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THE VOLUME ESTIMATES AND THEIR APPLICATIONS

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Dedicated to S. A. Telyakovskii on the occasion of his 70th birthday

We prove new estimates for the entropy numbers of classes of multivariate functions with bounded mixed derivative. It is known that the investigation of these classes requires development of new techniques comparing to the univariate classes. In this paper we continue to develop the technique based on estimates of volumes of sets of the Fourier coefficients of trigonometric polynomials with frequencies in special regions. We obtain new volume estimates and use them to get right orders of decay of the entropy numbers of classes of functions of two variables with a mixed derivative bounded in the L_1-norm. This is the first such result for these classes. This result essentially completes the investigation of orders of decay of the entropy numbers of classes of functions of two variables with bounded mixed derivative. The case of similar classes of functions of more than two variables is still open.

1. Introduction

We obtain in this paper new estimates of volumes of sets of the Fourier coefficients of trigonometric polynomials of two variables with frequencies in a *hyperbolic layer*. Section 2 is devoted to this kind of estimates. Then, in Section 3 we use results of Section 2 to prove lower estimates for the entropy numbers of classes of functions with bounded mixed derivative. The new volume estimates from Section 2 allowed us to essentially complete the investigation of orders of decay of the entropy numbers $\epsilon_m(W^r_{q,\alpha}, L_p)$ for all $1 \leq q, p \leq \infty$ and large enough r in the case of classes of functions of two variables. In Section 4 we apply results of Section 2 to the following problem of discretization. We look for a set of points with a property: (E) the

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uniform norm on this set of points is equivalent to the uniform norm for any trigonometric polynomial with frequencies from a given hyperbolic layer. We give in Section 4 other proof of the following surprising result from [KT3], [KT4]: the cardinality of a set with the property (E) must grow at least as N^{1+a}, $a > 0$, where N is the dimension of the corresponding subspace of trigonometric polynomials. The proof of the above result from [KT3], [KT4] made use of extremal properties of the multivariate normal distribution and an inequality for the trigonometric polynomials from [T2]. Later, other inequality for the trigonometric polynomials has been obtained in [T6]. Our proof in this paper uses this new inequality and the volume estimates from Section 2. This allowed us to simplify the original proof (see [KT4]).

We now formulate the main results of the paper. Let $s = (s_1, \dots, s_d)$ be a vector with non-negative integer coordinates ($s \in \mathbb{Z}^d_+$) and

$$\rho(s) := \{k = (k_1, \dots, k_d) \in \mathbb{Z}^d_+ : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}$$

where $[a]$ denotes the integer part of a number a. For any natural number n, let us denote

$$Q_n := \bigcup_{\|s\|_1 \leq n} \rho(s); \qquad \Delta Q_n := Q_n \setminus Q_{n-1}$$

with $\|s\|_1 = s_1 + \dots + s_d$ for $s \in \mathbb{Z}^d_+$. We call the set ΔQ_n a *hyperbolic layer*. For a finite set $\Lambda \subset \mathbb{Z}^d$ denote

$$\mathcal{T}(\Lambda) := \left\{ f : f(x) = \sum_{k \in \Lambda} c_k e^{i(k,x)} \right\}.$$

We assign to each $f = \sum_{k\in\Lambda} \hat{f}(k) e^{i(k,x)} \in \mathcal{T}(\Lambda)$ a vector

$$A(f) := \{(\mathrm{Re}\hat{f}(k), \mathrm{Im}\hat{f}(k)), \quad k \in \Lambda\} \in \mathbb{R}^{2|\Lambda|}$$

where $|\Lambda|$ denotes the cardinality of Λ and define

$$B_\Lambda(L_p) := \{A(f) : f \in \mathcal{T}(\Lambda), \quad \|f\|_p \leq 1\}.$$

The volume estimates of the sets $B_\Lambda(L_p)$ and related questions have been studied in a number of papers by the authors: the case $\Lambda = [-n, n]$, $p = \infty$ in [K1]; the case $\Lambda = [-N_1, N_1] \times \dots \times [-N_d, N_d]$, $p = \infty$ in [T2], [T3]; the case of arbitrary Λ and $p = 1$ in [KT1]. In particular, the results of [KT1] imply (see Theorem 2.4 of this paper) for $d = 2$ and $1 \leq p < \infty$ that

$$(\mathrm{vol}\,(B_{\Delta Q_n}(L_p)))^{(2|\Delta Q_n|)^{-1}} \asymp |\Delta Q_n|^{-1/2} \asymp (2^n n)^{-1/2}.$$

We shall prove in Section 2 (see Theorem 2.5) that in the case $p = \infty$ the volume estimate is different:

$$(\mathrm{vol}\,(B_{\Delta Q_n}(L_\infty)))^{(2|\Delta Q_n|)^{-1}} \asymp (2^n n^2)^{-1/2}.$$

We note that in the case $\Lambda = [-N_1, N_1] \times \cdots \times [-N_d, N_d]$ the volume estimate is the same for all $1 \le p \le \infty$. We discuss this in more detail in Section 2.

In Section 3 we apply the volume estimates to the problem of asymptotic behavior of the entropy numbers of the classes $W^r_{1,\alpha}$. We now give the corresponding definitions. For a natural number m and a compact set F in a Banach space X with the unit ball $B(X)$ we define the mth entropy number as

$$\epsilon_m(F, X) := \inf\left\{\epsilon : \exists f_1, \dots, f_{2^m} \in X : \ F \subset \bigcup_{j=1}^{2^m} (f_j + \epsilon B(X))\right\}.$$

Let $r > 0$ and $\alpha \in \mathbb{R}$. Define

$$F_r(t, \alpha) := 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos(kt - \alpha\pi/2), \quad t \in [0, 2\pi].$$

We define for $x = (x_1, x_2)$ and $\alpha = (\alpha_1, \alpha_2)$

$$F_r(x, \alpha) := F_r(x_1, \alpha_1) F_r(x_2, \alpha_2).$$

Finally, we define

$$W^r_{q,\alpha} := \{f : f = F_r(\cdot, \alpha) * \varphi, \quad \|\varphi\|_q \le 1\}$$

where $*$ means convolution. The problem of estimating $\epsilon_m(W^r_{q,\alpha}, L_p)$ has a long history. We shall mention some results only in the case $d = 2$. The first result on the right order of $\epsilon_m(W^r_{q,\alpha}, L_p)$ in the case $p = q = 2$ has been obtained by Smolyak [Sm] in 1960. Here is a list of further contributions: Dinh Dung [D], 1985, the case $1 < q = p < \infty$; Temlyakov [T1], [T2], 1988, the case $1 < q, p < \infty$, $r > 1$ and the case $p = 1, 1 < q < \infty$; Belinskii [B], 1990, the case $p = 1$, $q = \infty$, $r > 1/2$; Kashin, Temlyakov [KT1], [KT2], 1994–1995, the case $p = 1$, $q = \infty$, $r > 0$; Kuelbs, Li [KL], 1993, the case $p = \infty$, $q = 2$, $r = 1$; Temlyakov [T5], 1995, the case $p = \infty$, $1 < q < \infty$; Temlyakov [T6], 1998, the case $p = \infty$, $q = \infty$, $r > 1/2$.

In Section 3 we study the case left open: $q = 1$, $1 \le p \le \infty$. We prove that (see Theorem 3.1):

$$\begin{gathered} \epsilon_m(W^r_{1,\alpha}, L_p) \asymp m^{-r} (\log m)^{r+1/2}, \\ 1 \le p < \infty, \quad r > \max(1/2, 1 - 1/p) \end{gathered} \tag{1.1}$$

and

$$\epsilon_m(W^r_{1,0}, L_\infty) \asymp m^{-r} (\log m)^{r+1}, \quad r > 1.$$

It is interesting to compare these estimates with the case $1 < q, p < \infty$ where we have

$$\epsilon_m(W^r_{q,\alpha}, L_p) \asymp m^{-r} (\log m)^r. \tag{1.2}$$

We note that when we turn from $q > 1$ to $q = 1$ the exponent for $\log m$ jumps from r in (1.2) to $r + 1/2$ in (1.1).

2. The volume estimates

The main goal of this section is to prove new estimates for volumes of sets of the Fourier coefficients of trigonometric polynomials of two variables (dimension $d = 2$) with frequencies in a *hyperbolic layer*. In some cases we shall give estimates in arbitrary dimension d. It is well-known (see [K1], [T2], [T4], [KT1]) that the volume estimates of the above mentioned sets can be used in different problems of approximation theory including the problem of estimating the entropy numbers of function classes. We use the notations $\mathcal{T}(\Lambda)$, $A(f)$, and $B_\Lambda(L_p)$ introduced in Section 1. In the case

$$\Lambda = \Pi(N) := [-N_1, N_1] \times \cdots \times [-N_d, N_d], \quad N := (N_1, \ldots, N_d),$$

the volume estimates are known. We formulate it as a theorem.

Theorem 2.1. *For any $1 \le p \le \infty$ we have*

$$(\mathrm{vol}\,(B_{\Pi(N)}(L_p)))^{(2|\Pi(N)|)^{-1}} \asymp |\Pi(N)|^{-1/2},$$

with constants in $\asymp$ that may depend only on d.

We note that the most difficult part of Theorem 2.1 is the lower estimate for $p = \infty$. The corresponding estimate was proved in the case $d = 1$ in [K1] and in the general case in [T2] and [T3]. The upper estimate for $p = 1$ in Theorem 2.1 can be easily reduced to the volume estimate for an octahedron (see, for instance, [T4]).

The results of [KT1] imply the following estimate.

Theorem 2.2. *For any finite set $\Lambda \subset \mathbb{Z}^d$ and any $1 \le p \le 2$ we have*

$$\mathrm{vol}\,(B_\Lambda(L_p))^{(2|\Lambda|)^{-1}} \asymp |\Lambda|^{-1/2}.$$

The following result of Bourgain-Milman [BM] plays an important role in the volume estimates of finite dimensional bodies.

Theorem 2.3. *For any convex centrally symmetric body $K \subset \mathbb{R}^n$ we have*

$$(\mathrm{vol}\,(K)\mathrm{vol}\,(K^o))^{1/n} \asymp (\mathrm{vol}\,(B_2^n))^{2/n} \asymp 1/n$$

where K^o is a polar for K, that is

$$K^o := \{x \in \mathbb{R}^n : \sup_{y\in K}(x,y) \le 1\}.$$

Having in mind an application of Theorem 2.3 we define some other than $B_\Lambda(L_p)$ sets. Let

$$E^\perp_\Lambda(f)_p := \inf_{g\perp\mathcal{T}(\Lambda)} \|f-g\|_p, \qquad E^\perp_{\Lambda,R}(f)_p := \inf_{g\perp\mathcal{T}(\Lambda),\hat{g}(k)\in\mathbb{R}} \|f-g\|_p$$

and
$$B_\Lambda^\perp(L_p) := \{A(f) : f \in \mathcal{T}(\Lambda), \quad E_\Lambda^\perp(f)_p \le 1\}.$$
It is clear that
$$B_\Lambda(L_p) \subseteq B_\Lambda^\perp(L_p).$$
Moreover, if the orthogonal projector P_Λ onto $\mathcal{T}(\Lambda)$ is bounded as an operator from L_p to L_p, then we have

$$\operatorname{vol}(B_\Lambda(L_p))^{(2|\Lambda|)^{-1}} \asymp \operatorname{vol}(B_\Lambda^\perp(L_p))^{(2|\Lambda|)^{-1}}. \tag{2.1}$$

For example, it is the case when $\Lambda = \bigcup_{s\in A} \rho(s)$. We also consider

$$B_{\Lambda,R}(L_p) := \{A(f) : \|f\|_p \le 1, \quad f \in \mathcal{T}(\Lambda), \quad \hat{f}(k) \in \mathbb{R}\}$$

and

$$B_{\Lambda,R}^\perp(L_p) := \{A(f) : E_{\Lambda,R}^\perp(f)_p \le 1, \quad f \in \mathcal{T}(\Lambda), \quad \hat{f}(k) \in \mathbb{R}\}.$$

Using the Nikol'skii duality theorem and the basic properties of the de la Vallée-Poussin operators one can check that

$$\frac{1}{2} B_{\Lambda,R}^\perp(L_{p'}) \subset B_{\Lambda,R}(L_p)^o \subset 2B_{\Lambda,R}^\perp(L_{p'}), \quad p' = \frac{p}{p-1}. \tag{2.2}$$

We shall now show that the volumes of $B_\Lambda(L_p) \subset \mathbb{R}^{2|\Lambda|}$ and $B_{\Lambda,R}(L_p) \subset \mathbb{R}^{|\Lambda|}$ are closely related. First of all, if

$$\Big\|\sum_{k\in\Lambda} a_k e^{i(k,x)}\Big\|_p \le 1/2, \qquad \Big\|\sum_{k\in\Lambda} b_k e^{i(k,x)}\Big\|_p \le 1/2,$$

then
$$\Big\|\sum_{k\in\Lambda} (a_k + ib_k) e^{i(k,x)}\Big\|_p \le 1.$$

Therefore,

$$\Big(\frac{1}{2} B_{\Lambda,R}(L_p)\Big) \otimes \Big(\frac{1}{2} B_{\Lambda,R}(L_p)\Big) \subseteq B_\Lambda(L_p). \tag{2.3}$$

Next, let
$$f(x) = \sum_{k\in\Lambda} (a_k + ib_k) e^{i(k,x)}.$$

Then
$$\sum_{k\in\Lambda} a_k e^{i(k,x)} = \frac{1}{2}(f(x) + \bar{f}(-x))$$

and

$$i \sum_{k \in \Lambda} b_k e^{i(k,x)} = \frac{1}{2}(f(x) - \bar{f}(-x)).$$

This implies that

$$B_\Lambda(L_p) \subseteq B_{\Lambda,R}(L_p) \otimes B_{\Lambda,R}(L_p). \tag{2.4}$$

We get from (2.3) and (2.4) that

$$(\mathrm{vol}\,(B_\Lambda(L_p))^{(2|\Lambda|)^{-1}} \asymp (\mathrm{vol}\,(B_{\Lambda,R}(L_p))^{(|\Lambda|)^{-1}}. \tag{2.5}$$

Similarly we get

$$(\mathrm{vol}\,(B_\Lambda^\perp(L_p))^{(2|\Lambda|)^{-1}} \asymp (\mathrm{vol}\,(B_{\Lambda,R}^\perp(L_p))^{(|\Lambda|)^{-1}}. \tag{2.6}$$

This observation, Theorems 2.2 and 2.3 combined with (2.1) imply the following statement.

Theorem 2.4. *Let Λ have the form $\Lambda = \bigcup_{s \in S} \rho(s)$, where $S \subset \mathbb{Z}_+^d$ is a finite set. Then, for any $1 \leq p < \infty$, we have*

$$\mathrm{vol}\,(B_\Lambda(L_p))^{(2|\Lambda|)^{-1}} \asymp |\Lambda|^{-1/2}.$$

We now proceed to the main results of this section. First we denote $N := 2|\Delta Q_n|$.

Theorem 2.5. *In the case $d = 2$ we have*

$$(\mathrm{vol}\,(B_{\Delta Q_n}(L_\infty)))^{1/N} \asymp (2^n n^2)^{-1/2}; \tag{2.7}$$

$$(\mathrm{vol}\,(B_{\Delta Q_n}^\perp(L_1)))^{1/N} \asymp 2^{-n/2}. \tag{2.8}$$

It is interesting to compare the first relation in Theorem 2.5 with the following estimate for $1 \leq p < \infty$ that follows from Theorem 2.4

$$(\mathrm{vol}\,(B_{\Delta Q_n}(L_p)))^{1/N} \asymp (2^n n)^{-1/2}. \tag{2.9}$$

We see that in the case $\Lambda = \Delta Q_n$, unlike the case $\Lambda = \Pi(N_1, \dots, N_d)$, the estimate for $p = \infty$ is different from the estimate for $1 \leq p < \infty$.

Proof of Theorem 2.5. We begin with the proof of the lower estimate in (2.7). We shall formulate and prove it in a more general form.

Lemma 2.1. *Let $\Lambda \subseteq [-2^n, 2^n]^d$ and $N := 2|\Lambda|$. Then*

$$(\mathrm{vol}\,(B_\Lambda(L_\infty)))^{1/N} \geq C(d)(Nn)^{-1/2}.$$

Proof. We use the following result of Gluskin [G].

Theorem 2.6. *Let* $Y = \{y_1, \dots, y_M\} \subset \mathbb{R}^N$, $\|y_i\| = 1$, $i = 1, \dots, M$, *and*

$$W(Y) := \{x \in \mathbb{R}^N : |(x, y_i)| \le 1, \quad i = 1, \dots, M\}.$$

Then

$$(\mathrm{vol}\,(W(Y)))^{1/N} \ge C(\log(M/N))^{-1/2}.$$

Consider the following lattice on $\mathbb{T}^d$:

$$G_n := \{x(l) = (l_1, \dots, l_d)\pi 2^{-n-1}, \quad 1 \le l_j \le 2^{n+2}, \quad l_j \in \mathbb{N}, \quad j = 1, \dots, d\}.$$

It is clear that $|G_n| = 2^{d(n+2)}$. It is well-known that for any $f \in \mathcal{T}([-2^n, 2^n]^d)$ one has

$$\|f\|_\infty \le C_1(d) \max_{x \in G_n} |f(x)|.$$

Thus, for any $\Lambda \subseteq [-2^n, 2^n]^d$ we have

$$\{A(f) : f \in \mathcal{T}(\Lambda), \quad |f(x)| \le C_1(d)^{-1}, \quad x \in G_n\} \subseteq B_\Lambda(L_\infty). \tag{2.10}$$

Further

$$\begin{aligned} |f(x)|^2 &= \Big|\sum_{k \in \Lambda} \hat{f}(k) e^{i(k,x)}\Big|^2 \\ &= \Big(\sum_{k \in \Lambda} \mathrm{Re}\hat{f}(k)\cos(k,x) - \mathrm{Im}\hat{f}(k)\sin(k,x)\Big)^2 \\ &\quad + \Big(\sum_{k \in \Lambda} \mathrm{Re}\hat{f}(k)\sin(k,x) + \mathrm{Im}\hat{f}(k)\cos(k,x)\Big)^2. \end{aligned}$$

We associate with each point $x \in G_n$ two vectors $y^1(x)$ and $y^2(x)$ from $\mathbb{R}^N$:

$$\begin{aligned} y^1(x) &:= \{(\cos(k,x), -\sin(k,x)), \quad k \in \Lambda\}, \\ y^2(x) &:= \{(\sin(k,x), \cos(k,x)), \quad k \in \Lambda\}. \end{aligned}$$

Then

$$\|y^1(x)\|^2 = \|y^2(x)\|^2 = |\Lambda|$$

and

$$|f(x)|^2 = (A(f), y^1(x))^2 + (A(f), y^2(x))^2.$$

It is clear that the condition $|f(x)| \le C_1(d)^{-1}$ is satisfied if

$$|(A(f), y^i(x))| \le 2^{-1/2} C_1(d)^{-1}, \quad i = 1, 2.$$

Let now

$$Y := \{y^i(x)/\|y^i(x)\|, \quad x \in G_n, \quad i = 1, 2\}.$$

Then $M = 2^{d(n+2)+1}$ and by Theorem 2.6

$$(\mathrm{vol}\,(W(Y)))^{1/N} \gg (\log(M/N))^{-1/2} \gg n^{-1/2}. \tag{2.11}$$

Using the fact that the condition

$$|(A(f), y^i(x))| \leq 1$$

is equivalent to the condition

$$|(A(f), y^i(x)/\|y^i(x)\|)| \leq (N/2)^{-1/2},$$

we get from (2.10) and (2.11)

$$(\mathrm{vol}\,(B_\Lambda(L_\infty)))^{1/N} \gg (Nn)^{-1/2}.$$

This completes the proof of Lemma 2.1

Lemma 2.1 implies immediately the lower estimate in (2.7) because in this case $|\Delta Q_n| \asymp 2^n n$. We emphasize that Lemma 2.1 shows that a lower estimate similar to (2.7) holds for any $\Lambda \subseteq [-2^n, 2^n]^2$ with $|\Lambda| \asymp |\Delta Q_n|$ and therefore it does not depend on the geometry of Λ.

We now proceed to the proof of the upper estimate in (2.7). This proof uses the geometry of ΔQ_n. Comparing the estimate (2.7) with Theorem 2.1 we conclude that the upper estimate in (2.7) cannot be generalized for all $\Lambda \subseteq [-2^n, 2^n]^2$ with $|\Lambda| \asymp |\Delta Q_n|$. We prove first the lower estimate in (2.8). We shall use the following lemma that follows directly from Lemma 2.4 in [T6].

Lemma 2.2. *Let $d = 2$. For any $f \in \mathcal{T}(\Delta Q_n)$ satisfying*

$$\|\delta_s(f)\|_\infty \leq 1, \quad \|s\|_1 = n, \quad \delta_s(f) := \sum_{k \in \rho(s)} \hat{f}(k) e^{i(k,x)},$$

we have

$$E^\perp_{Q_n}(f)_1 \leq C.$$

Let us denote

$$H_\infty(\Delta Q_n) := \{f \in \mathcal{T}(\Delta Q_n) : \|\delta_s(f)\|_\infty \leq 1\}$$

and

$$A(H_\infty(\Delta Q_n)) := \{A(f) : f \in H_\infty(\Delta Q_n)\}.$$

Lemma 2.2 implies that ($N = 2|\Delta Q_n|$)

$$(\mathrm{vol}\,(B^\perp_{\Delta Q_n}(L_1)))^{1/N} \gg (\mathrm{vol}\,(A(H_\infty(\Delta Q_n))))^{1/N}. \tag{2.12}$$

Using Theorem 2.1 we get

$$
\begin{aligned}
(2.13) \qquad (\mathrm{vol}\,(A(H_\infty(\Delta Q_n))))^{1/N} &= \Big(\prod_{\|s\|_1=n} \mathrm{vol}(A(\mathcal{T}(\rho(s))_\infty))\Big)^{1/N} \\
&\gg 2^{-n/2},
\end{aligned}
$$

where

$$\mathcal{T}(\rho(s))_\infty := \{t \in \mathcal{T}(\rho(s)) : \|t\|_\infty \le 1\}.$$

The lower estimate in (2.8) follows from (2.12) and (2.13).

Using Theorem 2.3 and relations (2.2), (2.5), and (2.6) we complete the proof of Theorem 2.5.

3. Estimates of the ϵ-entropy

In this section we use the results from Section 2 for obtaining new lower estimates for the ϵ-entropy of the classes $W^r_{1,\alpha}$ in L_p, $1 \le p \le \infty$. We confine ourselves to the case of functions of two variables. We prove the following theorem here.

Theorem 3.1. *The following relations hold*

$$
\begin{aligned}
(3.1) \qquad &\epsilon_m(W^r_{1,\alpha}, L_p) \asymp m^{-r}(\log m)^{r+1/2}, \\
&1 \le p < \infty, \quad r > \max(1/2, 1 - 1/p);
\end{aligned}
$$

$$(3.2) \qquad \epsilon_m(W^r_{1,0}, L_\infty) \asymp m^{-r}(\log m)^{r+1}, \quad r > 1.$$

Proof. We first prove the upper estimates. We shall prove the estimates for a bigger class H^r_1. We remind the definition of this class. Let, for a positive integer l, $\Delta^l_t f(y), t, y \in [0, 2\pi]$ denote the l-th difference of f with step t, and

$$\Delta^l_{(t_1,t_2)} f(x_1, x_2) = \Delta^l_{t_2}(\Delta^l_{t_1} f(x_1, x_2))$$

be the mixed l-th difference with the step t_j in the variable x_j, $j = 1, 2$. Define

$$
\begin{aligned}
H^r_1 = \Big\{f : \|f\|_1 \le 1, \|\Delta^l_{t_1} f(\cdot, x_2)\|_1 \le |t_1|^r, \|\Delta^l_{t_2} f(x_1, \cdot)\|_1 \le |t_2|^r, \\
\|\Delta^l_{(t_1,t_2)} f(x_1, x_2)\|_1 \le |t_1 t_2|^r\Big\},
\end{aligned}
$$

with $l = [r] + 1$. For the embedding of the classes W^r_1 into H^r_1, see [Te].

It proved to be useful in studying approximation of functions with a bounded mixed derivative to consider along with the L_p-norms the Besov type norms. Let $V_n(t)$ be the de la Vallée-Poussin polynomials, $t \in [0, 2\pi]$. We define

$$\begin{aligned} \mathcal{A}_0(t) &:= 1, \\ \mathcal{A}_1(t) &:= V_1(t) - 1, \\ \mathcal{A}_n(t) &:= V_{2^{n-1}}(t) - V_{2^{n-2}}(t), \quad n \geq 2, \end{aligned}$$

and for $x = (x_1, x_2)$, $s = (s_1, s_2)$

$$\mathcal{A}_s(x) := \mathcal{A}_{s_1}(x_1)\mathcal{A}_{s_2}(x_2).$$

Consider the convolution operator A_s with the kernel $\mathcal{A}_s(x)$,

$$A_s(f) := f * \mathcal{A}_s,$$

and define the $B_{p,\theta}$-norm as follows

$$\|f\|_{B_{p,\theta}} := \left(\sum_s \|A_s(f)\|_p^\theta\right)^{1/\theta}, \quad 1 \leq \theta < \infty.$$

It is proved in [T2] that

$$\epsilon_m(H_1^r, B_{\infty,2}) \ll m^{-r}(\log m)^{r+1/2}, \quad r > 1. \tag{3.3}$$

The corresponding proof from [T2] implies also that

$$\epsilon_m(H_1^r, B_{\infty,1}) \ll m^{-r}(\log m)^{r+1}, \quad r > 1. \tag{3.4}$$

Using the obvious estimate

$$\|f\|_\infty \leq \|f\|_{B_{\infty,1}} \tag{3.5}$$

we get the upper estimate in (3.2) from (3.4) and (3.5).

We now proceed to the upper estimate in (3.1). The proof in [T2] was based on the following known estimate of the entropy numbers of an octahedron B_1^n in ℓ_∞^n (see [H], [M]).

Lemma 3.1. *The following estimates hold*

$$\epsilon_m(B_1^n, \ell_\infty^n) \ll \begin{cases} m^{-1}(\log(n/m))^2, & 2m \leq n \\ n^{-1}2^{-m/n}, & 2m > n. \end{cases}$$

One can use instead of Lemma 3.1 the following result (see [S]).

Lemma 3.2. *The following estimates hold*

$$\epsilon_m(B_1^n, \ell_p^n) \ll \begin{cases} m^{1/p-1}(\log(1+n/m))^{1-1/p}, & m \le n \\ n^{1/p-1}2^{-m/n}, & m > n. \end{cases}$$

Then, similarly to the proof in [T2], one gets instead of (3.3) the estimate

$$\epsilon_m(H_1^r, B_{\infty,2}) \ll m^{-r}(\log m)^{r+1/2}, \quad r > 1 - 1/p. \tag{3.6}$$

Next, we use the well-known corollary of the Littlewood-Paley inequality (see, for instance [Te])

$$\|f\|_p \ll \|f\|_{B_{p,2}}, \quad 2 \le p < \infty. \tag{3.7}$$

The upper estimate in (3.1) for $2 \le p < \infty$ follows from (3.6) and (3.7). The corresponding upper estimate for $1 \le p < 2$ follows from already considered case $p = 2$.

We now proceed to the lower estimates. We begin with the lower estimate in (3.1) for $p = 2$. We use the following simple well-known fact on a minimal ϵ-covering (see [P, p.57]). Let a Banach space E be the $\mathbb{R}^d$ equipped with a norm $\|\cdot\|_E$. Denote the corresponding unit ball by B_E. Let $N_\epsilon(F, E)$ be the minimal number of balls of radius ϵ needed for covering F. Then, for any body F with existing $\mathrm{vol}(F)$, we have

$$N_\epsilon(F, E) \ge \epsilon^{-d} \frac{\mathrm{vol}(F)}{\mathrm{vol}(B_E)}. \tag{3.8}$$

For a fixed natural number n we consider the orthogonal projector $S_{\Delta Q_n}$ onto $\mathcal{T}(\Delta Q_n)$. Then, for any m,

$$\epsilon_m(W_{1,\alpha}^r, L_2) \ge \epsilon_m(S_{\Delta Q_n}(W_{1,\alpha}^r), L_2 \cap \mathcal{T}(\Delta Q_n)). \tag{3.9}$$

Next, it is easy to understand that

$$S_{\Delta Q_n}(W_{1,\alpha}^r) = \Big\{ f \in \mathcal{T}(\Delta Q_n) : \ f = F_r(\cdot, \alpha) * \varphi(\cdot), \\ \varphi \in \mathcal{T}(\Delta Q_n), \quad E_{\Delta Q_n}^{\perp}(\varphi)_1 \le 1 \Big\}.$$

We observe that the operator of convolution with $F_0(x, \alpha)$ defined on $\mathcal{T}(\Delta Q_n)$ induces an orthogonal operator in the space $\mathbb{R}^{2|\Delta Q_n|}$ of Fourier coefficients $A(f)$. Therefore,

$$\mathrm{vol}(\{A(f) : f \in S_{\Delta Q_n}(W_{1,\alpha}^r)\})^{(2|\Delta Q_n|)^{-1}} \gg 2^{-rn}(\mathrm{vol}((B_{\Delta Q_n}^{\perp}(L_1)))^{(2|\Delta Q_n|)^{-1}}$$

Applying Theorem 2.5 we get

$$\mathrm{vol}(\{A(f) : f \in S_{\Delta Q_n}(W_{1,\alpha}^r)\})^{(2|\Delta Q_n|)^{-1}} \gg 2^{-n(r+1/2)}. \tag{3.10}$$

Further,

$$(3.11)\qquad (\operatorname{vol}\{A(f) : f \in \mathcal{T}(\Delta Q_n),\quad \|f\|_2 \le 1\})^{(2|\Delta Q_n|)^{-1}} \ll (2^n n)^{-1/2}.$$

Thus, the relations (3.8)–(3.11) imply

$$(3.12)\qquad N_\epsilon(W^r_{1,\alpha}, L_2)^{(2|\Delta Q_n|)^{-1}} \gg \epsilon^{-1} 2^{-rn} n^{1/2}.$$

Specifying $m = 2|\Delta Q_n|$ we get from (3.12)

$$\epsilon_m \gg 2^{-rn} n^{1/2} \asymp m^{-r} (\log m)^{r+1/2}.$$

It is clear that the case of general m follows from the special case $m = 2|\Delta Q_n|$, $n \in \mathbb{N}$, which has been considered above. So, we have established the lower estimate in (3.1) for $p = 2$. It implies the corresponding lower estimate for all $p \ge 2$.

Let us prove the lower estimate in (3.1) for $p = 1$. We use the following interpolation inequality for the entropy numbers (see [Pi])

$$(3.13)\qquad \epsilon_{2m-1}(W^r_{1,\alpha}, L_2) \le 2\epsilon_m(W^r_{1,\alpha}, L_1)^{\frac{p-2}{2(p-1)}} \epsilon_m(W^r_{1,\alpha}, L_p)^{\frac{p}{2(p-1)}}$$

with $p > 2$ such that $1 - 1/p < r$. The lower estimate for the left hand side of (3.13) and the upper estimate for $\epsilon_m(W^r_{1,\alpha}, L_p)$, $r > 1 - 1/p$, have already been proved above. Substituting these estimates into (3.13) we obtain the required lower estimate for $\epsilon_m(W^r_{1,\alpha}, L_1)$. This completes the proof of the lower estimate in (3.1).

We now proceed to the lower estimate in (3.2). Let $M_\epsilon(F, E)$ denote the maximal number of points $x_i \in F$ such that $\|x_i - x_j\|_E \ge \epsilon$, $i \ne j$. The following simple inequality is well-known

$$(3.14)\qquad N_\epsilon(F, E) \le M_\epsilon(F, E) \le N_{\epsilon/2}(F, E).$$

Alike the above case we shall carry out the proof for m of a special form: $m = 2|\Delta Q_n|$. Using Theorem 2.5 and the relation (3.8) we shall get the following analog of (3.12):

$$(3.15)\qquad N_\epsilon(\mathcal{T}(\Delta Q_n)^\perp_1, L_2)^{(2|\Delta Q_n|)^{-1}} \gg \epsilon^{-1} n^{1/2},$$

where

$$\mathcal{T}(\Delta Q_n)^\perp_1 = \{f \in \mathcal{T}(\Delta Q_n) : E^\perp_{\Delta Q_n}(f)_1 \le 1\}.$$

By (3.14) and (3.15) we conclude that there are 2^m polynomials $\{t_j\}_{j=1}^{2^m}$ from $\mathcal{T}(\Delta Q_n)$ such that

$$(3.16)\qquad E^\perp_{\Delta Q_n}(t_j)_1 \le 1, \quad j = 1, \dots, 2^m;$$

$$\|t_i - t_j\|_2^2 \gg n, \quad i \neq j. \tag{3.17}$$

Let $t_j^\perp \in \mathcal{T}(\Delta Q_n)^\perp$, $j = 1, \dots, 2^m$, be such that

$$\|t_j - t_j^\perp\|_1 \leq 2. \tag{3.18}$$

Consider the following collection of functions

$$\varphi_j := (t_j - t_j^\perp)/2, \quad f_j := F_r(\cdot, 0) * \varphi_j(\cdot), \quad j = 1, \dots, 2^m.$$

Then

$$f_j \in W^r_{1,0}, \quad j = 1, \dots, 2^m.$$

We now estimate from below the quantities $\|f_i - f_j\|_\infty$ for $i \neq j$. Consider the inner products

$$a_{ij} := \langle f_i - f_j, \varphi_i - \varphi_j \rangle.$$

On the one hand, by (3.18) we have

$$a_{ij} \leq 2\|f_i - f_j\|_\infty. \tag{3.19}$$

On the other hand,

$$a_{ij} = \sum_k \hat{F}_r(k, 0)|\hat{\varphi}_i(k) - \hat{\varphi}_j(k)|^2 \gg 2^{-rn}\|t_i - t_j\|_2^2. \tag{3.20}$$

Thus, by (3.17), (3.19), and (3.20), we get

$$\|f_i - f_j\|_\infty \gg 2^{-rn} n, \quad i \neq j.$$

Therefore,

$$\epsilon_m(W^r_{1,0}, L_\infty) \gg 2^{-rn} n \asymp m^{-r}(\log m)^{r+1}.$$

This completes the proof of Theorem 3.1.

4. The discrete L_∞-norm for polynomials from $\mathcal{T}(\Lambda)$

We begin with the following conditional statement.

Theorem 4.1. *Assume that a finite set $\Lambda \subset \mathbb{Z}^d$ has the following properties:*

$$(\mathrm{vol}\,(B_\Lambda(L_\infty)))^{1/N} \leq K_1 N^{-1/2}, \quad N := 2|\Lambda|, \tag{4.1}$$

and a set $\Omega = \{x^1, \dots, x^M\}$ satisfies the condition

$$\forall f \in \mathcal{T}(\Lambda) \qquad \|f\|_\infty \leq K_2\|f\|_{\infty,\Omega}, \quad \|f\|_{\infty,\Omega} := \max_{x \in \Omega}|f(x)|. \tag{4.2}$$

Then, there exists an absolute constant $C > 0$ such that

$$M \geq N e^{C(K_1K_2)^{-2}}.$$

Proof. Using the assumption (4.2) we derive from Theorem 2.6, in the same way as we proved Lemma 2.1, the following volume estimate

$$(\mathrm{vol}\,(B_\Lambda(L_\infty)))^{1/N} \geq C_1 K_2^{-1}(N\log(M/N))^{-1/2} \tag{4.3}$$

with an absolute constant $C_1 > 0$. Comparing (4.3) with the assumption (4.1) we get

$$M \geq N e^{C(K_1K_2)^{-2}}, \quad C = C_1^2.$$

Theorem 4.1 is proved.

We now give some corollaries from Theorem 4.1.

Theorem 4.2. *Assume a finite set $\Omega \subset \mathbb{T}^2$ has the following property:*

$$\forall t \in \mathcal{T}(\Delta Q_n) \qquad \|t\|_\infty \leq K_2 \|t\|_{\infty,\Omega}. \tag{4.4}$$

Then

$$|\Omega| \geq 2|\Delta Q_n| e^{Cn/K_2^2}$$

with an absolute constant $C > 0$.

Proof. By Theorem 2.5 (see (2.7)) we have

$$(\mathrm{vol}\,(B_{\Delta Q_n}(L_\infty)))^{1/N} \leq C(2^n n^2)^{-1/2} \leq C n^{-1/2} N^{-1/2}$$

with an absolute constant $C > 0$. Using Theorem 4.1 we obtain

$$|\Omega| \geq 2|\Delta Q_n| e^{Cn/K_2^2}.$$

This proves Theorem 4.2.

Remark 4.1. In the particular case $K_2 = bn^\alpha$, $0 \leq \alpha \leq 1/2$, Theorem 4.2 gives

$$|\Omega| \geq 2|\Delta Q_n| e^{Cb^{-2}n^{1-2\alpha}}.$$

Corollary 4.1. *Let a set $\Omega \subset \mathbb{T}^d$ have the property:*

$$\forall t \in \mathcal{T}(\Delta Q_n) \qquad \|t\|_\infty \leq bn^\alpha \|t\|_{\infty,\Omega}$$

with some $0 \leq \alpha < 1/2$. Then

$$|\Omega| \geq C_3 2^n n e^{Cb^{-2}n^{1-2\alpha}} \geq C_1(b,d,\alpha)|Q_n| e^{C_2(b,d,\alpha)n^{1-2\alpha}}$$

Corollary 4.2. *Let a set $\Omega \subset \mathbb{T}^2$ be such that $|\Omega| \le C_5|Q_n|$. Then*

$$\sup_{f\in\mathcal{T}(Q_n)} \|f\|_\infty/\|f\|_{\infty,\Omega} \ge Cn^{1/2}.$$

Proof. Denote

$$K_2 := \sup_{f\in\mathcal{T}(Q_n)} \|f\|_\infty/\|f\|_{\infty,\Omega}.$$

Then the condition (4.4) of Theorem 4.2 is satisfied with this K_2. Therefore, by Theorem 4.2,

$$2|\Delta Q_n|e^{Cn/K_2^2} \le |\Omega| \le C_5|Q_n|.$$

This implies that

$$K_2 \gg n^{1/2}.$$

Remark 4.2. One can derive from the known results on recovery of functions from the classes W^r_∞ (see [T7], [T8]) that for any n there is a set $\Omega_n \subset \mathbb{T}^d$ such that $|\Omega_n| \le C|Q_n|$ and

$$\sup_{f\in\mathcal{T}(Q_n)} (\|f\|_\infty/\|f\|_{\infty,\Omega_n}) \ll n^{d-1}.$$

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