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A Note on the Description of Frames of General Form

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Let H be a Hilbert space. A system of nonzero elements $\Phi = \{\varphi_j : j \in \mathbb{N}\} \subset H$ is called a *frame* if the following inequalities hold:

$$A\|g\|_H^2 \le \sum_{j=1}^{\infty} |(g, \varphi_j)|^2 \le B\|g\|_H^2 \qquad \forall g \in H, \tag{1}$$

where $0 < A \le B < \infty$ are absolute constants and $\|\cdot\|$ and $(\cdot\,,\,\cdot)$ are the norm and the inner product in H. The constants B and A are respectively called the *upper* and *lower bounds* of the frame Φ, and their ratio $\kappa = B/A$ is called the *condition number* and is denoted by $\kappa(\Phi)$. For the case in which $\kappa(\Phi) = 1$, i.e., $A = B$, the frame Φ is said to be *tight*. Frames were introduced in 1952 by Duffin and Schaeffer [1] (see also [2]), but some implicit results about frames were obtained earlier. In particular, in quantum information theory and in several areas of functional analysis, the following result has been known as Naimark's theorem since 1940.

Theorem A (see [3, Chap. 10; 4; 9]). *For any tight frame $\Phi = \{\varphi_j : j \in \mathbb{N}\}$ in a Hilbert space H, there exist a Hilbert space $H' \supset H$ and an orthonormal basis $\Psi = \{\psi_j : j \in \mathbb{N}\}$ of the space H' such that*

$$\varphi_j = \pi_{H' \to H}(\psi_j), \qquad j = 1, 2, \dots,$$

where $\pi_{H' \to H}$ is the orthogonal projection operator from H' on H.

Some results close to Theorem A are also contained in Kozlov's work written in 1948, where, in fact, tight frames were studied (see [5], as well as [6] and [7, Theorem 8.3]). Recently, some results adjacent to Theorem A were obtained by Lukashenko [8].

In the last few years, frames have been widely used in applied mathematics, in particular, in the construction of algorithms for image compression. In this connection, it seems to be natural to describe frames in general form, similarly to Theorem A. We have not found the corresponding result in the literature and so present our result in this short paper. We believe that the proof given below is simpler than the well-known proofs of Naimark's theorem (the argument given below under the assumption that $\kappa(\Phi) = 1$ is similar to verifying the following elementary fact from linear algebra: an orthonormal set of vectors $\{v_i,\ i = 1, 2, \dots, s\} \subset \mathbb{R}^n$, $1 \le s < n$, can be completed to an orthonormal basis in $\mathbb{R}^n$; the finite-dimensional version of Theorem 1 established below is also very transparent for the case in which $\kappa(\Phi) > 1$).

Prior to stating and proving our result, we introduce some notation. Let Λ be a countable set whose elements are arranged in some given order (in other words, we assume that a one-to-one mapping of Λ onto the set of positive integers $\mathbb{N}$ is chosen). By $\ell^2(\Lambda)$ we denote the Hilbert space of number sets $\{c_\omega,\ \omega\in\Lambda\}$ indexed by elements from Λ; this space is endowed with the norm

$$\Bigl(\sum_{\omega\in\Lambda}|c_\omega|^2\Bigr)^{1/2}.$$

The classical Hilbert space $\ell^2=\ell^2(\mathbb{N})$ corresponds to the case in which $\Lambda=\mathbb{N}$ and the elements of Λ are arranged in the natural order. In this case, the jth element a_j of the sequence $a=\{a_j : j\in\mathbb{N}\}\in\ell^2$ is also denoted by $(a)_j$. A system of elements $\Psi=\{\psi_\omega:\omega\in\Lambda\}$ of the Hilbert space H' is called the *Riesz basis* in H' if Ψ is complete in H' and the following estimates are satisfied for any set of numbers $\{c_\omega:\omega\in\Lambda\}\in\ell^2(\Lambda)$:

$$A^{1/2}\Bigl(\sum_{\omega\in\Lambda}|c_\omega|^2\Bigr)^{1/2}\le\Bigl\|\sum_{\omega\in\Lambda}c_\omega\psi_\omega\Bigr\|_{H'}\le B^{1/2}\Bigl(\sum_{\omega\in\Lambda}|c_\omega|^2\Bigr)^{1/2}, \tag{2}$$

where $0<A\le B<\infty$ are absolute constants (about Riesz bases, see, e.g., [7, p. 17]; usually, $\Lambda=\mathbb{N}$; we also note that the property of a system to be a Riesz basis is independent of the order of elements in this system).

Theorem 1. *For a system $\Phi=\{\varphi_j:j\in\mathbb{N}\}$ of elements of a Hilbert space H to be a frame with bounds A and B, $A\le B$, it is necessary and sufficient that there exist a Hilbert space H' containing H and a Riesz basis $\Psi=\{\psi_j:j\in\mathbb{N}\}$ in H' with properties* (2) *(for $\Lambda=\mathbb{N}$) such that*

$$\varphi_j=\pi_{H'\to H}(\psi_j),\qquad j=1,2,\dots.$$

In what follows, we assume for definiteness that H is a real Hilbert space. The following assertion is, in fact, well known.

Proposition 1. *Suppose that Λ is a countable set, $V=\{v_\omega:\omega\in\Lambda\}$ is a Riesz basis in a Hilbert space H, and moreover, for all $c\in\ell^2(\Lambda)$,*

$$A^{1/2}\Bigl(\sum_{\omega\in\Lambda}c_\omega^2\Bigr)^{1/2}\le\Bigl\|\sum_{\omega\in\Lambda}c_\omega v_\omega\Bigr\|_{H}\le B^{1/2}\Bigl(\sum_{\omega\in\Lambda}c_\omega^2\Bigr)^{1/2}, \tag{3}$$

where $A>0$ and B are absolute constants. Then the system $V^*=\{v_\mu^*:\mu\in\Lambda\}$ dual to V is a Riesz basis in the dual space H^* and the following estimate holds:

$$\frac{1}{B^{1/2}}\Bigl(\sum_{\mu\in\Lambda}\beta_\mu^2\Bigr)^{1/2}\le\Bigl\|\sum_{\mu\in\Lambda}\beta_\mu v_\mu^*\Bigr\|_{H^*}\le\frac{1}{A^{1/2}}\Bigl(\sum_{\mu\in\Lambda}\beta_\mu^2\Bigr)^{1/2}. \tag{4}$$

(Recall that the dual system is uniquely determined by the relations

$$\langle v_\mu^*,v_\omega\rangle=\begin{cases}1 & \text{if } \mu=\omega,\\ 0 & \text{if } \mu\ne\omega,\end{cases}$$

where $\langle f,g\rangle$ is the value of the functional $f\in H^*$ on the element $g\in H$.)

Proof. The fact that the system V^* is complete and minimal follows from general results about bases (see [7, Chap. 1]). Let us verify inequalities (4). We have

$$\begin{aligned}\Bigl\|\sum_{\mu\in\Lambda}\beta_\mu v_\mu^*\Bigr\|_{H^*}&=\sup_{\{c_\omega\}\,:\,\|\sum_{\omega\in\Lambda}c_\omega v_\omega\|_H\le1}\Bigl\langle\sum_{\mu\in\Lambda}\beta_\mu v_\mu^*,\sum_{\omega\in\Lambda}c_\omega v_\omega\Bigr\rangle\\&=\sup_{\{c_\omega\}\,:\,\|\sum_{\omega\in\Lambda}c_\omega v_\omega\|_H\le1}\sum_{\omega\in\Lambda}\beta_\omega c_\omega\ge\frac{1}{B^{1/2}}\Bigl(\sum_{\omega\in\Lambda}\beta_\omega^2\Bigr)^{1/2},\end{aligned}$$

since, by virtue of (3), the inclusion

$$\Big\{\{c_\omega\} : \Big\|\sum c_\omega v_\omega\Big\|_H \le 1\Big\} \supset \Big\{\{c_\omega\} : \Big(\sum_{\omega\in\Lambda} c_\omega^2\Big)^{1/2} \le \frac{1}{B^{1/2}}\Big\}$$

holds. Similarly, we have

$$\Big\|\sum_{\mu\in\Lambda} \beta_\mu v_\mu^*\Big\|_{H^*} = \sup_{\{c_\omega\} : \|\sum_{\omega\in\Lambda} c_\omega v_\omega\|_H \le 1} \sum_{\omega\in\Lambda} \beta_\omega c_\omega ,$$

and, in view of the inclusion

$$\Big\{\{c_\omega\} : \Big\|\sum_{\omega\in\Lambda} c_\omega v_\omega\Big\|_H \le 1\Big\} \subset \Big\{\{c_\omega\} : \Big(\sum_{\omega\in\Lambda} c_\omega^2\Big)^{1/2} \le \frac{1}{A^{1/2}}\Big\}$$

we obtain

$$\Big\|\sum_{\mu\in\Lambda} \beta_\mu v_\mu^*\Big\|_{H^*} \le \frac{1}{A^{1/2}}\Big(\sum_{\mu\in\Lambda} \beta_\mu^2\Big)^{1/2}.$$

The proof of Proposition 1 is complete. □

For any Riesz basis $V = \{v_\omega : \omega \in \Lambda\}$ with property (3) and for any element $f \in H^*$, Proposition 1 implies the relations

$$A\|f\|_{H^*}^2 \le \sum_{\omega\in\Lambda} \langle f, v_\omega\rangle^2 \le B\|f\|_{H^*}^2. \tag{5}$$

Indeed, using Proposition 1 to expand the functional f with respect to the basis $\{v_\mu^* : \mu \in \Lambda\}$,

$$f = \sum_{\omega\in\Lambda} \langle f, v_\omega\rangle v_\omega^*,$$

we obtain (5). Since the spaces H and H^* are isometric, it follows from (5) that, under the assumptions of Proposition 1, the inequality

$$A\|g\|_H^2 \le \sum_{\omega\in\Lambda} (g, v_\omega)^2 \le B\|g\|_H^2 \tag{6}$$

holds for any $g \in H$.

Proof of Theorem 1. *Sufficiency*. Suppose that $\Psi = \{\psi_j : j \in \mathbb{N}\}$ is a Riesz basis in H', $H' \supset H$, and relations (2) are satisfied for $\Lambda = \mathbb{N}$. Suppose also that $\varphi_j = \pi_{H'\to H}(\psi_j)$, $j = 1, 2, \dots$. Then for any element $g \in H$ we have $(g, \varphi_j) = (g, \psi_j)$, $j = 1, 2, \dots$, and hence for any $g \in H$ we can write

$$A\|g\|_H^2 \le \sum_{j=1}^{\infty} (g, \varphi_j)^2 \le B\|g\|_H^2,$$

i.e., the system $\{\varphi_j : j \in \mathbb{N}\}$ is a frame.

Necessity. Without loss of generality, we assume that $H = \ell^2$. We construct an infinite matrix R whose jth column coincides with the sequence $\varphi_j \in \ell^2$, $j = 1, 2, \dots$. Let v_i, $i = 1, 2, \dots$, be rows of the matrix R. Then inequalities (1) means that the estimates

$$A\|\alpha\|_{\ell^2}^2 \le \Big\|\sum_{i=1}^{\infty} \alpha_i v_i\Big\|_{\ell^2} \le B\|\alpha\|_{\ell^2}^2 \tag{7}$$

hold for any sequence $\alpha = \{\alpha_i : i = 1, 2, \dots\} \in \ell^2$. Let L be the closure (with respect to the norm of ℓ^2) of the linear span of the system $V = \{v_i : i \in \mathbb{N}\}$. It follows from (7) that $\dim L = \infty$ and V is a Riesz basis in L. We consider the orthogonal complement $L^\perp$ of L in ℓ^2. Let $W = \{w_\nu : \nu \in M\}$ be an arbitrary orthogonal basis in $L^\perp$, and let

$$\forall \nu \in M \qquad A \le \|w_\nu\|_{\ell^2}^2 \le B, \tag{8}$$

where M is some countable, finite, or empty (depending on the dimension of the subspace $L^\perp$) set of indices, $M \cap \mathbb{N} = \varnothing$, and the elements of the system W are "numbered" by points from M. Suppose that $\Lambda = \mathbb{N} \cup M$ and the system $\Gamma = \{\gamma_\omega : \omega \in \Lambda\} \subset \ell^2$ is determined by the relations

$$\gamma_\omega = \begin{cases} v_i & \text{if } \omega = i \in \mathbb{N}, \\ w_\nu & \text{if } \omega = \nu \in M. \end{cases} \tag{9}$$

Then it is easy to see that Γ is a Riesz basis in ℓ^2 and, moreover,

$$\biggl(A \sum_{\omega\in\Lambda} c_\omega^2\biggr)^{1/2} \le \biggl\| \sum_{\omega\in\Lambda} c_\omega \gamma_\omega \biggr\|_{\ell^2} \le \biggl(B \sum_{\omega\in\Lambda} c_\omega^2\biggr)^{1/2}. \tag{10}$$

Indeed, for any element $g \in \ell^2$ we have the decomposition

$$g = g_L + g_{L^\perp}, \qquad \text{where} \quad g_L = \pi_{\ell^2\to L}(g), \quad g_{L^\perp} = \pi_{\ell^2\to L^\perp}(g).$$

If

$$g_L = \sum_{i\in\mathbb{N}} \alpha_i v_i, \qquad g_{L^\perp} = \sum_{\nu\in M} \beta_\nu w_\nu,$$

then the fact that g_L and $g_{L^\perp}$ are orthogonal and inequalities (7) and (8) imply

$$\|g\|^2 = \|g_L\|_{\ell^2}^2 + \|g_{L^\perp}\|_{\ell^2}^2 \le B \sum_{i\in\mathbb{N}} \alpha_i^2 + \sum_{\nu\in M} \beta_\nu^2 \|w_\nu\|_{\ell^2}^2 \le B \biggl(\sum_{i\in\mathbb{N}} \alpha_i^2 + \sum_{\nu\in M} \beta_\nu^2\biggr),$$
$$\|g\|^2 = \|g_L\|_{\ell^2}^2 + \|g_{L^\perp}\|_{\ell^2}^2 \ge A \sum_{i\in\mathbb{N}} \alpha_i^2 + \sum_{\nu\in M} \beta_\nu^2 \|w_\nu\|_{\ell^2}^2 \ge A \biggl(\sum_{i\in\mathbb{N}} \alpha_i^2 + \sum_{\nu\in M} \beta_\nu^2\biggr),$$

which proves (10).

Choosing an arbitrary order of elements of the set Λ, we consider the "matrix"

$$\widetilde{R} = \{r_{\omega,j} : \omega \in \Lambda,\ j \in \mathbb{N}\}, \qquad \text{where} \quad r_{\omega,j} = \begin{cases} (v_\omega)_j & \text{if } \omega \in \mathbb{N}, \\ (w_\omega)_j & \text{if } \omega \in M \end{cases}$$

for all $\omega \in \Lambda$ and $j \in \mathbb{N}$. Let ψ_j, $j = 1, 2, \dots$, be the "jth column of the matrix $\widetilde{R}$,", i.e., the set of numbers $\{r_{\omega,j} : \omega \in \Lambda\} \subset \ell^2(\Lambda)$. Suppose also that

$$\pi_0 = \pi_{\ell^2(\Lambda)\to\ell^2(\mathbb{N})}$$

is the orthogonal projection operator from the Hilbert space $\ell^2(\Lambda)$ on its subspace $\ell^2(\mathbb{N})$. Then it is clear that $\pi_0(\psi_j) = \varphi_j$, $j = 1, 2, \dots$, and it remains to verify that the system $\{\psi_j\}$ is a Riesz basis in $H' = \ell^2(\Lambda)$ and inequalities (2) hold. Inequalities (10) mean that

$$A\|c\|_{\ell^2(\Lambda)}^2 \le \sum_{j=1}^{\infty} (c, \psi_j)^2 \le B\|c\|_{\ell^2(\Lambda)}^2 \tag{11}$$

for any element $c = \{c_\omega\} \in \ell^2(\Lambda)$. It follows from (11) that the system $\{\psi_j\}$ is complete in $\ell^2(\Lambda)$. Next, for any sequence $\rho = \{\rho_j\} \in \ell^2(\mathbb{N})$, we consider the series

$$\sum_{j=1}^{\infty} \rho_j \psi_j$$

in $\ell^2(\Lambda)$. Then (see (9)) we have

$$\left\| \sum_{j=1}^{\infty} \rho_j \psi_j \right\|_{\ell^2(\Lambda)}^2 = \sum_{\omega \in \Lambda} (\rho, \gamma_\omega)^2. \tag{12}$$

Since $\Gamma = \{\gamma_\omega\}$ is a Riesz basis in ℓ^2 and inequalities (10) hold, we can use relation (6) and, taking (12) into account, obtain

$$A \sum_{j=1}^{\infty} \rho_j^2 \le \left\| \sum_{j=1}^{\infty} \rho_j \psi_j \right\|_{\ell^2(\Lambda)}^2 \le B \sum_{j=1}^{\infty} \rho_j^2.$$

The proof of Theorem 1 is complete. □

In conclusion, we wish to make an addition to Theorem 1 from our recent paper [10]. As in [10], it is possibe to find lower bounds for the quantities

$$\gamma_n(W, \Phi, D) = \sup_{f \in W} \inf_{\Lambda_n, \{c_j\}_{j \in \Lambda_n} : (\sum c_j^2)^{1/2} \le D\|f\|} \left\| f - \sum_{j \in \Lambda_n} c_j \varphi_j \right\|_H,$$

where $W \subset H$, $\Phi = \{\varphi_j\}$ is a frame in H, and Λ_n runs through the family of all n-element subsets of $\mathbb{N}$.

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