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BRIEF COMMUNICATIONS

On the Validity for Frames of a Result Concerning Orthogonal Systems

B. S. Kashin and T. Yu. Kulikova

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In this paper, we establish lower bounds for canonical n-term approximations with respect to frames of general form of a family of functions which is a natural one from the point of view applications. The results proved in [1] for orthogonal bases in the spaces $L^2(0,1)$ are generalized. The expediency of such a step is connected mainly with the systematic use of frames and nonorthogonal bases in applied problems of image compression. Let us introduce the necessary definitions.

Definition 1. Suppose that H is a separable Hilbert space. A system of nonzero elements $\Phi = \{\varphi_j\}_{j=1}^{\infty} \subset H$ is called a *frame* if the following inequalities are valid:

$$A\|v\|^2 \le \sum_{j=1}^{\infty}(v,\varphi_j)^2 \le B\|v\|^2 \qquad \forall\, v \in H,$$

where $0 < A \le B < \infty$ are absolute constants, and $\|\cdot\|$ and $(\cdot\,,\,\cdot)$ are the norm and the inner product, respectively. The constants B and A are called, respectively, the *upper* and the *lower bound* of the frame Φ.

Obviously, any complete orthonormal system in H, as well as an arbitrary Riesz basis, is a frame. For a criterion for a system Φ to be a frame, see [2]. For each frame $\Phi = \{\varphi_j\}_{j=1}^{\infty}$, the dual frame $\widetilde{\Phi} = \{\widetilde{\varphi}_j\}_{j=1}^{\infty}$ is defined, and each element $f \in H$ can be expressed as the series convergent in norm

$$f = \sum_{j=1}^{\infty}(f,\widetilde{\varphi}_j)\varphi_j.$$

This series is called the *canonical expansion* of an element $f \in H$ with respect to the frame Φ (for more details, see [3]).

Definition 2. By the *best canonical n-term approximation* $(n = 1, 2, \dots)$ of an element $f \in H$ with respect to the frame Φ we mean

$$e_n(f,\Phi,H) = \inf_{\Lambda\subset\mathbb{N},\ \#\Lambda\le n}\Big\|f - \sum_{j\in\Lambda}(f,\widetilde{\varphi}_j)\varphi_j\Big\|$$

(here $\Lambda \subset \mathbb{N}$ is a subset of the natural numbers and $\#$ is the number of elements in the set).

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Further, if K is a subset of H, then by the *best canonical n-term approximation* of class K we mean

$$e_n(K, \Phi, H) = \sup_{f \in K} e_n(f, \Phi, H).$$

Consider a one-parameter family $\mathbb{X} = \{\chi_t\}$, $t \in [0, a]$, of characteristic functions of the subsets of the unit cube I^d in the space $\mathbb{R}^d$ such that

$$\begin{gathered} \chi_t = \chi_{\Omega_t} = \begin{cases} 1, & x \in \Omega_t, \\ 0, & x \notin \Omega_t, \end{cases} \\ \mu_d\{\Omega_t\} = t, \qquad \Omega_{t_1} \subset \Omega_{t_2} \quad \text{for } 0 \le t_1 < t_2 \le a \end{gathered} \tag{1}$$

(here μ_d is the Lebesgue measure of a set in I^d).

As an example of a family with property (1), we can point out the set of characteristic functions of the intervals $(0, t)$, $0 \le t \le 1$, lying in $L^2(0, 1)$ or the family of characteristic functions of concentric balls in $L^2(I^d)$. Our problem consists in determining lower bounds for the best canonical n-term approximations for the classes with property (1).

Theorem 1. *Suppose that $\Phi = \{\varphi_j\}_{j=1}^{\infty} \subset L^2(I^d)$ is a frame with bounds A and B. Then the canonical n-term approximation with respect to the frame Φ of any class of functions $\mathbb{X} = \{\chi_t\}$, $t \in [0, a]$, with property* (1) *has the following estimate from below*:

$$e_n(\mathbb{X}, \Phi, L^2(I^d)) \ge \big(C(A, B, a)\big)^{-n}, \qquad n = 1, 2, \dots,$$

where $C(A, B, a) > 0$ is a constant depending only on A, B, and a.

Theorem 2. *Suppose that $\Phi = \{\varphi_j\}_{j=1}^{\infty}$ is a Riesz basis in $L^2(I^d)$ such that*

$$\|\varphi_j\|_{L^\infty(I^d)} \le M, \quad \|\widetilde{\varphi}_j\|_{L^\infty(I^d)} \le \widetilde{M}, \qquad j = 1, 2, \dots . \tag{2}$$

Then, for $n = 1, 2, \dots$, the canonical n-term approximation with respect to the basis Φ of any class $\mathbb{X}$ with property (1) *satisfies the estimate*

$$e_n(\mathbb{X}, \Phi, L^2(I^d)) \ge \frac{C(\Phi, a)}{n^{1/2}} > 0,$$

where $C(\Phi, a)$ is a constant depending only on Φ and a.

As Theorem 2 shows, the use of bases “resembling” a trigonometric system for approximation does not ensure an order of approximation better than $n^{-1/2}$ even for such a “thin” class of functions as a family with property (1). This fact implicitly recommends the use of systems of wavelet type in the applications.

The proof of Theorems 1 and 2 is obtained by modifying the proofs given in [1]. An essentially new feature is the nonuniqueness of expansions with respect to the frame. This manifests itself, in particular, in that the canonical expansion of a polynomial with respect to a frame is not always a polynomial. This results, for example, in that the lower bound from Theorem 2 deteriorates from $\dots \ge cn^{-1/2}$ to $\dots \ge cn^{-1}$ if one uses the same method for arbitrary frames with property (2).

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(B. S. Kashin) V. A. Steklov Mathematics Institute,
Russian Academy of Sciences, Moscow
E-mail: kashin@mi.ras.ru
(T. Yu. Kulikova) Actuarial Financial Research Center
E-mail: koulikt@mi.ras.ru