{30} N^{1/2} \log N.$$

The proof of Theorem 1 is complete. □

An argument quite similar to that used in the proof of Theorem 1 yields the following result.

Theorem 1′. *Let $A = \{a_{ij}\}_{i,j=1}^{N} \in O^N$ and let $\{c_j\}_{j=1}^{N}$ be a set of numbers such that*

$$\#\{j : |c_j| \geq \alpha\} \geq \beta N$$

for some $\alpha, \beta > 0$.[1] *Then for $c = c(\alpha, \beta) > 0$ we have*

$$\sum_{i=1}^{N} \max_{1 \leq n \leq N} \left| \sum_{j=1}^{n} c_j a_{ij} \right| \geq c N^{1/2} \log N.$$

To state the following result, it is convenient to "normalize" the counting measure on $\{1, \ldots, N\}$; this leads to spaces L_p^N with the norm

$$\|x\|_{L_p^N} = \left(\frac{1}{N} \sum_{j=1}^{N} |x_j|^p \right)^{1/p}, \qquad x = \{x_j\}_{j=1}^{N},$$

where elements of L_p^N are functions on the set $\{1, \ldots, N\}$.

Corollary 1. *Let $\{\varphi_k\}_{k=1}^{N}$ be an orthonormal basis in L_2^N such that $\|\varphi_k\|_{L_1^N} \geq \alpha$, $k = 1, \ldots, N$, for some $\alpha > 0$. Then we have*

$$\#\left\{ y \in \{1, \ldots, N\} : \left\| S_N^* \left(\sum_{k=1}^{N} \varphi_k(y) \varphi_k \right) \right\|_{L_1^N} \geq c(\alpha) \log N \right\} \geq c'(\alpha) N.$$

The following assertion is well known (e.g., see [2, p. 42]).

Lemma 2. *Let f be a measurable function on some probability space and*

$$\|f\|_{L^2} \leq 1, \quad \|f\|_{L^1} \geq \alpha, \qquad \alpha > 0.$$

Then the measure $|E|$ of the set $E = \{x : |f(x)| \geq \alpha/2\}$ has the lower bound $|E| \geq \alpha^2/4$. Similarly, if $\|f\|_{L^\infty} \leq 1$ and $\|f\|_{L^1} \geq \alpha$, then $|E| \geq \alpha/2$.

Proof of Corollary 1. It follows from Lemma 2 that

$$\#\left\{ y : \#\left\{ k : \varphi_k(y) \geq \frac{\alpha}{2} \right\} \geq \frac{\alpha^2}{8} N \right\} \geq \frac{\alpha^2}{8} N,$$

and it remains to apply Theorem 1′. □

Now let us proceed to a more general problem that is motivated, in particular, by problems related to multiple orthogonal series. Let $\{\varphi_k(x)\}_{k \in I}$ be an o.n.s., where I is an arbitrary indexing set (e.g., $I = \{1, \ldots, N\}$, $I = \mathbb{N}$, or $I = \{1, \ldots, N\}^d$, and so on). Furthermore, let Ω be a family of subsets of I. For the orthogonal expansion $f(x) \sim \sum c_k \varphi_k(x)$, we define the "$\Omega$-majorant of partial sums" by the relation

$$S_\Omega^* f(x) = \sup_{\Lambda \in \Omega} \left| \sum_{k \in \Lambda} c_k \varphi_k(x) \right|. \tag{11}$$

In the "ordinary" case, we have $I = \{1, 2, \ldots, N\}$ and $\Omega = \{\{1, \ldots, k\},\ 1 \leq k \leq N\}$ (similar relations hold for $I = \mathbb{N}$ and $I = \mathbb{Z}$). When introducing the majorant (11), the case in which $I = \{1, 2, \ldots, N\}^d$ and

$$\Omega = \{\{1, \ldots, k_1\} \times \{1, \ldots, k_2\} \times \cdots \times \{1, \ldots, k_d\},\ 1 \leq k_j \leq N,\ j = 1, 2, \ldots, d\}, \tag{12}$$

which corresponds to the passage to the maximum of "rectangular" partial sums of an orthogonal series of multiplicity d, plays the main role for us.

Let I and Ω be given, where $\#I < \infty$. Let us define $\widetilde{\Omega} \subset \mathbb{R}^I$ by setting $\widetilde{\Omega} = \{\chi_\Lambda,\ \Lambda \in \Omega\}$, where we identify the Euclidean space $\mathbb{R}^I$ of dimension $\#I$ with the set of real functions on I.

The proof of the following assertion is completely similar to that of Theorem 1.

[1] By $\#A$ we denote the cardinality of the finite set A.

Theorem 2. *Let $\{\varphi_k\}_{k\in I}$ be an o.n.s. in $\mathbb{R}^I$, and let Ω be a family of subsets of I. Then*

$$\left\| S^*_\Omega\Big(\sum_{k\in I}\varphi_k\Big)\right\|_{\ell_1} \geq c\sum_{m\geq 1}\frac{d_{m-1}(\widetilde{\Omega})}{\sqrt{m}},$$

where the widths $d_m(\cdot)$ are defined in (8) *and $c>0$ is an absolute constant.*

To apply Theorem 2 successfully, we must use sufficiently precise estimates of the widths $d_m(\widetilde{\Omega})$. In some cases, estimates of this kind can be obtained, and Theorem 2 leads to order-exact results. In particular, we have the following assertion.

Corollary 2. *Suppose that $I=\{1,\dots,N\}^d$, the family Ω is defined in* (12), *and $\{\varphi_k\}_{k\in I}$ is an orthonormal basis of $\mathbb{R}^I$. Then*

$$\left\| S^*_\Omega\Big(\sum_{k\in I}\varphi_k\Big)\right\|_{\ell_1} \geq c(d)N^{d/2}(\log N)^d.$$

Remark. a) The example of discrete multiple trigonometric systems shows that Corollary 2 is order-exact.

b) Since the precise multiple analog of estimate (2) is not established, Corollary 2 gives new results also in the "uniformly bounded case." (For orthonormal systems in $\mathbb{R}^I$, the uniform boundedness condition reduces to the estimate $\|\varphi_i\|_\infty \leq C(\#I)^{-1/2}$.)

Proof of Corollary 2. Let $T\colon \mathbb{R}^I \to L^2(0,1)^d$ be the operator that takes each vector $a=\{a_k\}_{k\in I}$ to the piecewise constant function $f\in L^2(0,1)^d$ such that $f(x)=a_{k_1,\dots,k_d}$ for $x=\{x_j\}_{j=1}^d$, where $(k_j-1)/N < x_j \leq k_j/N$, $1\leq j\leq d$. It is easy to see that in this case $\|f\|_{L^2(0,1)^d} = N^{-d/2}\|a\|_{\ell_2^I}$. Let

$$F=\{T(\chi_\Lambda),\ \Lambda\subset\Omega\}\subset L^2(0,1)^d.$$

Then we have

$$d_m(\widetilde{\Omega}) = d_m(\widetilde{\Omega},\ell_2^I) = N^{d/2}d_m\big(F, L^2(0,1)^d\big), \qquad m=0,1,\dots. \tag{13}$$

Furthermore, let

$$G=\left\{\chi_P,\ P=\prod_{j=1}^d (a_j,b_j]\subset(0,1)^d\right\}$$

be the set of characteristic functions of intervals belonging to $(0,1)^d$ and let H be the subset of G that corresponds to the intervals with vertices of the form $(k_1/N,\dots,k_d/N)$, $1\leq k_j\leq N$, $j=1,\dots,d$. Note that for each function $\chi_P\in G$ there is a $\chi_{P'}\in H$ with

$$\|\chi_P-\chi_{P'}\|_{L^2} \leq \frac{c(d)}{N^{1/2}}. \tag{14}$$

Taking into account (14) and the fact that each function from H can be represented as a linear combination $\leq 2^d$ from F with coefficients ±1, we have

$$d_m(F,L^2) \geq 2^{-d}d_m(H,L^2) \geq 2^{-d}d_m(G,L^2) - c(d)N^{-1/2}. \tag{15}$$

Let us find a lower bound for the width $d_m(G,L^2)$ with the help of the theorem on the widths of the family of translations of a periodic function (see R. S. Ismagilov [6]; moreover, note that an approach to estimates of the widths that we need is based on the classical theorem of E. Schmidt; this approach was applied in [7, Theorem 3.1.1]). Let $g(x)$ be a function defined on $\mathbb{R}^d$ such that it is 1-periodic with respect to each variable and such that for $x \in (0,1)^d$ we have

$$g(x) = \begin{cases} 1, & x \in \left(\frac{1}{4}, \frac{3}{4}\right]^d, \\ 0, & x \notin \left(\frac{1}{4}, \frac{3}{4}\right]^d. \end{cases}$$

In $L^2\big((0,1)^d\big)$ consider the family of functions $Q = \{g(\cdot - \tau),\ \tau \in \mathbb{R}^d\}$. Each function from Q can be represented in the form of $\le 2^d$ functions from G, and therefore,

$$d_m(Q, L^2) \le 2^d d_m(G, L^2). \tag{16}$$

Combining (15) and (16), we obtain

$$d_m(F, L^2) \ge 2^{-2d} d_m(Q, L^2) - c(d) N^{-1/2}. \tag{17}$$

Furthermore,

$$\chi_{(1/4,3/4]}(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{\substack{k\,:\,|k|=2s+1 \\ s=0,1,\dots}} \frac{(-1)^s e^{2\pi i k x}}{|k|},$$

and hence the expansion of the function g into Fourier series has the form

$$g = \frac{1}{\pi^d} \sum_{k \in A} \frac{(-1)^{\alpha(k)}}{|k_1 \cdot \ldots \cdot k_d|} e^{2\pi i k x} + \sum_{k\,:\,k_1 \cdot \ldots \cdot k_d = 0} \widehat{g}_k e^{2\pi i k x}, \tag{18}$$

where $A = 2\mathbb{Z}^d + (1, \dots, 1)$ and $\alpha(k) = \sum_{j=1}^d s_j$ for $k = (k_1, \dots, k_d)$, $|k_j| = 2s_j + 1$, $j = 1, \dots, d$, and $\widehat{g}_k$ is the Fourier coefficient of the function g.

It follows from Theorem 1 in [6] (see also Theorem 4 and the remark at the end of [6]) that

$$d_m(Q, L^2) \ge \left(\sum_{j=m+1}^{\infty} \gamma_j^2 \right)^{1/2}, \tag{19}$$

where $\{\gamma_j\}_{j=1}^{\infty}$ is a nonincreasing permutation of the number set $\{|\widehat{g}_k|\}_{k \in \mathbb{Z}^d}$. Since for $R \ge C(d)$ we have

$$\#\Big\{s : \big|(2s_1+1) \cdot \ldots \cdot (2s_d+1)\big| \le R\Big\} \ \ge \#\Big\{s : s_j > 0,\ s_1 \cdot \ldots \cdot s_d \le \frac{R}{3^d}\Big\} \ge c'(d) R (\log R)^{d-1},$$

it follows from (19) and (18) that

$$d_{\beta(R)}(Q, L^2) \ge c(d) \left[\frac{R (\log R)^{d-1}}{R^2} \right]^{1/2} \ge c''(d) \left[\frac{\log^{d-1} R}{R} \right]^{1/2} \tag{20}$$

for $R \ge c(d)$ and $\beta(R) \equiv \big[(1/2) c_4(d) R (\log R)^{d-1}\big]$. For a given m, we apply the bound (20) for R such that $\beta_{R-1} < m \le \beta_R$, and obtain, for $m \ge c_5(d)$,

$$d_m(Q, L^2) \ge c_6(d) \frac{(\log m)^{d-1}}{m^{1/2}}. \tag{21}$$

It follows from Theorem 2 and from relations (13), (17), and (21) that for $N \ge c_7(d)$ we have

$$\begin{aligned} \left\| S_\Omega^* \Big(\sum_{k \in I} \varphi_k \Big) \right\|_{\ell_1} &\ge c \sum_{m \ge 1} \frac{d_{m-1}(\widetilde{\Omega})}{m^{1/2}} = c N^{d/2} \sum_{m \ge 1} \frac{d_{m-1}(F, L^2)}{m^{1/2}} \\ &\ge c N^{d/2} \sum_{m=c_8(d)}^{N} \frac{c_9(d) (\log m)^{d-1}}{m} \ge c_9(d) N^{d/2} (\log N)^d. \end{aligned}$$

For $N \le c_7(d)$ the assertion of Corollary 2 follows from the inequality

$$\Big\|\sum_{k\in I}\varphi_k\Big\|_{\ell_1} \ge \Big\|\sum_{k\in I}\varphi_k\Big\|_{\ell_2} = N^{d/2}.$$

This proves Corollary 2. □

The final result can be regarded as a finite-dimensional quantitative analog of the Olevskii theorem, which states that for each complete o.n.s. $\{\varphi_k\}$ in $L^2(0,1)$ with $\varphi_k \in L^\infty(0,1)$, $k = 1, 2, \dots$ there exists a function $f \in L^1$ such that $S^* f \notin L^1$.

Theorem 3. *Let $\{\varphi_k\}_{k=1}^N$ be an o.n.s. in L_2^N such that for some constant M we have*

$$\sum_{k=1}^{n}\varphi_k^2(x) \le Mn$$

for all $x \in \{1, 2, \dots, N\}$ and $n = 1, 2, \dots, N$. Then there exists a set $E \subset \{1, \dots, N\}$, $|E| \ge c(M)$, such that for $y \in E$ we have

$$\Big\|S_N^*\Big(\sum \varphi_k(y)\varphi_k\Big)\Big\|_{L_1^N} \ge c'(M)\log N,$$

where $c(M) > 0$, $c'(M) > 0$, and $|E| \equiv (\#E)/N$.

We need two simple lemmas.

Lemma 3. *Assume that for some A, $\alpha > 0$ and a function $g \ge 0$ we have $\|g\|_\infty \le A$ and $\|g\|_2^2 \ge \alpha A$. For $k \in \mathbb{N}$ we set $I_k = (2^{-k}A, 2^{-k+1}A]$ and $F_k = g^{-1}(I_k)$. Then there exists a number $k \in \mathbb{N}$ such that*

$$\int_{F_k} g \ge \frac{k\alpha}{4}.$$

Proof. By the conditions of the lemma, we have

$$\begin{aligned}\alpha A &\le \int g^2 = \sum_{k\ge1}\int_{F_k} g^2 \le \sum_{k\ge1} 2^{-k+1}A\int_{F_k} g\\ &= A\sum_{k\ge1} k2^{-k+1}\Big(\frac1k\int_{F_k} g\Big) \le A\Big(\sum_{k\ge1} k2^{-k+1}\Big)\sup_{k\ge1}\Big(\frac1k\int_{F_k} g\Big),\end{aligned}$$

and it remains to compare the first expression with the last one. □

Lemma 4. *Let $A > 0$, let $I_k = (2^kA, 2^{k+1}A]$ for $k \in \mathbb{Z}$, and let h_k, $k \in \mathbb{Z}$, be functions such that all nonzero values of h_k belong to I_k. Then*

$$\int \sup_k h_k \ge \frac13\sum_k\int h_k.$$

Proof. For $k \in \mathbb{Z}$ and $t > 0$, we set $\lambda_k(t) = \big|\{h_k > t\}\big|$; then we have

$$\int h_k = \int_0^\infty \lambda_k(t)\,dt.$$

Since $|\{\sup_k h_k > t\}| \geq \sup_k \lambda_k(t)$ for $t \in (0, \infty)$, it suffices to show that

$$\int_0^\infty \sup_j \lambda_j(t)\, dt \geq \frac{1}{3} \sum_{k \in \mathbb{Z}} \int_0^\infty \lambda_k(t)\, dt. \tag{22}$$

Note that for $t \in I_k$ we have $\sup_j \lambda_j(t) \geq \max\{\lambda_k(t), \lambda_{k+1}(2^{k+1}A)\}$. By applying this inequality to estimate the left-hand side in (22) below and by taking into account the fact that

$$\int_0^\infty \lambda_k(t)\, dt = \int_0^{2^{k+1}A} \lambda_k(t)\, dt = \int_{I_k} \lambda_k(t)\, dt + \lambda_k(2^k A) 2^k A \equiv a_k + b_k,$$

we reduce (22) to the inequality

$$\sum_{k \in \mathbb{Z}} \max\left\{a_k, \frac{1}{2} b_{k+1}\right\} \geq \frac{1}{3} \sum_{k \in \mathbb{Z}} (a_k + b_k).$$

It remains to use the fact that $\max\{a, \frac{1}{2}b\} \geq \frac{1}{3}(a+b)$ for $a, b \geq 0$; this proves (22) and hence, completes the proof of Lemma 4. □

Proof of Theorem 3. Note that for each $m \leq N/2$ we have

$$\int \sum_{k=m+1}^{2m} |\varphi_k|^2 = m \qquad \text{and} \qquad \sum_{k=m+1}^{2m} |\varphi_k|^2 \leq 2Mm.$$

By Lemma 2 (applied to $f = (2Mm)^{-1} \sum_{k=m+1}^{2m} |\varphi_k|^2$), if we have

$$E_m = \left\{y : \sum_{k=m+1}^{2m} |\varphi_k(y)|^2 \geq \frac{m}{2}\right\},$$

then $|E_m| \geq |4M|^{-1}$. Let us now assume that $y \in E_m$ for some m and set

$$g = \left| \sum_{k=m+1}^{2m} \varphi_k(y) \varphi_k \right|;$$

then we have $\|g\|_2 \geq \sqrt{m/2}$ and $\|g\|_\infty \leq 2Mm$.

If $y \in E_{2^{s_j}}$, $j = 1, \dots, t$, for some positive integers $s_1 < s_2 < \cdots < s_t$, then, by Lemma 3, there exist functions

$$g_j = \left| \sum_{i=2^{s_j}+1}^{2^{s_j+1}} \varphi_i(y) \varphi_i \right|,$$

positive integers k_j, and intervals $I_j = (M \cdot 2^{s_j - k_j + 1}, M \cdot 2^{s_j - k_j + 2}]$ such that

$$\int_{g_j^{-1}(I_j)} g_j \geq \beta k_j, \tag{23}$$

where $\beta = (16M)^{-1}$. If some interval I occurs r $(r > 1)$ times in the sequence $I_1, \dots, I_t$, then the corresponding indices k_j are pairwise distinct and one of the integrals (23) corresponding to I must be greater than or equal to βr. In any case, there exists a subsequence of the collection $g_1, g_2, \dots, g_t$ for which the corresponding intervals I_j are disjoint and the sum of the corresponding integrals (23) is at least βt. Then Lemma 4 implies

$$\int \max_{j\le t} g_j \ge \frac{1}{3}\beta t = (48M)^{-1}t.$$

Since we obviously have

$$S^*\left(\sum_{k=1}^{N} \varphi_k(y)\varphi_k\right) \ge \frac{1}{2}\max_{j\le t} g_j,$$

Theorem 3 will be proved if we show that "sufficiently many" points y belong to at least $c_2 \log N$ distinct sets E_{2^s}, where $c_2 = c_2(M)$. But this can readily be derived from the inequality $|E_{2^s}| \ge (4M)^{-1}$ by the argument that has already been used in the proof of Corollary 1. □

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