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BRIEF COMMUNICATIONS

On a Certain Norm and Related Applications

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§1. Introduction

Let μ be the normed Lebesgue measure on the unit circle. For a function $f \in L^1(d\mu)$ with Fourier series

$$f \sim \sum_{k=0}^{\infty} \delta_k(f,x), \qquad \delta_0 = \int_0^{2\pi} f\,d\mu, \quad \delta_k = \sum_{2^{k-1} \le |n| < 2^k} \widehat{f}(n) e^{i(n,x)}, \quad k = 1,2,\dots,$$

we consider the value

$$\|f\|_{\mathrm{QC}} \equiv \int_0^1 \left\| \sum_{k=0}^{\infty} r_k(\omega)\delta_k(f,x) \right\|_{L^\infty(d\mu)} d\omega, \tag{1}$$

where $\{r_k(\omega)\}_{k=0}^{\infty}$ is the Rademacher system. By the *space of quasicontinuous functions* (hence the notation $\|\cdot\|_{\mathrm{QC}}$) we mean the closure of the set of trigonometric polynomials with respect to the norm (1).

Spaces of quasicontinuous functions can also be introduced in the multidimensional case. This may be done in a number of ways. In what follows, we shall consider one of the variants: the closure of the set of trigonometric polynomials of d variables ($d = 2,3,\dots$) with respect to the norm

$$\|f\|_{\mathrm{QC}} \equiv \big\| \|f(\cdot\,,x^1)\|_{\mathrm{QC}} \big\|_\infty, \tag{2}$$

where, by definition, for $x = (x_1,\dots,x_d) \in \mathbb{T}^d$ we take $x^1 = (x_2,\dots,x_d) \in \mathbb{T}^{d-1}$. In other words, identity (2) involves the QC-norm with respect to the variable x_1 and the sup-norm with respect to the other variables.

In the paper inequalities for real-valued trigonometric polynomials in the norms (1), (2) are established. Note that our interest in these norms is related, first of all, to their possible applications to the study of the approximation properties of classes of functions of several variables (in particular, see §3). In addition to the study of the norms (1), (2), a comparison is carried out in this paper (see §4) between the norm $\|t\|_\infty$ and the discrete norm

$$\|t\|_{\infty,\Omega} \equiv \max_{x\in\Omega} |t(x)| \tag{3}$$

($\Omega \subset \mathbb{T}^d$ is a finite set of points) on the subspaces $T(Q_k)$ of trigonometric polynomials with spectrum in the staircase hyperbolic crosses:

$$Q_k \equiv \bigcup_{\|s\|_1 \le k} \rho(s), \qquad \rho(s) \equiv \big\{ n = (n_1,\dots,n_d) \subset \mathbb{Z}^d : [2^{s_j-1}] \le |n_j| < 2^{s_j},\ j = 1,\dots,d \big\}.$$

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§2. Inequalities

Theorem 1. *For any $f \in L^1(d\mu)$, the following inequality is valid:*

$$\|f\|_{QC} \geq \frac{1}{8} \sum_{s=0}^{\infty} \|\delta_{2s}(f, x)\|_{L^1(d\mu)}. \tag{4}$$

Remark 1. It is readily seen that in the proof of Theorem 1 we can confine ourselves to the case in which f is a trigonometric polynomial (we assume that (4) necessarily holds if $\|f\|_{QC} = \infty$).

Remark 2. It follows from Theorem 1 and Grigoriev's results [1] that

$$\sup_{t \in T(2^k)} \frac{\|t\|_{QC}}{\|t\|_\infty} \geq c\sqrt{k}.$$

On the other hand, it is readily seen from results on gap series that

$$\sup_{t \in T(2^k)} \frac{\|t\|_\infty}{\|t\|_{QC}} \geq c_1\sqrt{k};$$

here $T(m)$ is the space of real trigonometric polynomials of degree $\leq m$.

Remark 3. In the two-dimensional case the following inequality is valid (see [2]):

$$\Big\|\sum_{s \in Y_k^2} \delta_s(f)\Big\|_\infty \geq c \sum_{s \in Y_k^2} \|\delta_s(f)\|_1, \tag{5}$$

where, by definition, for even k we have

$$Y_k^d = \{s = (2k_1, \dots, 2k_d),\ k_1 + k_2 + \dots + k_d = k/2\}, \quad d = 1, 2, \dots, \qquad \delta_s(f) = \sum_{n \in \rho(s)} \hat{f}(n) e^{i(n,x)}.$$

In [1] (see also Remark 2 above) an example showing that there is no analog of inequality (5) in the one-dimensional case was constructed. The problem of the validity of the corresponding analogs of the estimate (5) for $d \geq 3$ remains open (see the discussion of this problem in [3]). For the d-dimensional case, let us cite an inequality similar to (5) but with norm $\|\cdot\|_{QC}$ instead of $\|\cdot\|_\infty$.

Theorem 2. *Suppose that for an even k the following polynomial in d variables ($d = 2, 3, \dots$) is given:*

$$f = \sum_{s_1 \in Z_k} \sum_{\|s^1\|_1 = k - s_1} \delta_s(f), \qquad s^1 = (s_2, \dots, s_d), \quad Z_k = \{2l\}_{l=0}^{k/2}$$

with the property: for some $G \subset Z_k$

1) $\|\delta_s(f)\|_4 \leq 1$ *if* $s_1 \in G$;
2) *the following estimate is valid:*

$$\sum_{s_1 \in G} \sum_{\|s^1\|_1 = k - s_1} \|\delta_s(f)\|_2^2 \geq bk^{d-1},$$

where $b > 0$ is an absolute constant.

Then

$$\|f\|_{QC} \geq ck^{d/2}, \qquad c = c(b) > 0.$$

The following result shows that also in the one-dimensional case under additional constraints on the polynomial f its uniform norm admits a lower bound similar to (4).

Theorem 3. *For any polynomial of the form*

$$f = \sum_{k=l+1}^{2l} p_k(x)\cos 4^k x,$$

where $p_k \in T(2^l)$, $k = l+1, \dots, 2l$, *the following inequality is valid:*

$$\|f\|_\infty \geq c \sum_{k=l+1}^{2l} \|p_k\|_1, \qquad c > 0.$$

§3. Estimates of entropy numbers and Kolmogorov widths

In what follows, we preserve the notation and definitions used in our joint paper [4], in which the approximation characteristics of classes of functions of d variables were studied.

Theorem 4. *For* $r > \max(1/q, 1/2)$, *the following relations are valid:*

$$\varepsilon_m(H_q^r, \mathrm{QC}) \asymp m^{-r}(\log m)^{r(d-1)+d/2}, \quad \varepsilon_m(W_q^r, \mathrm{QC}) \asymp m^{-r}(\log m)^{r(d-1)+1/2}, \qquad 1 < q \leq \infty.$$

Theorem 5. *For* $r > 1/2$ *and* $2 \leq q \leq \infty$, *the following relations are valid:*

$$d_m(H_q^r, \mathrm{QC}) \asymp m^{-r}(\log m)^{r(d-1)+d/2}, \qquad d_m(W_q^r, \mathrm{QC}) \asymp m^{-r}(\log m)^{r(d-1)+1/2}.$$

Inequality (5) was used in [2] to obtain lower bounds for the entropy numbers of function classes. With the help of Theorem 2, similar arguments yield the lower bounds in Theorem 4. The lower bounds in Theorem 5 follow from Theorem 4 and well-known inequalities connecting the entropy numbers ε_m and the Kolmogorov widths d_m (e.g., see [5]). The upper bounds in Theorems 4, 5 can be established similarly to the corresponding upper bounds in the metric L^∞ from [6].

Theorem 3 allows us to obtain the correct order of the entropy numbers and Kolmogorov widths of the classes LG^r of functions of a single variable with smoothness of logarithmic type. Let us define the classes LG^r, $r > 0$, by the following condition on the binary blocks of the Fourier series of their members:

$$LG^r = \{f \in L^\infty : \|\delta_s(f)\|_\infty \leq (1+s)^{-r},\ s = 0, 1, \dots\}.$$

Theorem 6. *Let* $r > 1$. *The following relations are valid* ($m \to \infty$):

$$\varepsilon_m(LG^r, L^p) \asymp d_m(LG^r, L^p) \asymp \begin{cases} (\log m)^{-r+1} & \text{for } p = \infty, \\ (\log m)^{-r+1/2} & \text{for } 1 \leq p < \infty. \end{cases}$$

In particular, Theorem 6 shows that the order of $\varepsilon_m(\cdot\, L^p)$ and $d_m(\cdot\, L^p)$ changes by a jump under the transition from $p < \infty$ to $p = \infty$. A similar phenomenon in the two-dimensional case occurs for the classes H_∞^r (see [2, 3]).

§4. The discrete L^∞-norm for polynomials in d variables from $T(Q_k)$

It is well known that for the space $T(\Pi)$ of trigonometric polynomials in d variables with spectrum in a parallelepiped Π there exists a finite set Ω such that the number $|\Omega|$ of elements in Ω has the same order as the dimension of $T(\Pi)$ and the following equivalence holds:

$$\|t\|_{\infty,\Omega} \asymp \|t\|_\infty, \qquad t \in T(\Pi)$$

(see also (3)).

Theorem 7 (stated below) shows that the situation is different for the spaces $T(Q_k)$: the equivalence of the norms $\|t\|_{\infty,\Omega}$ and $\|t\|_\infty$ for polynomials from $T(Q_k)$ can occur only if the number of points in Ω is much larger than the dimension $\dim T(Q_k) \asymp 2^k k^{d-1}$: $|\Omega| \geq 2^{(1+\gamma)k}$, $\gamma > 0$.

Theorem 7. *Suppose that the set* $\Omega \subset \mathbb{T}^d$ *possesses the following property: for any polynomial* $t \in T(Q_k)$,

$$\|t\|_\infty \leq bk^\alpha \|t\|_{\infty,\Omega}, \qquad 0 \leq \alpha \leq \frac{1}{2}.$$

Then

$$|\Omega| \geq c_1 |Q_k| \exp(ck^{1-2\alpha}), \qquad c = c(b) > 0, \quad c_1 = c_1(b) > 0.$$

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